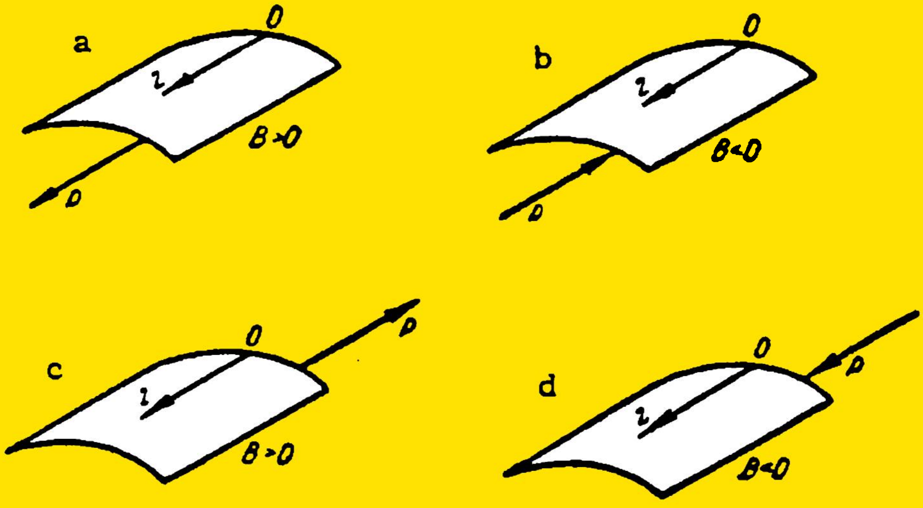
V.Z. VLASOV

THIN-WALLED ELASTIC BEAMS



V. Z. VLASOV

THIN-WALLED ELASTIC BEAMS

(Tonkostennye uprugie sterzhni)

Second edition
revised and augmented

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VASILII ZAKHAROVICH VLASOV

(1906 – 1958)

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This second edition of "Thin-walled Beams" is published posthumously. Vasilii Zakharovich Vlasov passed away on 7 August 1958. Structural mechanics has thereby lost one of its most brilliant representatives. Engineering was deprived of a prominent engineer and science of an outstanding scientist. The death of V. Z. Vlasov is a heavy loss to world science.

Vasilii Zakharovich was born on 24 February 1906, in the village of Kareevo in the Tarusa District of the Kaluga Region. In 1924 he entered the Faculty of Geodesy of the Moscow Land Survey Institute from which, in 1926, he passed to the Civil Engineering Faculty of the Moscow Higher Engineering College (MVTU). In 1930 he graduated from the Higher Civil Engineering College (VISU)–which had separated from the MVTU–as a Structural Engineer for bridges and constructions. Immediately after leaving the VISU (later renamed the Moscow Civil Engineering College–MISI) he began to lecture in the College of Civil Engineering and also engaged in scientific research at the All-Union Institute of Construction (presently the Central Scientific Research Institute for Industrial Construction-TsNIPS). Vasilii Zakharovich lectured at the MISI till the end of his life. He worked at the TsNIPS up to 1951. From 1932 to 1942 Vasilli Zakharovich lectured in the V. V. Kuibyshev Military-Engineering Academy. From 1946 he headed the Civil Engineering section of the Mechanics Institute of the Academy of Sciences of the USSR.

In 1937 Vasilii Zakharovich was awarded the degree of Doctor of Technical Sciences for his inaugural thesis "The Structural Mechanics of Shells" (Stroitel'naya mekhanika obolochek.– Moskva, Stroiizdat, 1936) presented at the MISI.

In 1943 Vasilii Zakharovich was elected a member of the Moscow Mathematical Society, and in 1953 he was elected Corresponding Member of the Academy of Sciences of the USSR.

V. Z. Vlasov devoted an entire lifetime of scientific activity to the theory of thin-walled structures. The thin-walled structure is the most modern and optimal, designed for minimum weight and maximum stiffness. This may be a roof of an industrial building, the main girder of a bridge, the wing and fuselage of an airplane, the hull of a surface vessel or a submarine, as well as of a rocket.

Vlasov's outstanding achievement was his formulation of an approximate theory of shells which can be easily applied to the design of structures. By a successful combination of the methods of the mathematical theory of elasticity, the theory of strength of materials and structural mechanics he succeeded in obtaining exceedingly simple and clear results in the theory of shells.

The main results of Vlasov's investigations are contained in his books "Tonkostennye uprugie sterzhni" (Thin-walled Elastic Beams) (First Edition, 1940), "Stroitel'naya mekhanika tonkostennykh prostranstvennykh sistem" (The Structural Mechanics of Thin-walled Spatial Systems) (1949) and "Obshchaya teoriya obolochek i ee prilozheniya v tekhnike" (The General Theory of Shells and its Application in Engineering) (1949).

The first of these books was awarded a Stalin Prize, First Class, in 1941 and the two others a Stalin Prize, Second Class, in 1950. The earlier monographs "Novyi metod rascheta tonkostennykh prizmaticheskikh skladchatykh pokrytii i obolochek" (A New Method of Designing Thin-walled Prismatical Hipped Roofs and Shells) (1933) and "Stroitel'naya mekhanika obolochek" (The Structural Mechanics of Shells) (1936) were essentially absorbed in the last two books.

The most important results were obtained by V. Z. Vlasov in the theory of cylindrical shells of intermediate length whose contour is either curved or of the form of a broken line (hipped systems). Vlasov introduces an exceptionally simple design model in which the shell is replaced by a spatial system of an infinite number of curved arcs, connected by ties which transmit the forces but not bending or torsional moments. In other words, the shell is momentless [in a "membrane state"] in the longitudinal direction and may deflect transversely. This is the key to the behavior of a cylindrical shell of intermediate length, as Vasilii Zakharovich so elegantly demonstrated. Subsequent checking of his hypotheses has verified their complete validity.

V. Z. Vlasov reduces the design of a cylindrical shell to that of a discrete-continuous system, and the partial differential equations of the shell to a system of ordinary equations. Vlasov's variational method for the reduction of partial to ordinary differential equations is important in itself. Vlasov assigns the shell a finite number of degrees of freedom in transverse displacement and an infinite number in longitudinal displacement. The calculation of the transverse displacements is then elementary and for the longitudinal displacements we obtain differential equations of a type familiar from the theory of structures.

Such methods were worked out by Vasilii Zakharovich Vlasov for the design of shells and hipped systems of open and closed section, and the design for stiffness of cylindrical shells with one or several ribs.

The theory of thin-walled beams may be obtained from the above-mentioned theory. The basic features of the design of thin-walled structures were known before V. Z. Vlasov. It had been established that the Euler-Bernoulli technical theory of the bending of beams was not applicable to thin-walled beams because of the distortion (warping) of the section which has to do with the way of applying the statically equivalent loads to the end faces etc. The statement of the problem and its solution are very fully described in Vlasov's treatise on thin-walled beams. Design models for beams are lucidly described. In the expression for the normal stress there appears, besides the three usual terms, a term determined by the law of sectorial areas.

The theory permitted a complete solution of the problem of flexural-torsional instability and of the vibrations of thin-walled elastic beams, and also the development of methods for the design of beams with elastic and rigid connections and of beams under transverse loads.

A series of important results were obtained by Vlasov from the theory of momentless [membrane-state] shells. He gave a method of designing membrane-like shells in the form of surfaces of revolution and also shells shaped like quadric surfaces. In the latter case V. Z. Vlasov reduced the problem to the solution of a Laplace equation. Later, Vlasov examined the possibility of designing a shell by the membrane-state theory in view of its geometrical variability which leads to well-defined boundary-value problems for the initial equations (elliptic or hyperbolic).

Vlasov's last monograph "Obshchaya teoriya obolochek" (General Theory of Shells) sets forth a variant of the theory of shells which is free of kinematical hypotheses. On introducing the corresponding assumptions, a theory of thin shells is obtained from this theory. The practically very important theory of sloping shells (1944) follows as a particular case from the general symmetrical form of the equations which contain no high order terms. This theory regards the curvature as constant in the examined portion of the shell, the shell itself as nearly plane and the variations of the curvature as depending only on the displacements along the normal. In that case, the solution of the problem is reduced to a system of two fourth-order equations each involving Airy's stress function and the normal deflection. V. Z. Vlasov applied these equations to the design of shells for stability and to the design of vibrating cylindrical and spherical shells.

Of no less importance are the equations of the nonlinear theory for finite deflections, advanced by Vlasov, which allow the study of the behavior of a shell under postcritical conditions. The linear and the nonlinear equations have found an exceptionally large application in diverse problems.

V. Z. Vlasov also obtained a series of important results in the theory of elasticity. He developed the method of initial functions for the solution of spatial problems in the theory of elasticity, in particular for the solution of the problem of the thick plate. His study "Uravnenie nerazryvnosti deformatsii v krivolineinykh koordinatakh" (The Continuity [compatibility] Equations for the Strains in Curvilinear Coordinates) was published in 1950.

It is difficult to overestimate the influence of Vlasov's ideas and methods on the development of the structural mechanics of thin-walled spatial systems. By dint of a subtle engineering intuition he could pinpoint the main factor in a problem and, discarding all the secondary factors, construct a convincing design model giving all the main features of the interplay of forces in the structure; this, and his splendid mastery of mathematical technique, were the qualities by virtue of which V. Z. Vlasov obtained such lucid and practically useful results. The many diverse studies, devoted to the verification of Vlasov's main hypotheses in the theory of thin-walled beams and hipped systems, and in the theory of sloping shells, have confirmed their correctness. The results he obtained have found application in almost every field of engineering: in the design of structures, in the design of composite beams and in the design of airplane wings.

We have lost a brilliant scientist, an outstanding, sympathetic, original and very dedicated man.

E. I. Grigolyuk

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From the Foreword to the First Edition

The present book expounds the general theory of thin-walled beams widely used in construction, ship building and aircraft engineering. A thin-walled beam which has in its natural (unloaded) condition the shape of a cylindrical shell or a prismatic hipped section is considered in this theory as a continuous spatial system composed of plates capable of bearing, in each point of the middle surface, not only axial (normal and shear) stresses but moments as well. The deformation of the beam is not analyzed on the basis of the usual hypothesis of plane sections. In its stead the author uses the more general and natural hypotheses of an inflexible section contour and absence of shear stresses in the middle surface, which constitute the basis for a new law of distribution of longitudinal stresses in the cross section. This law, which the author calls the law of sectorial areas and which includes the law of plane sections as a particular case, permits the computation of stresses in the most general cases of flexural-torsional equilibrium of a beam.

Apart from theoretical investigations, the book presents results of the experimental studies designed to check the validity of this theory, conducted in 1938 at TsNIPS by D.V. Bychkov, N.G. Dobudoglo, A.R.Rzhanitsyn and S.I. Stel'makh. Further details on these experimental studies and practical calculation methods, including graphs and tables, will be published as a separate collection of reports on work done at the TsNIPS laboratory for structural mechanics (in press).

It should be stressed that the author's consideration of thin-walled beams as spatial hipped systems in flexural-torsional equilibrium compelled him to invent many new terms in order to express the new statical and geometrical concepts related to the law of sectorial areas. The author apologizes for this terminology which is in the most part arbitrary and by no means definitive.

In conclusion the author would like to thank Prof. A.A. Gvozdev, Prof. I.M. Rabinovich and Prof. N.S. Streletskii for their invaluable remarks and help in the revision of the manuscript, and the members of the research staff at the TsNIPS–D.V. Bychkov, A.L. Gol'denveizer, A.K. Mroshchinskii, Yu.V. Repman and A.R. Rzhanitsyn–for editing the manuscript and checking the formulas; he is indebted to F.A. Pern and B.P. Sinitsyna for their technical aid in the layout of the book.

V.Z. Vlasov

Moskva, 1940.

Foreword to the Second Edition

The second edition of the monograph "Thin-Walled Beams", prepared by the author 18 years after the first edition was published, includes scores of new problems in strength of materials, structural mechanics and applied elasticity such as: thin-walled beam-shells of open section consolidated by "bimomental" cross-ties; beams of closed section in which warping and deformation of the section contour are attended by shearing strain; solid beams; the bimomental theory of prestressed beams; the bimomental theory of thermal stresses; the equilibrium of beam-shells under combined loading; the problem of spatial stability; vibration of thin-walled beams and the like. All these problems are solved on the basis of the bimomental theory of warping whereby the static problem is handled by the variational method of reduction to ordinary differential equations.

The theory also applies to such structural elements which, judged by their dimensions, can be classified as beams.

The author has formulated a more general theory which permits the design of composite spatial systems of the cylindrical type and of prismatic orthotropic shells of intermediate length. This is treated in a separate work entitled "Thin-walled spatial systems".

The author would like to express his profound gratitude to V.V. Vlasov, A.K. Mroshchinskii and F.A. Pern for their great help in the preparation of the manuscript for the second edition.

V.Z. Vlasov

Moskva, 1958.

Chapter I

THEORY OF THIN-WALLED BEAM-SHELLS OF OPEN CROSS SECTION

§ 1. Classification of structural elements according to their spatial character

In the general field of modern stuctural engineering, the diagrams of basic elements can be divided according to the spatial character of these elements into four classes: (1) massive bodies, (2) plates and shells, (3) solid beams, and (4) thin-walled beams.

The first class comprises bodies whose three dimensions are comparable, such as a sphere, a cube, an ellipsoid or a parallelepided, an elastic continuum filling all space or a half-space, as well as various structural or machine parts, that are subject to local loads or thermal stresses. Problems concerning the determination of stresses and deformations in solid bodies, and methods for their exact solution are dealt with in the mathematical theory of elasticity.

The second class comprises bodies having one dimension [width] small compared with the other two [height and length] which are of the same order of magnitude. Examples of such bodies are plates, thin slabs (round, rectangular, trapezoidal, etc), shells of various shapes (cylindrical, conical, spherical, elliptical, hyperbolic, and others), and in general, most of the thin-walled structural members so widely used in structural engineering, aviation, ship building, instrument building, and other fields of engineering.

The general theory of plates and shells, which includes problems of strength and vibration, is based on geometrical hypotheses, valid for thin deformable bodies within a certain degree of approximation. The exact and approximate methods of calculation for plates and shells form the subject of an extensive separate field of modern structural engineering, which can be called the structural engineering of thin-walled three-dimensional systems.

The third class comprises bodies characterized in that two of their dimensions are of the same order of magnitude and very small compared with the third dimension. We call such bodies solid beams. In particular, a slender parallelepiped may be considered as a solid beam when its two cross-sectional dimensions (width and height of the rectangle) do not differ much.

In investigations of the phenomena of flexure and of tension in solid

beams, their particular form allows to introduce a number of geometrical hypotheses which simplify calculations. On the basis of such hypotheses, in the case of a beam deformed by transverse bending and longitudinal extension, of the six compoennts of the strain tensor only one is preserved, corresponding to elongation in the direction parallel to the beam axis. The other five components of the strain tensor are taken to vanish. The law of plane sections, which forms the basis of the elementary theory of the bending of beams, is a consequence of these geometrical hypotheses. This law states that any cross section of a beam which is initially plane stays plane after deformation.

When investigating the phenomenon of torsion we assume the so-called theory of pure torsion to be valid. This theory is based on the hypotheses that there is no extension or shearing strain in the plane of the cross section, and furthermore that there is no longitudinal extension. This theory allows to determine only the tangential stresses arising in the cross sections of the beam.

The elementary theory of extension (compression), flexure, and pure torsion is included in the general engineering theory of solid beams, which is essentially based on the application of Saint-Venant's principle. According to this principle and the adopted hypotheses, the internal forces acting on the cross sections of the beam (within the elastic limits) lead to one resultant, which is given as a vector in a 6-dimensional space of the three force components and the three components of the moment. This resultant may be replaced by an equivalent system of statically equipollent forces without changing the stress or state of strain of the mathematical model adopted for the solid beam.

Design diagrams of the third class comprise all beam systems (plane, three-dimensional, statically determinate or indeterminate) whose elements are beams that in bending obey the law of plane sections and undergo torsion according to the law of pure torsion, i. e., mainly solid beams. Courses of structural engineering of beam systems include the theory and methods of calculation of such systems.

The fourth class of design diagrams and structural elements comprises bodies which have the form of long prismatic shells. These bodies are characterized by the fact that their three dimensions are all of different order of magnitude. The thickness of the shell is small compared with any characteristic dimension of the cross section, and the cross sectional dimensions are small compared with the length of the shell. The author calls these bodies "thin-walled beams". Rolled, welded, or riveted metallic beams, columns, elements of girders and frames, etc, which are widely used in structural engineering, are examples of thin-walled beams. Many spatial engineering structures have such proportions that they can be considered as thin-walled beam structures. Among such structures there are certain types of girder-bridges and arched bridges with sufficiently rigid cross ties, suspension bridges having a roadway with a channel or I-cross section, long reinforced-concrete ribbed, cylindrical or prismatic shell-arches, bunkers, pipe lines, etc. Metal stringers, frames, and other structural members of aircraft and of ships can also be considered as thin-walled beams.

The distinctive feature of thin-walled beams is that they can undergo longitudinal extension as a result of torsion. Consequently, longitudinal

normal stresses proportional to these strains are created, which lead to an internal equilibrium of the longitudinal forces in each cross section. These complementary longitudinal normal stresses, which arise as a result of the relative warping of the section and which are not examined in the theory of pure torsion, can attain very large values in thin-walled beams with open (rigid or flexible) cross sections and also in beams with closed flexible cross sections.

In conclusion, it should be mentioned that the classification of structural elements according to their spatial features is far from being exhausted, even after systems of the thin-walled beam type have been grouped in a new independent class*.

These classifications are of a rather qualitative character. They are made according to the hypotheses which lie at the basis of the design diagrams of each class.

It is impossible to draw sharp limits between the elements of the design diagrams described above. The same structure can be classified differently depending on the external loading conditions, on the manner in which the structure is to be employed in the given structural problem, and on the degree of accuracy demanded from the calculation. Thus, for example, a thin-walled beam which has a rigid closed cross section can, in many cases, be considered as belonging to the class of solid beams insofar as its behavior under combined flexure and torsion is concerned. The complementary longitudinal normal stresses which appear in such a beam have a local character. According to the Saint-Venant principle, these stresses fall off very rapidly along the beam. If the thin-walled beam has a closed flexible (deformable) cross section, it must be regarded as a thin-walled long shell of cylindrical or prismatic form. It is then necessary to allow for complementary stresses in the shell associated with the distortion of the section, due to the bending deformation of the thin-walled beam.

In the same way, it is necessary, when solving certain special problems in the theory of solid beams, to introduce additional factors connected with the distortion of such beams and to modify, in a sense, the methods used in the theory of thin-walled beams, by generalizing the physical conception of this method to suit solid beams. Examples of such problems are rails, beams of rectangular cross section, beams and plates resting on elastic foundations, etc.

§ 2. Fundamental hypotheses. Calculation models. Flexural torsion.

1. The general theory of thin-walled beams which have rigid (open or closed) flexible cross sections was evolved by the author from an engineering theory he had worked out earlier for orthotropic cylindrical and prismatic semi-momentless [partial membrane-state] shells.

* In structural engineering courses, the principal diagrams for design and construction are divided into the following three classes according to the spatial features: (1) massive bodies, (2) plates and shells, and (3) beams and beam systems. Bodies which we call thin-walled beams, and which in effect constitute a new class with regard to design and construction, were previously included in the last class even though they basically differ from the solid beams.

We shall examine a thin-walled structure of a cylindrical or prismatic shell type, having an arbitrary cross section and consisting of a finite number of thin (flat or curved) narrow plates (Figure 1). We shall assume that the component plates of such a shell are rigidly connected along their lines of contact so that no plate is free to move with respect to its neighbor at any point of the joint. Let δ be the thickness of the shell, d any characteristic dimension of the cross section (its width or height), and l its length. When the proportions satisfy the relations

$$\frac{\delta}{d} \leqslant 0.1; \quad \frac{d}{l} \leqslant 0.1. \tag{1}$$

we can classify the given structures as long cylindrical shells. We called such shells thin-walled beams, despite the form or geometrical dimensions of the cross section.

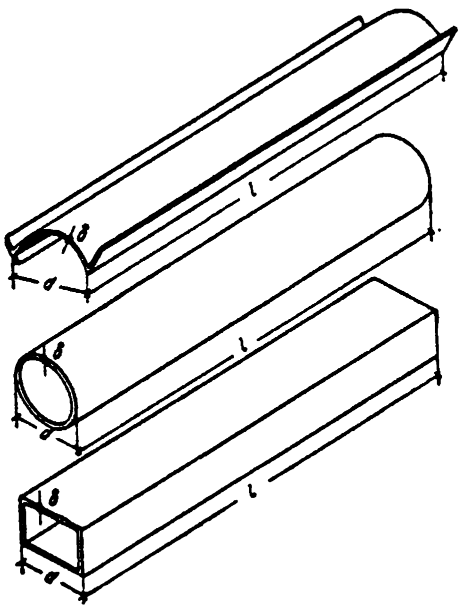


Figure 1

2. In our theory of thin-walled beams (as in general shell theory) an important role is played by the so-called middle surface of the beam, i.e., the surface lying midway through the plates composing the beam. The middle surface of a thin-walled straight beam, composed in the general case of flat plates and cylindrical shells, belongs to the class of cylindrical surfaces. Straight lines lying on the middle surface and parallel to the beam axis are the generators of this surface. The intersection of the middle surface with a plane P normal to the generators is called the profile line [contour line].

Adopting a generator and a profile line as coordinate lines, we have an orthogonal coordinate system in which the position of any point on the middle surface is uniquely specified. Henceforward we shall denote the coordinates of any point M along the generator and profile line by z and s respectively (Figure 2).

Any plane perpendicular to the beam axis (or to a generator) can be taken as the origin of the coordinate z. We shall usually use the terminal plane of the beam. In Figure 2 the origin plane, P_0, is the plane farthest from the observer. The direction from this plane to the observer is taken as the positive direction of the coordinate z. Any generator can be taken as the origin of the coordinate s. We shall usually use as origin a generator which lies in a symmetry plane (in the case of symmetrical cross

sections), or the generator coinciding with the lengthwise edge of any element of the beam profile.

3. As is known, normal and tangential stresses which arise on the cross sectional areas of the beam are an inherent feature of beams undergoing extension (or compression), bending, or torsion. These stresses are the calculated static quantities according to which the basic dimensions of the structure are determined. Normal stresses in longitudinal sections (i. e., stresses parallel to tangents of the profile line) do not play an essential role when thin-walled beams form a spatial system.

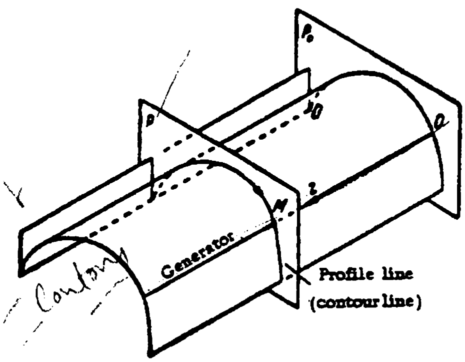


Figure 2

The author's theory of beams of open section is based on the following geometrical hypotheses:

a) a thin-walled beam of open section can be considered as a shell of rigid (undeformable) section;

b) the shearing deformations of the middle surface (characterizing the change in the angle between the coordinate lines $z =$ const, and $s =$ const) can be assumed to vanish.

By considering the beam section as rigid, we postulate that the stresses (normal or tangential) on the cross sections of the beam do not change when the external transverse load on the beam element included between the sections $z=$const and $z+dz=$const is replaced by another load, statically equivalent to the first. In other words, we assume that under an external load acting in the plane of the cross section and which results in a system of forces statically equivalent to zero for each elementary transverse slice, the beam behaves as a rigid body and, consequently, the normal and tangential stresses on the cross sections due to this load are equal to zero.

Indeed, let a load consisting of two balanced transverse forces Q be applied at two different points of an arbitrary elementary strip (Figure 3a) of a thin-walled beam of open section, stiffened at the ends by transverse plates. This kind of load on an elastic beam of flexible cross section gives rise to transverse bending moments as well as transverse and normal tension in the longitudinal sections of the beam. The vertical edge plates to which the transverse forces Q are applied are bent in their own planes. It is easy to see that this is associated with a deformation of the beam profile. According to our general theory of prismatic and cylindrical shells, this deformation will cause elongation of the longitudinal ribs of the thin-walled beam. Normal stresses in the cross sections are related to these deformations in the elastic beam. These normal stresses in their turn cause tangential stresses in the beam plates, which are determined from the equilibrium conditions for an element of a thin-walled beam. These normal and tangential stresses are in this case directly proportional to the strains which characterize the change in form and dimensions of the beam section.

If we assume that these deformations vanish, i. e., if we consider

the beam as a shell reinforced by rigid diaphragms, then the self-balancing transverse load performs no work. The beam section remains unchanged and, consequently, the stresses vanish in the beam sections.

The second hypothesis (absence of shearing strain) amounts to the assumption that the coordinate lines z=const, s=const, which are initially orthogonal, remain orthogonal after deformation, i.e., we neglect the small change in the angle between these lines, considering it a geometrical factor of minor influence on the stresses in a thin-walled beam of open section.

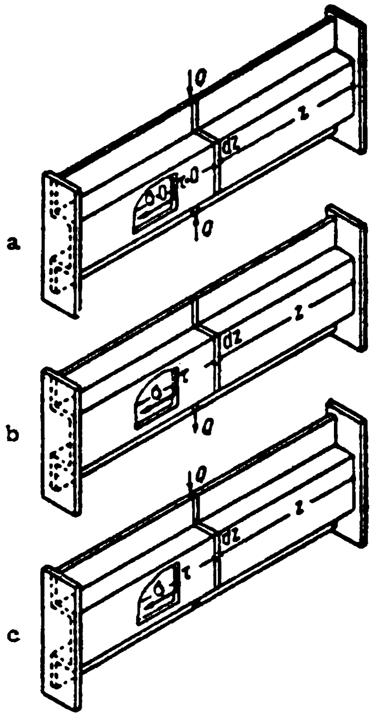


Figure 3

4. The above considerations can be generalized. The conclusion is that an external transverse load, consisting of any number of forces applied to various points of the profile line of the section z=const, can be replaced by a single statically equivalent force with a well-defined line of action in the same plane z=const (Figure 3b, c) for the purpose of determining the normal and tangential stresses σ and τ which in a thin-walled beam arise only in the cross sections. This resultant, which is represented in the general case as a specific transverse load (the load on a transverse strip of unit width), may not be displaced parallel to itself. The shift of the transverse load from one longitudinal plane of action to another is accompanied by a change in the stress and strain state of the thin-walled beam. The transverse loads, shown in Figure 4, which act on the free end of a cantilever I-beam cause various states of strain in this beam. The load shown in Figure 4a causes flexure, the load of Figure 4b causes torsion, and the load of Figure 4c, which acts in the plane of one flange only, gives rise to a compound state of strain. By the principle of independent action of different forces, these deformations can be obtained as a superposition of two states: *flexure* (Figure 4a) and *flexural torsion* (Figure 4b).

The state of flexure is described by the law of plane sections. In this state both flanges of the I-beam undergo the same bending deformation. The upper fibers of each flange are extended while the lower are compressed.

The state of flexural torsion is typical of restrained torsion, i.e., torsion in which the separate longitudinal elements (plates) of the beam undergo bending in addition to torsion. Normal stresses arise in the cross sections, in addition to the tangential stresses. The state of flexural torsion is already described in our theory by a separate law which differs essentially from the law of plane sections. In the considered example the

flexural torsion is made apparent in that both flanges of the I-beam, which are encastered at one end, are bent in their planes as a result of the torsion. This bending takes place in opposite directions for the two flanges. Under the loading shown in Figure 4b the uppermost fiber of the right flange is extended and the lowest one compressed. The uppermost fiber of the left flange is compressed and the lowest extended. As a result of this deformation, the cross sections of the I-beam do not remain plane.

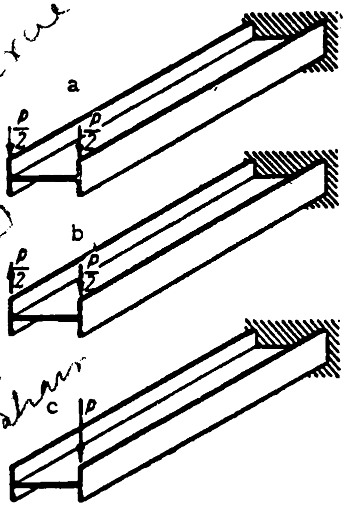


Figure 4

The distortion of the plane section caused by longitudinal displacements of its points is called the *warping of the section.*

In the case of flexural torsion, the warping of the beam is accompanied by normal stresses appearing in the cross sections in addition to the tangential stresses. In each section of the beam the normal stresses lead to a new "generalized force" which has the form of a system of mutually balancing longitudinal tensions. We call this generalized force the *longitudinal bimoment of the beam.* In the present example the bimoment consists of bending moments for the flanges of the beam which for any cross section have the same magnitude but opposite signs.

5. The theory of equilibrium for the flexural torsion of a thin-walled beam is based essentially on the rejection of the hypothesis of plane sections and of treating the beam as a one-dimensional elastic system for which the boundary conditions can be given only by Saint-Venant's principle. The principal difference between our theory and the elementary theory of bending of beams is based on hypotheses which allow the flexural torsion of beams to be investigated in the same way as the torsion of a shell-like thin-walled system, by taking into consideration the warping of the section.

If we consider the elementary section of the beam enclosed between the planes $z=$const and $z+dz=$const as a rigid body with respect to displacements in the transverse plane, then by our hypotheses this section is considered a deformable body with respect to displacements away from this plane. Points of any section $z=$const of the beam may also have longitudinal displacements which are connected with the warping of the section in addition to any displacements indicated by the law of plane sections. As a result of warping *the external longitudinal load which consists of forces applied at points of any cross section $z=$ const of the beam, differs from the transverse load in that it is not replaceable by another statically equivalent system of longitudinal forces.* Any replacement of a longitudinal load by another load statically equivalent to it amounts to subjecting the beam to an additional self-balancing longitudinal load. Such a load, which would have been applied to the beam according to the elementary theory of beam bending, does not cause

additional strain and stresses, since the cross sections remain plane in the elementary theory. A balanced longitudinal load applied to points of a cross section of a thin-walled beam can distort the cross section. As a result of such warping additional normal stresses arise in the cross sections of the beam. These stresses in the planes $z=$const lead to an internal longitudinal bimoment. These bimoments will decay away from the section at which the external self-balancing longitudinal load (external bimoment) is applied, according to Saint-Venant's principle. This fall-off can be more or less sharp, depending largely on the relative proportions and shape of the beam section. The warping of a section caused by an external longitudinal self-balancing load has a local character for a solid beam and also for a thin-walled beam having a rigid closed cross section. This warping, as well as the associated internal stresses, diminish rapidly with increasing distance from the point of application of the load. In the case of a thin-walled beam with an open section, the warping of the section caused by the action of a longitudinal self-balancing load (which is a flexural torsion) as a rule falls off very slowly with the distance from the point of application of the load. The stresses of the internal bimoment and the flexural torsion due to them are a very essential factor for such beams. Application of Saint-Venant's principle to calculations concerning thin-walled beams subjected to external torsional loads can lead to gross errors.

We shall illustrate these points by an example. Figure 5a shows a cantilever I-beam. It is eccentrically loaded at the free end by a longitudinal force P. Such a force, according to the elementary theory of beam bending, can be replaced by a statically equivalent system of longitudinal tensions which cause a central extension and pure bending in the principal planes. A longitudinal load giving rise to an extension only is shown in Figure 5b. Longitudinal loads giving rise to pure bending only are shown in Figure 5c and d. Each of these loads is represented by longitudinal forces applied at the four extreme points of the beam cross section, all of the same magnitude.

For the case of central extension, the longitudinal load is symmetrical with respect to both principal axes of the section, Ox and Oy. For the case of pure bending in the principal plane Oxz, the longitudinal load will be symmetrical with respect to the Ox axis and skew-symmetrical with respect to the Oy axis. For the case of pure bending in the principal plane Oyz, the longitudinal load is symmetrical with respect to the Oy axis and skew-symmetrical with respect to the Ox axis.

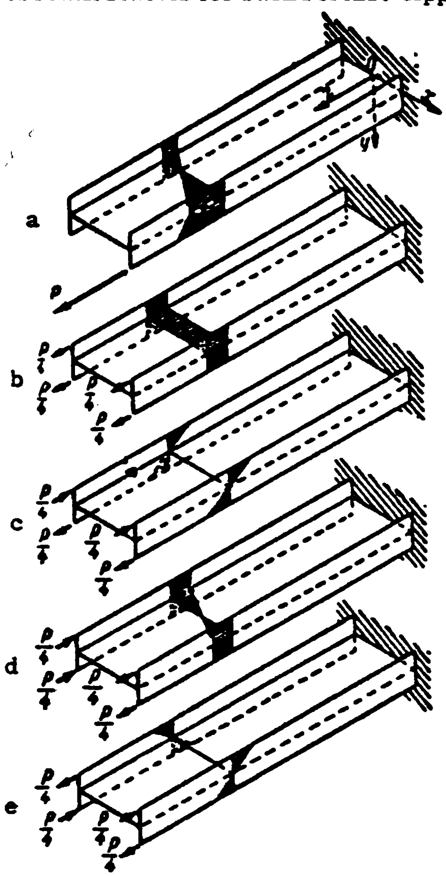


Figure 5

One component of the given load is still missing from this decomposition, done according to the plane section model. This is the system of four longitudinal forces (each of magnitude $P/4$) which have skew—symmetric positions with respect to both axes (Figure 5e). This load leads to two bending moments which have equal values and opposite signs. These moments act in the planes of the flanges of the I-beam and cause these flanges to bend in opposite directions. As a result of this bending, the cross sections of the I-beam are warped, and the warping diminishes with the distance from the point of application of the load. The warping extends farther along the I-beam the thinner its walls. As a result of such a pure bimomental longitudinal load, i.e., a load statically equivalent to zero, the thin-walled beam is twisted. Thus, torsion of a beam can take place not only under the action of transverse torsional forces (moments), but even under the sole action of longitudinal forces. This phenomenon for a thin-walled beam with an open cross section is closely allied to the warping of its sections. For this reason thin-walled beams cannot be studied by the methods of the elementary bending theory of beams, since these methods are based on the hypothesis of plane sections which expresses the application of the principle of Saint-Venant to these beams as well as to solid beams.

3. The displacements and strains. Law of sectorial areas. Generalized hypotheses for plane sections.

1. In the theory of strength of materials the line of centroids of the cross sections is usually called the axis of the beam. In the following, each line in space parallel to this axis shall be called an arbitrary axis of the beam. Let an arbitrary axis of the beam intersect the plane of the cross section z =const at the point O (Figure 6). The cross section of the beam forms part of a rectangular coordinate system Oxy with the origin at O. The Ox and Oy axes are oriented to form, together with the positive direction of the third axis Oz, a left-handed system of coordinates*. The coordinates x and y of an arbitrary point M on the profile line of a thin-walled beam are well-defined functions of the argument s.

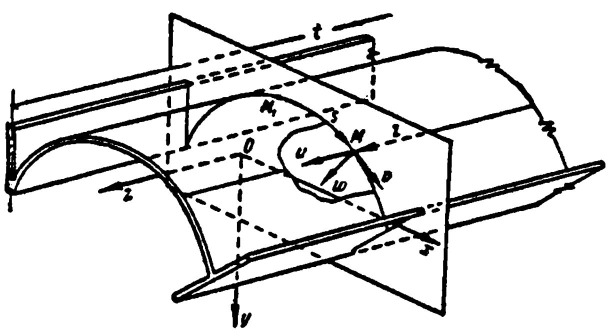


Figure 6

* A coordinate system is called left-handed if, for an observer facing the Oxy plane from the z-positive side, clockwise rotation through 90° is necessary to make Ox coincide with Oy.

2. Let the thin beam or shell undergo some deformation. As a result of this deformation any point M of the middle surface of the beam comes to occupy a new position in space. Our problem in fact is to determine the displacement of points of the middle surface, since the strained state of the beam is characterized by these displacements.

We first examine the transverse displacements, i. e., the displacements in the plane of the cross section. According to the first geometrical hypothesis of rigid cross sections, the beam is deformed so that the shape of the cross section and all its geometrical dimensions remain unchanged in its plane. We may consider the beam section to be displaced as an absolutely rigid body whose position is determined by three independent variables corresponding to the three degrees of freedom of a lamina lying on a plane.

We shall use a method similar to that used in the kinematics of a rigid body. The transverse displacement of the points of the section z = const shall be given by the displacement of some chosen point A which lies in the cross section, and by the angle of rotation of the whole section about this point. If the point A does not belong to the profile line, we assume it to be rigidly connected to it.

Let a_x and a_y be the coordinates of the point A, and $\xi(z)$ and $\eta(z)$ the corresponding projections on the Ox and Oy axes of the displacement of the point A. These projections, which are functions of z, determine a space curve. An axis of the beam with the coordinates $x=a_x$, $y=a_y$ (Figure 7) transforms into the above curve upon deformation of the beam. Let $\theta(z)$ be the angle of rotation of the section z = const in the Oxy plane about the same point A. As a function of z this angle determines the torsional angle along the beam. We shall consider the torsional angle $\theta(z)$ positive for clockwise rotation of the section z = const as observed from the positive part of the Oz axis.

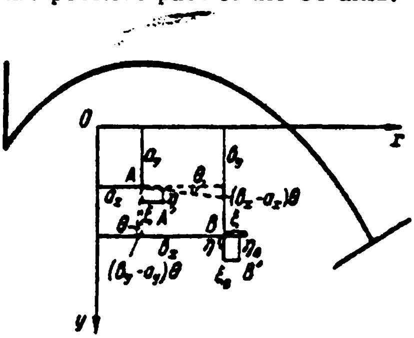


Figure 7

The displacements of any point B of the cross section of the beam in the direction of the Ox and Oy axes are denoted by ξ_B and η_B respectively. Arcs of circles are replaced by their tangents since the angle θ is small. These displacements are given by the expressions (Figure 7):

$$\left.\begin{aligned}\xi_B &= \xi-(b_y-a_y)\,\theta,\\ \eta_B &= \eta+(b_x-a_x)\,\theta,\end{aligned}\right\} \tag{3.1}$$

where b_x and b_y are the coordinates of the point B.

For small values of $\xi(z)$, $\eta(z)$ and $\theta(z)$, the displacement of the cross section of the beam in its own plane can be regarded as rotation about a certain point called the i n s t a n t a n e o u s c e n t e r o f r o t a t i o n. The position of this center in the Oxy plane is determined by the condition that it is the one point that is fixed. Identifying the arbitrary point B with the instantaneous center of rotation thus amounts to putting the

displacements ξ_B and η_B of this point equal to zero. Equation (3.1) becomes:

$$\left.\begin{aligned} \xi-(b_y-a_y)\theta=0, \\ \eta+(b_x-a_x)\theta=0, \end{aligned}\right\} \tag{3.2}$$

where b_x and b_y now denote the coordinates of the instantaneous center of rotation. In the following we shall call the instantaneous center of rotation the *torsion center*. From (3.2) we obtain the equations for the coordinates of the torsion center:

$$\left.\begin{aligned} b_x=a_x-\frac{\eta}{\theta}, \\ b_y=a_y+\frac{\xi}{\theta}. \end{aligned}\right\} \tag{3.3}$$

In the general case a spatial curve which can be called the *line of torsion centers* is determined by equations (3.3).

The position of the torsion center, in contrast to the centroid of the section, depends on the deformation of the beam, and consequently also on the external load which causes this deformation.

3. The full displacement of an arbitrary point M of the middle surface of a thin-walled beam is, naturally, a vector. It is determined by three space components. For these components we take:

1) the longitudinal displacement u; this displacement shall be considered positive if it is in the direction of increasing z.

2) The transverse tangential displacement v, directed along the tangent to the profile line. It is positive when the displacement is in the direction of increasing s.

3) The transverse normal displacement w. The positive direction of this displacement is determined by the condition that the positive directions of the displacements u, v and w from a left-handed coordinate system (see Figure 6).

In the general case all these displacements will be functions of the two independent variables z and s.

We shall now see how to determine the transverse displacements v and w. They are easily obtained from equation (3.1) which assumes the form

$$\left.\begin{aligned} \xi_s=\xi-(y-a_y)\theta, \\ \eta_s=\eta+(x-a_x)\theta, \end{aligned}\right\} \tag{3.4}$$

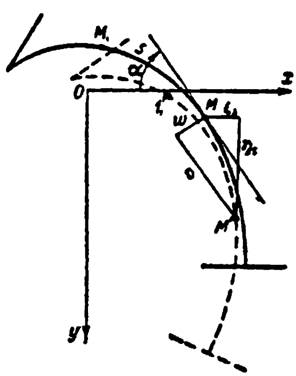


Figure 8

for an arbitrary point M of the profile line with the coordinates x and y. Here ξ_s and η_s are the displacements of the point M in the direction of the Ox and Oy axes.

Denoting by α the angle that the tangent to the profile line at the point M makes with the axis Ox, and projecting ξ_s and η_s on this tangent (Figure 8), we obtain for the tangential displacement $v(z, s)$ the expression

$$v(z, s) = \xi_s \cos\alpha + \eta_s \sin\alpha. \tag{3.5}$$

Analogously we obtain the expression for the normal component $w(z, s)$ of the total displacement,

$$w(z, s) = \eta_s \cos\alpha - \xi_s \sin\alpha. \tag{3.6}$$

Substituting in (3.5) and (3.6) the values of ξ_s and η_s from equations (3.4), we have:

$$\left.\begin{aligned} v(z, s) &= \xi\cos\alpha + \eta\sin\alpha + [(x - a_x)\sin\alpha - (y - a_y)\cos\alpha]\theta, \\ w(z, s) &= -\xi\sin\alpha + \eta\cos\alpha + [(x - a_x)\cos\alpha + (y - a_y)\sin\alpha]\theta. \end{aligned}\right\} \tag{3.7}$$

From Figure 9 it is seen that

$$\left\{\begin{aligned} (x - a_x)\sin\alpha - (y - a_y)\cos\alpha &= h(s), \\ (x - a_x)\cos\alpha + (y - a_y)\sin\alpha &= t(s), \end{aligned}\right\} \tag{3.8}$$

where $h(s)$ and $t(s)$ are, respectively, the lengths of the perpendiculars from the point A to the tangent and normal of the profile line at M.

Equations (3.7) and (3.8) can be used to give the expressions for $v(z, s)$ and $w(z, s)$ a more compact form:

$$v(z, s) = \xi(z)\cos\alpha(s) + \eta(z)\sin\alpha(s) + \theta(z)\,h(s), \tag{3.9}$$

$$w(z, s) = -\xi(z)\sin\alpha(s) + \eta(z)\cos\alpha(s) + \theta(z)\,t(s). \tag{3.10}$$

The last terms in equations (3.9) and (3.10) determine the displacements due to the rotation of the whole section with respect to the point A. For small angle of rotation each of these displacements, as we know from the kinematics of rigid bodies, equals the product of the rotation angle and the distance of the point A (taken as the rotation center) from the line passing through the point M in the direction of the given displacement.

We now determine the longitudinal displacement $u(z, s)$ of the point M. This is due to the deformation of the middle surface and is directed across the plane of the cross section. We can find this displacement by means of our second hypothesis concerning the absence of shearing strain in the middle surface.

This shearing strain is defined as the change, upon deformation, in the initially right angle between the coordinate lines $s =$ const and $z =$ const. For a cylindrical shell the shear is determined by the displacements $u = u(z, s)$ and $v = v(z, s)$.

Figure 10 shows the tangential displacements (occurring in the appropriate tangent planes) of the four vertices M, a, b and c of an elementary rectangle. A knowledge of the displacements of all four vertices allows us to determine the desired shearing strain. This strain at the point M is,

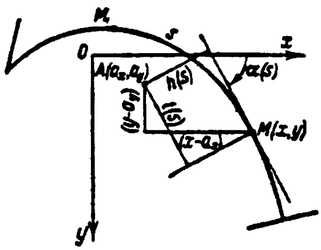


Figure 9

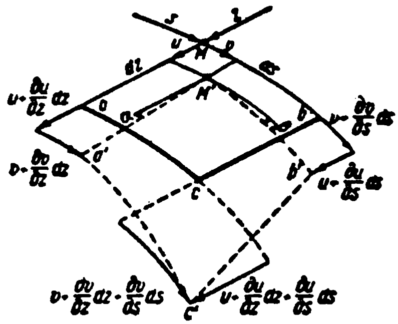


Figure 10

by definition, equal to the sum of the angles α and β, through which the sides Ma and Mb of the elementary rectangle rotate during deformation. Denoting the shearing strain by γ, we have

$$\gamma=\frac{\partial u}{\partial s}+\frac{\partial v}{\partial z}. \tag{3.11}$$

Assuming that the shearing strain vanishes for a thin-walled beam of open cross section, we may write:

$$\frac{\partial u}{\partial s}+\frac{\partial v}{\partial z}=0. \tag{3.12}$$

Since the displacement $v(z, s)$ is already determined by equation (3.9), we can now find the longitudinal displacement $u(z, s)$. Solving (3.12) for the desired function $u(z, s)$, we have

$$u(z, s)=\zeta(z)-\int_{M_1}^{M}\frac{\partial v}{\partial z}\,ds. \tag{3.13}$$

Here $\zeta(z)$ is an arbitrary function, depending only on z, which describes the longitudinal displacement of the point M_1, which serves as the origin of the coordinate s. The integral on the right-hand side of (3.13) is taken along the profile line with respect to the variable s from the origin M_1 to the point M for which the displacement $u(z, s)$ is sought.

Differentiating (3.9) with respect to the variable z and multiplying both sides by ds, we obtain

$$\frac{\partial v}{\partial z}ds = \xi'(z)\cos\alpha(s)\,ds + \eta'(z)\sin\alpha(s)\,ds + \theta'(z)\,h(s)\,ds. \tag{3.14}$$

From Figure 11 we have:

$$\left.\begin{aligned} \cos\alpha\cdot ds &= dx,\\ \sin\alpha\cdot ds &= dy,\\ h\cdot ds &= d\omega. \end{aligned}\right\} \tag{3.15}$$

In these equations ds is the arc element of the profile line and dx and dy are infinitesimal increments of the Cartesian coordinates corresponding to ds; $d\omega$ is twice the area of the elementary triangle (sector) whose base is the arc element ds, and whose height, h, is the length of the perpendicular dropped from A to the tangent to the profile line at M.

Substituting (3.14) and (3.15) in the right-hand side of (3.13) and performing the integration, we obtain

$$u(z, s) = \zeta(z) - \xi'(z)\,x(s) - \eta'(z)\,y(s) - \theta'(z)\,\omega(s), \tag{3.16}$$

where $x(s)$, $y(s)$ are the Cartesian coordinates of the point M, and $\omega(s)$ is twice the area of the sector enclosed between the arc M_1M of the profile line and the two lines AM_1, AM, joining the ends of this segment with A (Figure 12). The author chooses to call this the sectorial area.

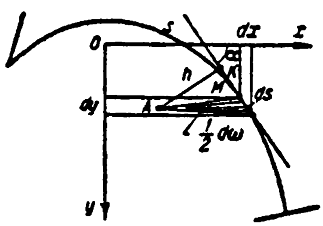


Figure 11

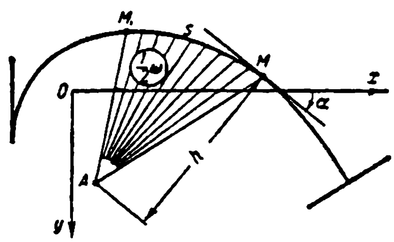


Figure 12

For a given profile line and specified points A and M_1, the area $\omega(s)$ is a well-defined function of the coordinate s, just as $x(s)$ and $y(s)$. We call the point A the pole of the sectorial areas and the point M_1 the sectorial origin.

The line AM_1, which serves as initial ray for the sectorial areas and which connects the pole A with a chosen point M_1 on the profile line, may be called the fixed radius vector. The line AM, which connects the pole A with the variable point M, i. e., with the point for which we calculated the area $\omega(s)$, shall be called the mobile radius vector.

We shall consider the sectorial area positive if the mobile radius vector AM moves clockwise when observed from the negative z direction.

Figure 13 is the diagram of sectorial areas for the cross section shown in Figure 6. The ordinates $\omega(s)$ reproduce the sectorial areas on a certain scale, and have positive values on the curved web-part of the cross section, adopting the above rule of signs. On the straight flange-parts of the profile line the sectorial areas have always rectilinear (in the general case trapezoidal) diagrams, since the area $\omega(s)$ in this case is always a linear function of the coordinate s.

Equation (3.16) describes the general law for the longitudinal displacement $u(z, s)$ of a thin-walled beam (shell) of open section. This law can be expressed as follows:

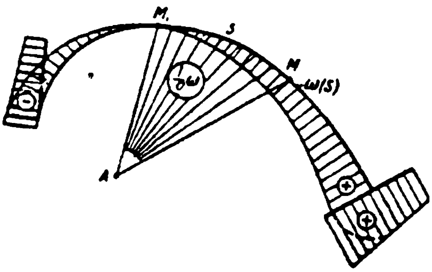


Figure 13

The longitudinal displacements $u(z, s)$ in the section $z = \text{const}$ of a thin-walled open shell of cylindrical or prismatic shape are made up of displacements linear in the Cartesian coordinates of the point on the profile line and displacements proportional to the sectorial area, providing there are no bending deformations of the cross section and the middle surface is free of shear.

The first three terms of equation (3.16) express the Bernoulli-Navier law. According to this law, any cross sections which were plane before deformation, remain plane after it. The longitudinal displacements given by the sum of the first three terms of equation (3.16) arise as a result of combined extension and bending in the Oxz and Oyz planes. The function $\zeta(z)$ determines the axial deformation. In such a deformation the cross sections only suffer a rectilinear displacement along a generator of the beam. The functions $\xi(z)$ and $\eta(z)$, which describe the flexure of an arbitrary axis of the beam $x = a_x$ and $y = a_y$ in the longitudinal planes Oxz and Oyz (Figure 14), characterize the bending deformation. In such a deformation those cross sections that remain plane rotate about the Ox and Oy axes (which lie in the plane of the cross section).

Now, the fourth term of equation (3.16) determines the part of the displacement that does not obey the law of plane sections and which arises as a result of torsion. We call this deviation from the law of plane sections the sectorial warping of the section. This warping is given by the law of sectorial areas.

The quantity $\theta'(z) = \frac{d\theta}{dz}$, which gives the [local] relative torsion angle, serves as a measure of the warping of the section. We shall refer to this quantity as the torsional warping of the beam. The longitudinal displacement of any point M on the middle surface through torsional deformation only is equal to minus the product of the torsional warping $\theta'(z)$ and the sectorial area $\omega(s)$. This area has a pole at the point A of of the plane cross section and an origin at the point M_1 of the profile line.

We note that in the determination of the longitudinal displacements $u(z, s)$ we did not make use of the hypothesis of plane sections. The Bernoulli-Navier law, which embodies this hypothesis and which is the basis of the modern theory of beam bending, is a particular case of the law (3.16).

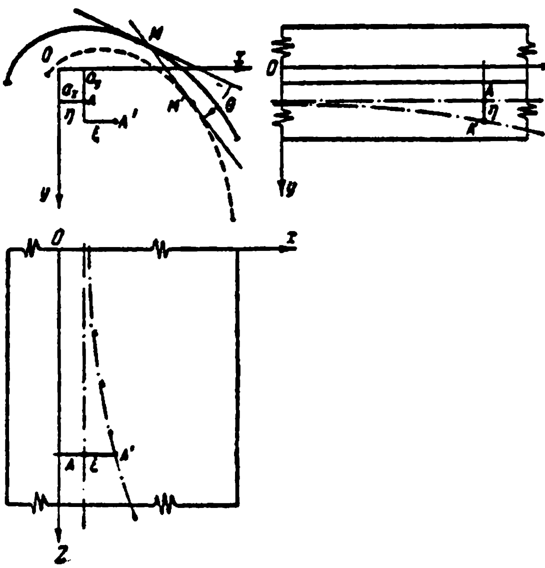


Figure 14

4. Knowing the displacements of the points of the middle surface of the beam, we can now also find the deformation of this surface at any point M. In the following we shall be interested in the longitudinal deformation $\varepsilon = \varepsilon(z, s)$, which is given as the relative extension of the linear element dz which passes through a point M of the surface and lies parallel to its generators. This extension is defined as the ratio of the difference between the longitudinal displacements $u + \frac{\partial u}{\partial z} dz$ and u of two neighboring points a and M, to the length of the element dz of the original (undeformed) surface (see Figure 10):

$$\varepsilon = \frac{\partial u}{\partial z}. \tag{3.17}$$

Differentiating (3.16) with respect to z and using (3.17), we obtain a general equation (which also has four terms) for the relative longitudinal extension

$$\varepsilon(z, s) = \zeta'(z) - \xi''(z)\,x(s) - \eta''(z)\,y(s) - \theta''(z)\,\omega(s). \tag{3.18}$$

Equation (3.18) shows that the relative longitudinal extensions $\varepsilon(z, s)$ at the profile line $z=$ const are made up of extensions linear in the coordinates $x(s)$ and $y(s)$ of the point on this line and obeying the law of plane sections, and extensions distributed along the profile line according to the law of sectorial areas which arise as a result of the warping of the section.

Equation (3.18) is quite general and allows the determination of the longitudinal extension at an arbitrary point of the middle surface of a thin-walled beam when the four functions $\zeta(z)$, $\xi(z)$, $\eta(z)$ and $\theta(z)$ are assigned. This equation is also a generalization of the equation of the theory of strength of materials for extensions caused by axial stretching and bending of a beam, there obtained from the law of plane sections.

5. Equation (3.18) shows that any cylindrical surface, considered as a middle surface of a thin-walled beam, can undergo a deformation even with the assumption that the contour of its transverse section remains unchanged and there is no shearing strain. The deformation of this surface is characterized by the fact that its generators (the lines $s=$const) undergo relative extensions which may have different values at various points of the surface.

The question may now be asked, can the surface still be deformed if we stipulate that the relative longitudinal extensions vanish at all its points (i. e., for any values of z and s). In other words, does a cylindrical surface lend itself to any deformation at all when our two foregoing geometrical hypotheses are in force and it is nondeformable in the longitudinal direction?

To solve this problem we should obviously first put $\varepsilon=0$.

From equation (3.18) we obtain

$$\zeta'(z)-\xi''(z)x(s)-\eta''(z)y(s)-\theta''(z)\omega(s)\equiv 0. \tag{3.19}$$

Since (3.19) is to be satisfied for any value of the variable s, and since the functions 1, $x(s)$, $y(s)$, $\omega(s)$ are linearly independent*, we obtain the equations:

$$\zeta'(z)=0; \quad \xi''(z)=0; \quad \eta''(z)=0; \quad \theta''(z)=0.$$

Integrating these equations, we obtain

$$\left.\begin{aligned}\zeta(z)&=\zeta_0,\\ \xi(z)&=\xi_0+\xi'_0 z,\\ \eta(z)&=\eta_0+\eta' z,\\ \theta(z)&=\theta_0+\theta'_0 z.\end{aligned}\right\}$$

The constants of integration are considered here as arbitrarily given quantities. The constants ζ_0, ξ_0, and η_0 give the translations of the initial section $z=0$ in three mutually perpendicular directions; ξ'_0, η'_0, θ_0

* The functions 1, $x(s)$, $y(s)$, $\omega(s)$ are linearly independent if none of them can be obtained as a linear combination of the other three for all admissible values of s.

give the angular displacements of the same section $z=0$ about three mutually perpendicular axes; θ_0' gives the generalized displacement which specifies the warping of the section $z=0$.

It is easy to see that the six independent quantities ζ_0, ξ_0, η_0, ξ_0', η_0', and θ_0, correspond to the six degrees of freedom of the surface when regarded as a rigid body in space. We need not take these quantities into account, since they refer to the displacement of the surface as a rigid body, and consequently have no bearing on the deformation of the surface. We can therefore assume that the translations ζ_0, ξ_0, and η_0 and the angular displacements ξ_0', η_0', and θ are all zero for the initial section. We are still left with one arbitrary quantity, the warping θ_0', which characterizes, through (3.16), the distortion of the cross sections of the surface according to the law of sectorial areas.

It follows from the last equation of (3.20) that the warping θ_0', prescribed for the section $z=0$, is accompanied by torsional strain of the beam. This strain, defined as the relative torsional angle, is constant along the beam (independent of z) for a thin-walled beam with a nondeformable middle surface

$$\theta' = \theta_0' = \text{const.}$$

We see that every unextensible cylindrical or prismatic surface having a rigid cross section can undergo a deformation, if this deformation is characterized by a torsional angle $\theta(z) = \theta_0' z$. This angle is proportional to the coordinate z with the coefficient θ_0', which is the sectorial warping of the initial section $z=0$. This kind of deformation is investigated in the theory of pure torsion of a thin-walled beam. In such torsion only tangential stresses arise in the cross sections of the beam and these stresses, which are proportional to the torsional deformation $\theta' = \theta_0'$, remain constant over the length of the beam.

Thus, the bending and the pure torsion of thin-walled beams as examined in the theory of strength of materials are special cases of our general theory for the said deformations. Our general theory of thin-walled beams is governed by the law of sectorial areas and is founded on two hypotheses: the shape of the beam cross section is invariable, and its middle surface is free of shear.

The Bernoulli-Navier hypotheses, which form the basis of the elementary theory of the bending of beams, and the Saint-Venant hypotheses, which concern the problem of pure torsion of a beam, are particular cases of our more general geometrical hypotheses. The latter afford to extend the modern theory of the strength of materials and provide an approach to a series of new problems concerning the strength, stability, and vibrations of thin-walled beams, considered as a cylindrical or prismatic shell-like three-dimensional structure of rigid cross section.

§ 4. The law of plane sections as a particular case of the law of sectorial areas

1. We have noted above that for the most general deformation of a thin-walled beam the relative extensions, determined by (3.18) for the section $z = \text{const}$, are made up of extensions determined by the law of plane sections and others, which vary according to the law of sectorial areas along the arc of the profile.

The diagram of the extensions $\varepsilon(z, s)$ for the section $z = \text{const}$ is obtained by superposing four diagrams. One diagram remains constant along the profile line. The second and third are respectively proportional to the distance from the given point of the profile to the axes Ox and Oy. The fourth is determined by the sectorial area which has a pole at the point A. We thus obtain four different expressions for the longitudinal extension ε as a function of the arc-length s.

However, it is possible to show that for a thin-walled beam having a cross section with an inflexible contour the variation (3.18) of ε as a function of the arc-length s can be given by the law of sectorial areas only. The law of plane sections, which is the basis of the conventional theory of bending, is a particular case of this more general law.

The law of sectorial areas can be formulated in the following way.

The relative longitudinal extensions $\varepsilon = \varepsilon(z, s)$ for the section $z = \text{const}$ vary according to the law of sectorial areas for a thin-walled beam of rigid cross section subject to simultaneous extension (compression), bending, and torsion. The sectorial areas have a pole and an origin at certain points of the cross section.

Before we start proving this proposition, we wish to determine the law of transformation of the sectorial area upon change of the pole and the origin.

Let ω_A and ω_D be sectorial areas having poles at the points A and D respectively (Figure 15). The differentials of these areas are

$$\left.\begin{aligned} d\omega_A &= (x - a_x)\,dy - (y - a_y)\,dx, \\ d\omega_D &= (x - d_x)\,dy - (y - d_y)\,dx, \end{aligned}\right\} \qquad (4.1)$$

where a_x and a_y are the coordinates of the point A, d_x and d_y those of D, and x and y are the coordinates of the profile point M.

From equations (4.1) we obtain an expression for the differential of the difference between the sectorial areas:

$$d(\omega_A - \omega_D) = (a_y - d_y)\,dx - (a_x - d_x)\,dy. \qquad (4.2)$$

Integrating both sides of equation (4.2) and expressing ω_A in terms of ω_D, we obtain

$$\omega_A = \omega_D + (a_y - d_y)\,x - (a_x - d_x)\,y + C, \qquad (4.3)$$

where C is a constant which depends on the origin of the coordinate system.

Denoting by x_0 and y_0 the coordinates of the point from which the arc s is measured, then if this point is also the origin of the sectorial areas ω_A and ω_D, i.e., assuming that for $s=0$ the sectorial areas $\omega_A=\omega_D=0$, we obtain the following expression for the integration constant C:

$$C=-(a_y-d_y)x_0+(a_x-d_x)y_0. \tag{4.4}$$

In equation (4.4) the constant C is expressed in terms of the coordinates of the arc-length origin M_0 which we are still free to choose.

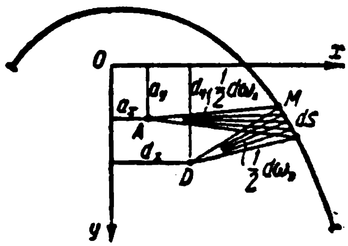


Figure 15

Substituting (4.4) in (4.3), we obtain the desired relation between ω_A and ω_D,

$$\begin{aligned}\omega_A=\omega_D+(a_y-d_y)(x-x_0)-\\-(a_x-d_x)(y-y_0).\end{aligned} \tag{4.5}$$

We can now turn to the proof of the above-stated theorem. The deformed state of the beam, as equation (3.18) shows, is determined by the functions ζ, ξ, η and θ. The function ζ, as has been shown, describes the longitudinal translations, which are the same for all points of the cross section. ξ and η are the projections on the Oxz and Oyz planes of the space curve into which the straight line, parallel to the generators of the cylinder and piercing the plane Oxy at the point A (coordinates a_x and a_y) is deformed. θ is the torsional angle which also depends on z. Given ζ, ξ, η and θ, the relative extensions ε are determined by equation (3.18) in which $\omega(s)$ denotes the sectorial areas with a pole at the point A. Passing from the pole A to the pole D and replacing $\omega(s)$ in (4.3) by ω_D, we obtain for ε,

$$\varepsilon=\zeta'-C\theta''-[\xi''+(a_y-d_y)\theta'']x-[\eta''-(a_x-d_x)\theta'']y-\theta''\omega_D. \tag{4.6}$$

We choose the arbitrary quantities C, d_x and d_y so that those deformations which are linear in the coordinates of the section point vanish. We obtain the following equations for the quantities C, d_x and d_y,

$$\left.\begin{aligned}\zeta'-C\theta''=0,\\ \eta''-(a_x-d_x)\theta''=0,\\ \xi''+(a_y-d_y)\theta''=0.\end{aligned}\right\} \tag{4.7}$$

From these equations we find:

$$\left.\begin{aligned} C &= \frac{\zeta'}{\theta''}, \\ d_x &= a_x - \frac{\eta''}{\theta''}, \\ d_y &= a_y + \frac{\xi''}{\theta''}. \end{aligned}\right\} \tag{4.8}$$

Imposing the conditions (4.7), equation (4.6) becomes

$$u = -\theta'(z)\,\omega_D(z, s), \tag{4.9}$$

which proves our theorem. For the general deformation of a beam the sectorial area $\omega_D(z, s)$ already depends here on the two variable z and s, since the coordinates of the pole for this area d_x and d_y and the quantity C, shown by (4.4) to be determined by the origin of the sectorial area, are functions of z.

In the following we shall call the point D, which serves as the pole of the sectorial area ω_D, the s e c t o r i a l w a r p i n g c e n t e r. For a general deformation this point (as can be seen from equations (3.3) and (4.8)) does not coincide with the torsion center B.

The points B and D do coincide when the deflections ξ and η of the point A and the torsional angle θ satisfy the equations

$$\frac{\xi}{\theta} = \frac{\xi''}{\theta''}; \qquad \frac{\eta}{\theta} = \frac{\eta''}{\theta''}. \tag{4.10}$$

Equations (4.10) are satisfied, for example, when the deformation of the beam follows a trigonometric law. Thus, if the variables ξ, η and θ have the form

$$\xi = \xi_0 \sin \lambda z, \quad \eta = \eta_0 \sin \lambda z, \quad \theta = \theta_0 \sin \lambda z,$$

where ξ_0, η_0, θ_0 and λ are constant quantities, equations (4.10) are satisfied for any z:

$$\frac{\xi}{\theta} = \frac{\xi''}{\theta''} = \frac{\xi_0}{\theta_0} = \text{const},$$
$$\frac{\eta}{\theta} = \frac{\eta''}{\theta''} = \frac{\eta_0}{\theta_0} = \text{const}.$$

In this case the sectorial warping center D coincides with the

torsion center B:

$$\left.\begin{aligned} d_x &= b_x = a_x - \frac{\eta_0}{\theta_0}, \\ d_y &= b_y = a_y + \frac{\xi_0}{\theta_0}. \end{aligned}\right\} \tag{4.11}$$

The coordinates of the points D and B, as can be seen from equations (4.11), do not depend on z. A line parallel to the z axis, which serves in the case of a sinusoidal deformation as axis of torsion, is determined by these coordinates. The cross sections of the beam remain unchanged and rotate with respect to the torsion axis through an angle $\theta = \theta_0 \sin \lambda z$, which varies along the beam as $\sin \lambda z$.

The law of sectorial areas given by equation (4.9) is a general law for the relative extensions ε of a thin-walled beam of rigid cross section.

If the deformation of the beam is such that $\theta'' = 0$, while ξ'' and η'' differ from zero, then the warping center lies at infinity, as can be seen from equations (4.8). The law of sectorial areas reduces in this case into the law of plane sections. The line joining the shear center with the origin of the sectorial area passes through the so-called neutral axis of the section. The extension ε for the section $z = \text{const}$ is proportional to the distance from the neutral axis.

This law, which we have formulated for relative extensions, holds true also for the longitudinal displacement u. The four-term equation (3.16) for the displacement u can be transformed by the same method into a single-term form:

$$u = -\theta'(z)\,\omega_E(z,s).$$

The displacements u are here determined only by the law of sectorial areas. The coordinates of the pole E, which we call the sectorial center of absolute warping, are

$$\left.\begin{aligned} e_x &= a_x - \frac{\eta'}{\theta'}, \\ e_y &= a_y + \frac{\xi'}{\theta'}. \end{aligned}\right\} \tag{4.12}$$

The sectorial origin is determined from the condition

$$C' = \frac{\zeta}{\theta'}. \tag{4.13}$$

Comparing equations (4.12) and (4.13) with equations (4.8), we see that for a general deformation, the warping center does not coincide with the center of absolute warping. The sectorial origins are different points.

2. We proceed to examine a geometrical method for constructing the sectorial warping center. First we note that the diagram of the relative extensions for the cross section of a beam is completely determined by four coordinates. Indeed, let x_i, y_i be the coordinates of four points of

the beam cross section for which the relative extensions ε_i, are given, and let ω_i ($i=1, 2, 3, 4$) be the sectorial areas which correspond to these points. Substituting these values for each point into equation (3.18), we obtain a system of four algebraic equations in the unknowns ζ', ξ'', η'' and θ''. If the points x_i, y_i are so chosen that the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 & \omega_1 \\ 1 & x_2 & y_2 & \omega_2 \\ 1 & x_3 & y_3 & \omega_3 \\ 1 & x_4 & y_4 & \omega_4 \end{vmatrix} \qquad (4.14)$$

is different from zero, they can be solved for the unknowns ζ', ξ'', η'' and θ''. The diagram of the relative extensions for the cross sections of the beam can then be constructed.

If the points x_i, y_i are so chosen that the determinant (4.14) vanishes, the equations are incompatible, and the diagram of the relative extensions cannot be constructed.

We mention two cases for which the determinant (4.14) vanishes.

1) The determinant vanishes if all four points are collinear. Indeed, if the determinant is expanded in elements of the last column the cofactors of these ω_i ($i = 1, 2, 3, 4$) must vanish, since they represent the areas of triangles having all three vertices on a straight line.

2) The determinant vanishes when three points lie on one rectilinear element of the beam section. As a matter of fact, the diagram of the relative extensions on a straight section is linear and, consequently, determined by two points on this section.

We examine a thin-walled beam of open section, composed of narrow rectangular plates. The cross section of the beam and the diagram of the relative extensions are shown in Figure 16. We start from the diagrams of relative extension for the part of the cross section consisting of the segments 1—2, 3—4, and 5—6, in order to construct the sectorial warping center, since the diagram of relative extension for the whole cross section of the beam can be determined by the four coordinates of the points 1, 2, 3, and 4 (Figure 16).

We note the three points in Figure 16: M_0, M'_0 and M''_0, for which the relative extensions vanish. The points M_0 and M''_0 belong to the profile line, while the point M'_0 is obtained as the intersection of the protracted 1 – 2 and 1' – 2' lines. In addition, we construct the line $M_0 — M'_0$. This line is the neutral axis for the 1 – 2 – 5 –M_0 part of the beam since the relative extensions for this part of the contour are proportional to the distance from the line $M_0 — M'_0$.

We shall show that it is possible to consider the diagram of relative extension for the 1 – 2 – 5 –M_0 part of the beam as a sectorial area diagram, with a pole at an arbitrary point K of the line passing through point 5 and parallel to $M_0 — M'_0$. The sectorial origin is placed at M_0. We conclude that the sectorial area vanishes at M'_0. Indeed, the sectorial area at M'_0 is the difference of twice the areas of the triangles KM_05 and KM'_05. The first sectorial area is positive (the radius-vector rotates clockwise) and the second is negative (the radius-vector rotates anti-clockwise). However,

the areas of these triangles are equal since they have a common height $K-L$ and a common base $K-5$. Our statement is proved since at the point $M_{\bullet}'$ the sectorial area is equal to zero, and on the part $5-M_{\bullet}$ the diagram varies according to the law of sectorial areas.

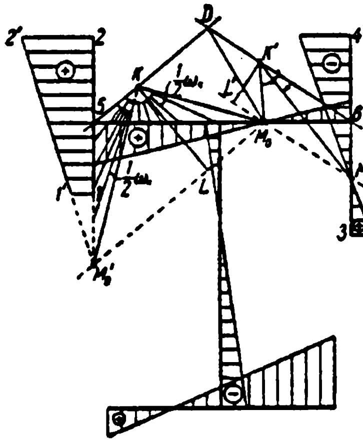


Figure 16

In exactly the same way it can be proved that the line $M_{\bullet}-M_{\bullet}''$ is the neutral axis for the part $3-4-6-M_{\bullet}$ of the profile line. It can also be proved that the relative extensions for this part vary according to the law of the sectorial areas. The pole is taken at an arbitrary point K' of the line parallel to $M_{\bullet}-M_{\bullet}''$ through the point 6. The sectorial origin is at the point $M_{\bullet}$.

The intersection of the protracted $K-5$ and $K'-6$ lines gives the sectorial warping center at point D. The relative extension of the cross section of our beam will vary according to the law of sectorial areas with a pole at the point D and a sectorial origin at $M_{\bullet}$.

3. We consider a beam which consists of one or more interconnected bundles of narrow rectangular plates. The cross section of such a beam (consisting of a single bundle) is shown in Figure 17.

The following statement is valid for any bundle.

The relative extensions within the bundle vary over the cross section according to the law of plane sections.

In order to prove this theorem it is sufficient to show that the sectorial area which has a pole and an origin at an arbitrary point of the cross section varies according to the law of plane sections within this bundle.

For convenience we choose the sectorial origin at the center of the bundle, i.e., at the point A. We choose the pole of the sectorial area at an arbitrary point K of the cross section of the beam (Figure 17).

We draw a straight line through these points. The line AK is the neutral axis of the stack A since the sectorial areas for the bundle points are proportional to the distances from these points to the line AK. Consequently, the sectorial diagram varies according to the law of plane sections within the limits of the bundle. This proves the theorem.

Our theorem has the following corollary: a thin-walled beam

consisting of a single bundle of very thin rectangular plates does not become warped. Its cross sections which are initially plane remain plane after deformation.

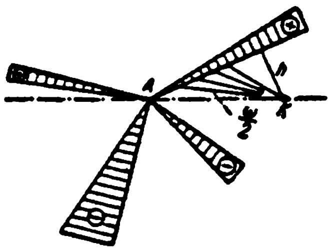


Figure 17

§ 5. Stress-strain relation

So far we have examined thin-walled beams from a purely geometrical point of view. By assuming inflexibility of the section contour and considering the shearing strain to vanish at the middle surface (i.e., that the coordinate lines z = const and s = const remain orthogonal after deformation), we obtained the general law (3.18) for the longitudinal deformations.

However, equation (3.18) does not fully determine the strain ε, since the functions $\zeta(z)$, $\xi(z)$, $\eta(z)$, and $\theta(z)$ are unknown.

This kinematical indeterminacy [kinematical, for it is involved in the description of the motion during deformation] is due to our not having used all the static conditions, namely, the conditions of equilibrium for an elastic body which undergoes a definite deformation. The problem becomes unambiguously determined if we use the physical relations holding under static conditions between stresses and strains as given by Hooke's law in addition to the kinematic conditions for the deformation, ε, expressed by the law of sectorial areas

When the beam is deformed, internal elastic forces arise in it. These forces represent normal and tangential stresses in the cross section z = const.

In our theory, which concerns beams consisting of thin plates, of all the stresses which arise in the cross section of the beam we consider only the normal stresses in the direction of the generator of the middle surface and the tangential stresses in the direction of the tangent to the profile line. In our theory the tangential stresses, which are directed along the normal to the profile line and which have very small values, are assumed to vanish.

In the following we shall assume that the normal stresses are constant over the thickness of the beam wall and that the tangential stresses over the beam wall vary according to a linear law (Figure 18, a, b).

We start with these statical hypotheses which refer to stresses acting over the cross section of the beam and we presuppose that the so-called longitudinal bending moments, which arise as a result of a nonuniform

distribution of longitudinal normal stresses over the thickness of the beam wall, are equal to zero*.

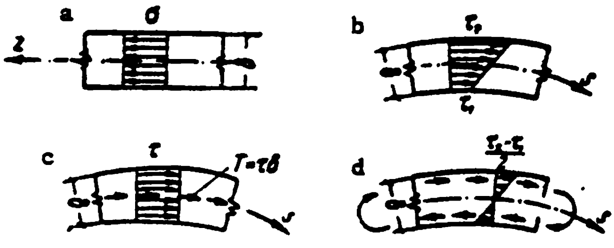


Figure 18

The normal stresses of the cross section multiplied by the thickness of the walls yield statically normal forces only, which act in the direction of the generator of the middle surface.

The tangential stresses lead to forces which act along the tangent to the arc of the contour (Figure 18c), and to torsional moments. The shear forces are obtained by deducting from the general trapezial diagram for the tangential stresses the rectangular diagram with a coordinate equal to half the sum of the coordinates at the extreme points of the trapezial diagram. The torsional moments appear as a result of the difference in the tangential stresses at the extreme points of the beam. These moments are statically equivalent to the tangential stresses, which vary over the thickness of the wall according to a skew-symmetrical triangular diagram. The values at the extreme points of the beam are equal to half the difference between the extreme coordinates of the trapezium (Figure 18d).

Thus, the state of stress in the cross section plane can be expressed by the normal stresses $\sigma(z, s)$, the average tangential stresses $\tau(z, s)$ and the torsional moments. The tangential and normal stresses $\tau(z, s)$ and $\sigma(z, s)$ are considered as functions of the two variables z and s. We replace the torsional moments per unit section (which depend on the difference of the tangential stresses at the extreme points of the wall) by a distribution of torsional moments $H_k(z)$ over the cross section. This moment will be a function of the variable z only.

We now agree on the signs of these quantities. An elementary area of the cross section shall be considered positive if its normal points in the direction of positive z (Figure 19). We shall consider the stresses acting on any point of this area as positive if they are in the direction of increasing coordinates for the point of the middle surface of the shell (Figure 19). We shall consider the torsional moment as positive if it causes the cross section to rotate clockwise when viewed from the positive side of the area.

By using the physical relation between the stresses and strains in the beam we can find the stresses σ, τ and the moments H_k from the strains.

Hooke's law gives for the relative extensions of an element of the

* Calculation of thin-walled beams with allowance for longitudinal bending moments are given in § 14, Chapter II.

middle surface of the beam in two perpendicular directions

$$\left.\begin{aligned} \varepsilon &= \frac{1}{E}(\sigma - \mu\sigma_1), \\ \varepsilon_1 &= \frac{1}{E}(\sigma_1 - \mu\sigma), \end{aligned}\right\} \tag{5.1}$$

where ε and ε_1 are the relative extensions of the beam in the longitudinal and transverse directions, σ and σ_1 are the normal stresses in the longitudinal and transverse directions respectively, E is Young's modulus, and μ is Poisson's ratio.

According to the supposed inflexibility of the contour, the extension ε_1 of the contour arc is equal to zero.

Thus

$$\sigma_1 - \mu\sigma = 0,$$

which leads to

$$\sigma_1 = \mu\sigma. \tag{5.2}$$

Introducing expression (5.2) into the first of equations (5.1), we obtain

$$\varepsilon = \frac{1-\mu^2}{E}\sigma. \tag{5.3}$$

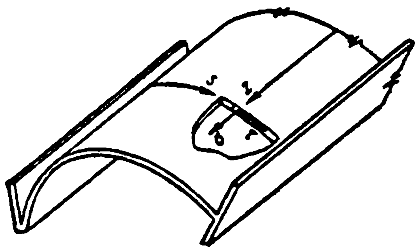


Figure 19

Equation (5.3) establishes the relation between the longitudinal extensions ε and the normal stresses σ. This relation can be presented in the form

$$\sigma = E_1\varepsilon, \tag{5.4}$$

where E_1 is the reduced Young's modulus of longitudinal elongation:

$$E_1 = \frac{E}{1-\mu^2}. \tag{5.5}$$

In the following we shall consider the quantity μ^2 negligible compared with unity, and take $E_1 = E$.

Substituting the expression for ε from equation (3.18) in equation (5.4), we obtain the general law for the distribution of normal stresses $\sigma = \sigma(z, s)$:

$$\sigma = E(\zeta' - \xi''x - \eta''y - \theta''\omega). \tag{5.6}$$

According to this law the normal stresses σ are obtained as a result of

the superposition of stresses distributed over the section $z=$ const and depend on the distance from a certain line, and of stresses which vary as a function of the arc-length s according to the law of sectorial areas. As in the case of the strains ε, the distribution of the stresses σ over the section $z=$const can be described by a sectorial area $\omega(z, s)$ with a pole at the warping center. Indeed, on the basis of equations (4.9) and (5.4) we have

$$\sigma(z, s)=-E\theta''(z)\,\omega_D(z, s).$$

As we see, equation (5.6) is a generalization of the three-term equation known from the theory of the strength of materials, which refers to the longitudinal normal stress in a beam which simultaneously undergoes an axial deformation and a bending in two perpendicular planes. The four-term equation (5.6) indicates that longitudinal normal stresses, as determined by the law of sectorial areas, may also arise in addition to the stresses in the cross sections of the thin-walled beam which are due to extension and bending of the beam and which are distributed over the section $z=$const according to the law of plane sections (i. e., which depend linearly on the coordinates x and y).

The difference between our theory and the theory of the so-called pure torsion is that the law of sectorial areas, as given by the general four-term equation (5.6), affords the determination of the normal stresses arising upon torsion of the beam. In the Saint-Venant hypothesis, which refers to pure torsion, these are assumed to vanish.

We now determine the bending moment H_k. As we have already noted, this moment arises in the cross section as a result of a nonuniform distribution of the tangential stresses over the thickness of the wall of the beam and refers to the case of pure torsion. In the theory of pure torsion, i. e., torsion in which only tangential stresses arise in the cross section, we have for the bending moment

$$H_k=GJ_d\theta', \tag{5.7}$$

where G is the rigidity modulus, θ' is the derivative of the angle of rotation, J_d is the moment of inertia for pure torsion as calculated for a thin-walled folded cross section according to the equation

$$J_d=\frac{\alpha}{3}\sum d\delta^3, \tag{5.8}$$

where d and δ are, respectively, the width and thickness of the plates which make up the beam and α is an empirical coefficient close to unity.

It now remains to determine the tangential stresses. We note that the normal stresses $\sigma(z, s)$, described by equation (5.6), were determined by using the relation (5.4) for the elastic beam and then assuming that the stresses are proportional to the corresponding extensions. We cannot use the analogous elastic equation to determine the average tangential stresses $\tau(z, s)$ shown in Figure 18c, since the shearing strains caused by the tangential stresses are taken to vanish in our theory.

To determine the tangential stresses $\tau(z, s)$ we shall not start from Hooke's law, but from the corresponding conditions of statics which express the equilibrium of the shell element in the longitudinal direction. These stresses are determined statically in the same way as in the elementary theory of bending.

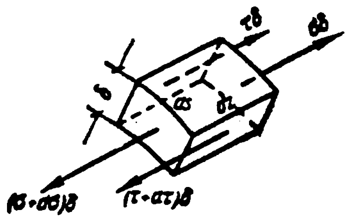


Figure 20

If the sum of the projections on the direction of the generator of all the forces acting on an infinitesimal beam element with the sides dz and ds (Figure 20) is set equal to zero, we have

$$d(\sigma\delta)\,ds + d(\tau\delta)\,dz + p_z dz\,ds = 0, \tag{5.9}$$

where $\delta = \delta(s)$ is the thickness of the shell which in general is a function of s and $p_z = p_z(z, s)$ is the projection of the external surface load on the axis z. In general, this projection depends on the variables z and s.

Dividing equation (5.9) by $dz\,ds$, we obtain

$$\frac{\partial(\sigma\delta)}{\partial z} + \frac{\partial(\tau\delta)}{\partial s} + p_z = 0. \tag{5.10}$$

In this equation $\sigma\delta$ and $\tau\delta$ are the normal and shear stresses in the cross section $z = \text{const}$, referred to a unit length. Solving equation (5. 10) with respect to the tangential stresses τ (realizing that the thickness of the wall δ does not depend on z) we find

$$\tau(z, s) = \frac{1}{\delta}\left[S_0(z) - \int_0^s p_z ds - \int_0^s \frac{\partial\sigma}{\partial z}\delta\,ds\right]. \tag{5.11}$$

Here $S_0(z)$ is an arbitrary function which depends on the variable z. It is easy to see the physical meaning of this function.

Setting in (5.11) $s = 0$ and noting that the definite integrals on the right-hand side of this equation vanish for $s = 0$, we obtain

$$\tau(z, 0) = \frac{1}{\delta(0)} S_0(z).$$

Whence

$$S_0(z) = \tau(z, 0)\,\delta(0). \tag{5.12}$$

It is seen from (5.12) that the function $S_0(z)$ gives the shear stresses $\tau\delta$ which act on the longitudinal section $s=0$.

Substituting (5.6) for σ in the right-hand side of (5.11) and denoting the differential of the section area δds by dF, we find

$$\tau=\frac{1}{\delta}[S_0-\int_0^s p_z ds - E(\zeta''\int_0^s dF-\xi'''\int_0^s x\,dF-$$
$$-\eta'''\int_0^s y\,dF-\theta'''\int_0^s \omega\,dF)]. \qquad (5.13)$$

where ζ'', ξ''', η''' and θ''' denote the second and third derivatives of the corresponding functions.

It is convenient to rewrite equation (5.13) in the following form:

$$\tau=\frac{1}{\delta}S_0-\frac{1}{\delta}\int_0^s p_z ds - E\zeta''(z)\frac{F(s)}{\delta}+E\xi'''\frac{S_y(s)}{\delta}+$$
$$+E\eta'''\frac{S_x(s)}{\delta}+E\theta'''\frac{S_\omega(s)}{\delta}, \qquad (5.14)$$

where $F(s)$, $S_x(s)$ and $S_y(s)$ are respectively the area and the static moments with respect to the x and y axes, given by the equation

$$F(s)=\int_0^s dF, \quad S_x(s)=\int_0^s y\,dF, \quad S_y(s)=\int_0^s x\,dF; \qquad (5.15)$$

$S_\omega(s)$ is a new geometrical characteristic defined by the equation

$$S_\omega=\int_0^s \omega\,dF. \qquad (5.16)$$

This expression for the function $S_\omega(s)$, which describes together with $F(s)$, $S_x(s)$ and $S_y(s)$ the distribution of the shear stresses over the section contour, is reminiscent of the equation for a static moment. The only difference is that the moment arm in the latter equation is replaced here by the sectorial area ω. Accordingly we shall henceforth call $S_\omega(s)$ the **sectorial static moment**.

The quantities $F(s)$, $S_x(s)$, $S_y(s)$ and $S_\omega(s)$ which are determined by equations (5.15) and (5.16), are calculated for that part of the section

which is included between the point of origin M_0 on the arc of the contour and the point M for which the tangential stress τ is determined by equation (5.14). If the origin of the coordinate s is the extreme point of the contour arc, then these geometrical factors are calculated only for that part of the section that is cut-off, e.g., in Figure 21 for the part lying above the point M of a definite arc s.

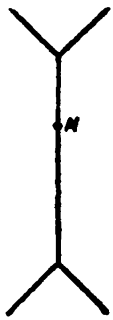


Figure 21

If there is no longitudinal load distribution on lateral edges or sides, equation (5.13) assumes the simpler form

$$\tau(z,\ s)=\frac{E}{\delta(s)}[-\zeta''(z)F(s)+\xi'''(z)S_y(s)+\eta'''(z)S_x(s)+\theta'''(z)S_\omega(s)]. \quad (5.17)$$

Equation (5.17) represents a particular case of the more general equation (5.13), which in its turn is a generalization of the equation, known from strength of materials, for the determination of the tangential stresses due to the transverse bending of a beam.

The first term of the general equation (5.17) determines the tangential stresses which appear as a result of the action on the beam by shear tractions on the lateral edges and caused only by extension. The second and third terms of equation (5.17) refer to tangential stresses which appear as a result of bending a thin-walled beam in two perpendicular longitudinal planes. The last term, which is similar in form to the second and third terms, determines the axial tangential stresses which appear in restrained torsion of a thin-walled beam and are also distributed uniformly across the wall (see Figure 18c).

§ 6. Differential equations of equilibrium for a beam in arbitrary coordinates

It is seen from equations (5.6), (5.7), and (5.14) that the stresses σ, τ, and the moment H_ω which determine the state of the internal stresses of the shell in the cross section $z=$ const, depend on the functions $\zeta(z)$, $\xi(z)$, $\eta(z)$ and $\theta(z)$. These functions are unknown as yet. We have to apply the equilibrium conditions (which still have not been used) in order to determine them.

We consider two sections, $z = \text{const}$ and $z + dz = \text{const}$, of the beam. They enclose a transverse strip of width dz. We substitute the action of the beam on this isolated strip by normal and shear stresses and by the bending moments which are distributed over the curved side of the strip. On passing from the section $z = \text{const}$ to the section $z + dz = \text{const}$ (Figure 22), these stresses and moments, which are functions of z, change by a certain increment (neglecting higher order infinitesimals) proportional to the differential dz. In addition to these stresses, which replace the action of the beam on the isolated element, an external load will also act on the element.

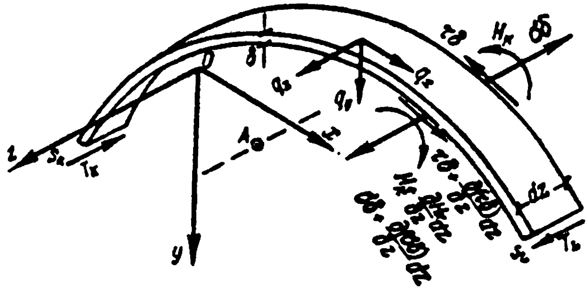


Figure 22

The equilibrium conditions for an elementary transverse strip can be given in the following form:

$$\left.\begin{aligned}
&\sum Z = 0, && \int_L \frac{\partial(\sigma\delta)}{\partial z} dz\,ds + (T_L - T_K + q_z)\,dz = 0;\\
&\sum X = 0, && \int_L \frac{\partial(\tau\delta)}{\partial z} \cos\alpha \cdot dz\,ds + q_x\,dz = 0;\\
&\sum Y = 0, && \int_L \frac{\partial(\tau\delta)}{\partial z} \sin\alpha \cdot dz\,ds + q_y\,dz = 0;\\
&\sum M_A = 0, && \int_L \frac{\partial(\tau\delta)}{\partial z} dz\,[(x - a_x)\sin\alpha - (y - a_y)\cos\alpha]\,ds +\\
& && + H'_K dz + m\,dz = 0.
\end{aligned}\right\} \quad (6.1)$$

The first equation embodies the condition that the sum of the projections on the direction of the generator of all forces acting on an isolated element is equal to zero. In this equation the integrand $\frac{\partial(\sigma\delta)}{\partial z} dz\,ds$, gives the difference between the longitudinal normal forces exerted on the elements $dz\,ds$, q_z is the longitudinal surface load directed along the z axis and exerted on an entire cross section of unit width, and $T_L = T_L(z)$ and $T_K = T_K(z)$ are the shear stresses acting on the lateral edges that correspond to the extreme points of the contour $s = s_K$ and $s = s_L$.

The second and third equations express the equilibrium conditions for the elementrary strip in the direction of the x and y axes of the

cross section. In these equations the integrands $\frac{\partial(\tau\delta)}{\partial z}\cos\alpha\cdot dz\,ds$ and $\frac{\partial(\tau\delta)}{\partial z}\sin\alpha\cdot dz\,ds$ describe the projections of the differential of the shear force on the x and y axes respectively (this force acts on elements of the shell $dz\,ds$); $q_x=q_x(z)$ and $q_y=q_y(z)$ are the magnitudes of the external transverse loads which act in the direction of the x and y axes respectively.

The last equation is obtained from the condition that the resultant moment, with respect to an arbitrary point A of the cross section, of all forces exerted on the isolated element is equal to zero. The first term of this equation gives the moment of the axial shear stresses on the section $z=$const and $z+dz=$ const. This moment is determined for the element dz as the moment of the projections on the x and y axes of the differentials of the shear forces along the tangents to the contour arcs. The second term gives the differential of the torsional moment H_z which acts on a strip of width dz. The last term gives the moment of the external transverse load.

The integrals (6.1) are calculated with respect to the variable s along the whole curve L of the cross section. Dividing equation (6.1) by dz and noting

$$ds\cos\alpha=dx,\quad ds\sin\alpha=dy,$$
$$[(x-a_x)\sin\alpha-(y-a_y)\cos\alpha]ds=d\omega$$

(see equation (4.1)), we find

$$\left.\begin{aligned}&\int_L\frac{\partial(\sigma\delta)}{\partial z}\,ds+T_L-T_K+q_z=0,\\&\int_L\frac{\partial(\tau\delta)}{\partial z}\,dx+q_x=0,\\&\int_L\frac{\partial(\tau\delta)}{\partial z}\,dy+q_y=0,\\&\int_L\frac{\partial(\tau\delta)}{\partial z}\,d\omega+H'_z+m=0.\end{aligned}\right\}\tag{6.2}$$

The thickness of the beam wall, δ, is considered a function of the variable s only. Consequently, the δ under the integral sign of the first equation of (6.2) behaves as a constant in the differentiation of the expression $(\sigma\delta)$.

Denoting the product $\delta\,ds$ by dF in these equations and integrating by parts, in the last three equations, we obtain

$$\left.\begin{aligned}&\int_F\frac{\partial\sigma}{\partial z}\,dF+T_L-T_K+q_z=0,\\&\left|\frac{\partial(\tau\delta)}{\partial z}x\right|_K^L-\int_L x\frac{\partial}{\partial s}\left[\frac{\partial(\tau\delta)}{\partial z}\right]ds+q_x=0,\\&\left|\frac{\partial(\tau\delta)}{\partial z}y\right|_K^L-\int_L y\frac{\partial}{\partial s}\left[\frac{\partial(\tau\delta)}{\partial z}\right]ds+q_y=0,\\&\left|\frac{\partial(\tau\delta)}{\partial z}\omega\right|_K^L-\int_L \omega\frac{\partial}{\partial s}\left[\frac{\partial(\tau\delta)}{\partial z}\right]ds+H'_z+m=0.\end{aligned}\right\}\tag{6.3}$$

In these equations the expressions between the vertical lines must be evaluated for the values of the argument s at the extreme points of the curve $z =$ const, i. e., for $s = s_K$ and $s = s_L$

The quantity $\tau\delta$ describes the shear stress T, which acts on a unit length of the longitudinal section of the middle surface. For $s = s_K$ and $s = s_L$, this quantity equals, respectively, the shear stresses $T_K(z)$ and $T_L(z)$, at the lateral edges of the beam. We have

$$\left|\frac{\partial(\tau\delta)}{\partial z}\right|_{s=s_K} = T'_K(z),$$
$$\left|\frac{\partial(\tau\delta)}{\partial z}\right|_{s=s_L} = T'_L(z).$$

Carrying out in (6.3) the operation indicated by the vertical lines, and interchanging the partial derivatives with respect to z and s of the function $(\tau\delta)$ in the integrands, we obtain

$$\left.\begin{aligned} \int_F \frac{\partial\sigma}{\partial z}\,dF + q_z + T_L - T_K = 0, \\ -\int_F x\frac{\partial}{\partial z}\left[\frac{\partial(\tau\delta)}{\partial s}\right] ds + q_x + T'_L x_L - T'_K x_K = 0, \\ -\int_F y\frac{\partial}{\partial z}\left[\frac{\partial(\tau\delta)}{\partial s}\right] ds + q_y + T'_L y_L - T'_K y_K = 0, \\ -\int_F \omega\frac{\partial}{\partial z}\left[\frac{\partial(\tau\delta)}{\partial s}\right] ds + H'_k + m + T'_L\omega_L - T'_K\omega_K = 0, \end{aligned}\right\} \tag{6.4}$$

where x_K, y_K, x_L and y_L are the coordinates of the initial and final points respectively of the cross section curve of the shell, and ω_K and ω_L are the sectorial areas for these points.

From equations (5.6), (5.14), and (5.7) we have

$$\left.\begin{aligned} &\frac{\partial\sigma}{\partial z}dF = \zeta''E\,dF - \xi'''Ex\,dF - \eta'''Ey\,dF - \theta'''E\omega\,dF, \\ &\frac{\partial}{\partial z}\left[\frac{\partial(\tau\delta)}{\partial s}\right] ds = -\frac{\partial p_z}{\partial z}ds - \zeta'''E\,dF + \xi^{\mathrm{IV}}Ex\,dF + \\ &\qquad\qquad + \eta^{\mathrm{IV}}Ey\,dF + \theta^{\mathrm{IV}}E\,dF, \\ &H'_k = GJ_d\theta''. \end{aligned}\right\} \tag{6.5}$$

Substituting now expressions (6.5) in equations (6.4), and noting that the derivatives of the desired functions $\zeta(z)$, $\xi(z)$, $\eta(z)$ and $\theta(z)$ can be put in front of the integral sign, we obtain

$$\left.\begin{aligned}
&\zeta''E\int_F dF-\xi'''E\int_F x\,dF-\eta'''E\int_F y\,dF-\\
&\qquad\qquad-\theta'''E\int_F \omega\,dF+q_z+T_L-T_K=0,\\
&\int_L \frac{\partial p_z}{\partial s}x\,ds+\zeta'''E\int_F x\,dF-\xi^{IV}E\int_F x^2dF-\\
&\qquad-\eta^{IV}E\int_F xy\,dF-\theta^{IV}E\int_F x\omega\,dF+q_x+T_L'x_L-T_K'x_K=0,\\
&\int_L \frac{\partial p_z}{\partial z}y\,ds+\zeta'''E\int_F y\,dF-\xi^{IV}E\int_F yx\,dF-\\
&\qquad-\eta^{IV}E\int_F y^2dF-\theta^{IV}E\int_F y\omega\,dF+q_y+T_L'y_L-T_K'y_K=0,\\
&\int_L \frac{\partial p_z}{\partial z}\omega\,ds+\zeta'''E\int_F \omega\,dF-\xi^{IV}E\int_F \omega x\,dF-\\
&\qquad-\eta^{IV}E\int_F \omega y\,dF-\theta^{IV}E\int_F \omega^2\,dF+\\
&\qquad\qquad+GJ_d\theta''+m+T_L'\omega_L-T_K'\omega_K=0.
\end{aligned}\right\}\qquad(6.6)$$

We obtain four differential equations for the four unknown functions $\zeta(z)$, $\xi(z)$, $\eta(z)$ and $\theta(z)$. The integrals which enter into the coefficients of these equations, are calculated over the whole cross section of the beam.

For greater clarity we present equations (6.6) in the following tabular form:

Table 1

$\zeta(z)$	$\xi(z)$	$\eta(z)$	$\theta(z)$	Terms depending on the load
$\int_F 1^2dF\cdot D^3$	$-\int_F 1x\,dF\cdot D^3$	$-\int_F 1y\,dF\cdot D^3$	$-\int_F 1\omega\,dF\cdot D^3$	$=-\frac{1}{E}(q_z+T_L-T_K)$
$-\int_F x1\,dF\cdot D^3$	$\int_F x^2dF\cdot D^4$	$\int_F xy\,dF\cdot D^4$	$\int_F \omega x\,dF\cdot D^4$	$=\frac{1}{E}\left(q_x+T_L'x_L-T_K'x_K+\int_L x\frac{\partial p_z}{\partial z}ds\right)$
$-\int_F y1\,dF\cdot D^3$	$\int_F yx\,dF\cdot D^4$	$\int_F y^2dF\cdot D^4$	$\int_F \omega y\,dF\cdot D^4$	$=\frac{1}{E}\left(q_y+T_L'y_L-T_K'y_K+\int_L y\frac{\partial p_z}{\partial z}ds\right)$
$-\int_F \omega 1\,dF\cdot D^3$	$\int_F \omega x\,dF\cdot D^4$	$\int_F \omega y\,dF\cdot D^4$	$\int_F \omega^2dF\cdot D^4-\frac{GJ_d}{E}D^2$	$=\frac{1}{E}\left(m+T_L'\omega_L-T_K'\omega_K+\int_L \omega\frac{\partial p_z}{\partial z}ds\right)$

The letter D in Table 1 denotes the operator of partial differentiation with respect to z,

$$D = \frac{\partial}{\partial z}$$

of the function given at the top of the corresponding column. The exponent of D denotes the order of the derivative. Thus, for example, the entry of the second equation which appears in the column under $\xi(z)$ is written in full as $\int\limits_F x^2 dF \cdot \frac{\partial^4 \xi}{\partial z^4}$, etc.

This presentation facilitates to observe that the system of differential equations for the equilibrium of an isolated strip of a thin-walled beam is symmetrical. The coefficients in Table 1 are symmetrical with respect to the diagonal which passes from the upper left to the lower right corner. The symmetrically placed coefficients are equal. Thus, only ten (four diagonal and six off-diagonal) of the 16 coefficients of the system are independent.

It is also easy to see the form of these coefficients. They all have the form of integrals taken over the area of the whole cross section of the beam. The integrands are composed of four known functions (Figure 23)

$$1;\ x(s);\ y(s);\ \omega(s) \tag{6.7}$$

by pairwise multiplication. Thus, the integrands of the coefficients of the first equation are obtained by multiplying the first function, 1, (constant throughout the section contour) by each of the functions (6.7). The integrands in the coefficients of the second equation are obtained by multiplying the function $x(s)$ by each of the functions (6.7). The integrands of the coefficients for the other equations are obtained similarly, multiplying the functions $y(s)$ and $\omega(s)$ by each of the functions (6.7).

For a given section contour these coefficients are completely determined. Since they depend on the form of the beam section, they are called the geometrical characteristics of the section. We have already encountered several of these coefficients in the theory of strength of materials. They have the following shortened notation:

a) static moments (dimension $[L^3]$, where $[L]$ is the unit of length):

$$S_x = \int\limits_F y\, dF, \quad S_y = \int\limits_F x\, dF; \tag{6.8}$$

b) moments of inertia (dimension $[L^4]$):
axial:

$$J_x = \int\limits_F y^2\, dF, \quad J_y = \int\limits_F x^2\, dF; \tag{6.9}$$

products of inertia:

$$J_{xy} = \int\limits_F xy\, dF. \tag{6.10}$$

In analogy with the designations and terminology adopted in the theory of strength of materials, we shall name and briefly describe the characteristics obtained as a result of our introduction of the new sectorial coordinate $\omega(s)$:

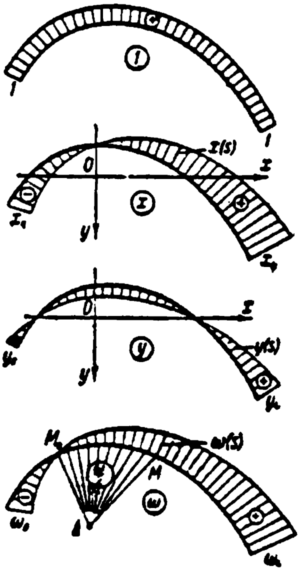


Figure 23

a) the sectorial static moment [first moment] (see § 5) (dimension $[L^4]$*):

$$S_\omega = \int_F \omega \, dF; \tag{6.11}$$

b) the sectorial moment of inertia (second moment) (dimension $[L^6]$):

$$J_\omega = \int_F \omega^2 \, dF; \tag{6.12}$$

and the sectorial products of inertia (dimension $[L^5]$):

$$J_{\omega x} = \int_F \omega x \, dF, \quad J_{\omega y} = \int_F \omega y \, dF. \tag{6.13}$$

With these notations the system of differential equations for the equilibrium of the beam is written in the following form

$$\left.\begin{aligned}
&F\zeta'' - S_y\xi''' - S_x\eta''' - S_\omega\theta''' = -\frac{1}{E}(q_z + T_L - T_K),\\
&-S_y\zeta''' + J_y\xi^{IV} + J_{xy}\eta^{IV} + J_{\omega x}\theta^{IV} =\\
&\qquad = \frac{1}{E}\left(q_x + T'_L x_L - T'_K x_K + \int_L x \frac{\partial p_z}{\partial z}\, ds\right),\\
&-S_x\zeta''' + J_{xy}\xi^{IV} + J_x\eta^{IV} + J_{\omega y}\theta^{IV} =\\
&\qquad = \frac{1}{E}\left(q_y + T'_L y_L - T'_K y_K + \int_L y \frac{\partial p_z}{\partial z}\, ds\right),\\
&-S_\omega\zeta''' + J_{\omega x}\xi^{IV} + J_{\omega y}\eta^{IV} + J_\omega\theta^{IV} - \frac{GJ_d}{E}\theta'' =\\
&\qquad = \frac{1}{E}\left(m + T'_L\omega_L - T'_K\omega_K + \int_L \omega \frac{\partial p_z}{\partial z}\, ds\right).
\end{aligned}\right\} \tag{6.14}$$

*Here and in the following, any new geometrical characteristics of the cross section bear subscripts showing the integrand function.

§ 7. Differential equations of equilibrium for a beam in principal coordinates

1. The system of differential equations of equilibrium for a beam-shell (6.14) is complicated and inconvenient in practice. This is explained by the fact that we chose the basic functions 1, $x(s)$, $y(s)$ and $\omega(s)$ arbitrarily. We referred the cross section to arbitrary rectangular axes, placed the pole of the sectorial area at an arbitrary point A, and chose as origin of the sectorial area an arbitrary point M_1 of the profile line.

Our arbitrariness is, consequently, characterized by the free choice, on the plane of the cross section of the beam, of the six quantities S_x and S_y, which determine the position of the coordinate origin; J_{xy}, which determines the direction of the coordinate axes; $J_{\omega x}$ and $J_{\omega y}$ which determine the position of the pole of the sectorial areas, and S_ω which determines the direction of the initial radius-vector (i.e., the position of the origin of the sectorial areas).

We shall choose the basic functions 1, $x(s)$, $y(s)$ and $\omega(s)$ so that they will be orthogonal. Orthogonality means the vanishing of all integrals of pair-products of these functions (but not their squares) taken over the entire section F,

$$\left.\begin{aligned} S_y &= \int_F 1x\,dF = 0, & S_\omega &= \int_F 1\omega\,dF = 0, \\ S_x &= \int_F 1y\,dF = 0, & J_{\omega x} &= \int_F x\omega\,dF = 0, \\ J_{xy} &= \int_F xy\,dF = 0, & J_{\omega y} &= \int_F y\omega\,dF = 0. \end{aligned}\right\} \tag{7.1}$$

We see that orthogonality of the basic functions finds its expression in that all the symmetric coefficients of equations (6.14) vanish.

The last-mentioned system falls into four separate equations when the conditions of orthogonality (7.1) for the basic functions are satisfied:

$$\left.\begin{aligned} EF\zeta'' &= -(q_z + T_L - T_K), \\ EJ_y\xi^{IV} &= q_x + T_L x_L - T_K x_K + \int_L x\frac{\partial p_z}{\partial z}\,ds, \\ EJ_x\eta^{IV} &= q_y + T_L y_L - T_K y_K + \int_L y\frac{\partial p_z}{\partial z}\,ds, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' &= m + T_L\omega_L - T_K\omega_K + \int_L \omega\frac{\partial p_z}{\partial z}\,ds. \end{aligned}\right\} \tag{7.2}$$

If the lateral edges of the beam are free from shear forces and the external load is composed only of transverse specific forces $q_x(z)$, $q_y(z)$ and a moment $m(z)$ (the case common in practice), equations (7.2) have even simpler form:

$$\left.\begin{aligned} EF\zeta'' &= 0, \\ EJ_y\xi^{\mathrm{IV}} &= q_x, \\ EJ_x\eta^{\mathrm{IV}} &= q_y, \\ EJ_\omega\theta^{\mathrm{IV}} - GJ_d\theta'' &= m. \end{aligned}\right\} \tag{7.3}$$

The first of equations (7.3) determines the longitudinal displacement, $\zeta(z)$, due to the longitudinal tension or pressure which is applied to the ends of the beam and uniformly distributed over the section.

The second and third equations refer to transverse bending of the beam and determine the displacements $\xi(z)$ and $\eta(z)$ for that point A of the cross section with respect to which the sectorial area function $\omega(s)$ is orthogonal to the functions $x(s)$ and $y(s)$.

The fourth equation refers to the torsion of the beam under the action of a transverse load which creates a bending moment, $m(z)$, with respect to the point A.

2. We call the functions 1, $x(s)$, $y(s)$ and $\omega(s)$, which satisfy the conditions of orthogonality (7.1) the principal generalized coordinates of the cross section of a thin-walled beam. The conditions of orthogonality (7.1) are interpreted physically as follows: the internal (external) forces do no work when the beam passes from any state of strain to another, both states corresponding to the displacements $\zeta(z)$, $\xi(z)$, $\eta(z)$ and $\theta(z)$, if the cross section is referred to the principal generalized coordinates 1, $x(s)$, $y(s)$ and $\omega(s)$.

We shall determine the principal generalized coordinates of the cross section.

We immediately note that the first three conditions of orthogonality (7.1) refer to the law of plane sections and express the conditions known from the theory of strength of materials for the determination of the principal central axis of the section. Consequently, the coordinates $x(s)$ and $y(s)$ are taken from the cprincipal central axis of the cross section and are the principal coordinates.

Our problem is to determine the principal sectorial coordinate ω. For that purpose we use the last three conditions of orthogonality (7.1). Of these conditions we need the last two for the determination of the sectorial origin.

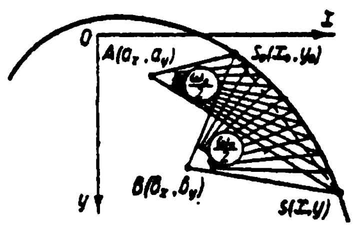


Figure 24

It is essential to note that in determining the pole and the origin of the principal sectorial coordinate, we may refer the cross section to an arbitrary coordinate system Oxy. We prove this below, but in order to simplify the calculations we shall assume that the coordinates system Oxy consists of the principal central axes.

We shall first find the coordinates of the desired pole. Rewriting our earlier equation (4.5) in the form

$$\omega_A = \omega_B + (a_y - b_y)\,x(s) - (a_x - b_x)\,y(s) + \\ + [(a_x - b_x)\,y(s_0) - (a_y - b_y)\,x(s_0)], \tag{7.4}$$

where ω_A, ω_B are the sectorial areas of a part of the contour with poles respectively at the points A and B; $x(s_0)$, $y(s_0)$ are the coordinates of the sectorial origin (the point for which $\omega_A = \omega_B = 0$); $x(s)$, $y(s)$ are the variable coordinates of the contour point s, for which the sectorial areas ω_A and ω_B are determined; a_x, a_y are the coordinates of the point A, and b_x, b_y of the point B (all the coordinates are referred to an arbitrary system Oxy).

We related equation (7.4) to the sectorial coordinates calculated from a common origin but with respect to two different poles located at arbitrary points A and B of the cross section (Figure 24).

We use the orthogonality conditions $J_{\omega x} = 0$ and $J_{\omega y} = 0$. We assume that A is a pole for which these conditions are fulfilled, and B is a certain auxiliary pole. We then substitute the conditions of orthogonality (7.1) in equation (7.4) and obtain the following two equations

$$\int\limits_F \omega_A x\,dF = \int\limits_F \omega_B x\,dF + (a_y - b_y)\int\limits_F x^2\,dF - (a_x - b_x)\int\limits_F xy\,dF + \\ + [(a_x - b_x)\,y(s_0) - (a_y - b_y)\,x(s_0)]\int\limits_F x\,dF = 0$$

$$\int\limits_F \omega_A y\,dF = \int\limits_F \omega_B y\,dF + (a_y - b_y)\int\limits_F xy\,dF - (a_x - b_x)\int\limits_F y^2\,dF + \\ + [(a_x - b_x)\,y(s_0) - (a_y - b_y)\,x(s_0)]\int\limits_F y\,dF = 0$$

Since the section is referred to the principal central axes,

$$\int\limits_F xy\,dF = J_{xy} = 0, \quad \int\limits_F x\,dF = S_y = 0, \quad \int\limits_F y\,dF = S_x = 0.$$

Using the expressions (6.9) and (6.13) for the moments of inertia, we obtain:

$$\left.\begin{aligned} a_x &= a_x - b_x = \frac{1}{J_x}\int\limits_F \omega_B y\,dF = \frac{J_{\omega_B y}}{J_x}, \\ a_y &= a_y - b_y = -\frac{1}{J_y}\int\limits_F \omega_B x\,dF = -\frac{J_{\omega_B x}}{J_y}. \end{aligned}\right\} \tag{7.5}$$

The point A with coordinates a_x, a_y is called the **principal sectorial pole**, or simply the **principal pole**.

We now verify that the coordinates of the point A which are determind by equations (7.5) do not depend on the position of the origin of the sectorial area ω_B, or, in other words, that the coordinates of these points are invariants with respect to changes of the origin of sectorial areas.

We first note the following obvious fact. The sectorial areas $\omega_B(s_1, s)$, and $\omega_B(s_0, s)$ which have a common pole at the point B but different sectorial origins s_0 and s_1, differ from one another by a constant $\omega_B(s_0, s_1)$, equal to twice the area of the sector enclosed between the radius-vectors connecting the pole with these points (Figure 25):

$$\omega_B(s_0, s) = \omega_B(s_1, s) + \omega_B(s_0, s_1). \tag{7.6}$$

Subtituting (7.6) in equation (7.5), we obtain

$$a_x - b_x = \frac{1}{J_x}\int_F y\omega_B(s_0, s)\,dF = \frac{1}{J_x}\left[\int_F y\omega_B(s_1, s)\,dF + \omega_B(s_1, s_0)\int_F y\,dF\right],$$

$$a_y - b_y = -\frac{1}{J_y}\int_F x\omega_B(s_0, s)\,dF =$$

$$= -\frac{1}{J_y}\left[\int_F x\omega_B(s_1, s)\,dF + \omega_B(s_1, s_0)\int_F x\,dF\right].$$

Since we are dealing with the principal axes,

$$\int_F x\,dF = S_y = 0, \quad \int_F y\,dF = S_x = 0,$$

and whence

$$a_x - b_x = \frac{1}{J_x}\int_F y\omega_B(s_0, s)\,dF = \frac{1}{J_x}\int_F y\omega_B(s_1, s)\,dF,$$

$$a_y - b_y = -\frac{1}{J_y}\int_F x\omega_B(s_0, s)\,dF = -\frac{1}{J_y}\int_F x\omega_B(s_1, s)\,dF,$$

which confirms our initial assertion.

It is seen from the foregoing that the principal pole is a special point of the beam cross section. The position of the pole (like the position of the centroid) depends only on the geometrical dimensions of the cross section.

It is easy to show that for shells and hipped cross sections with only one axis of symmetry, the principal pole lies on this axis. In the case of a section with two symmetry axes, the principal pole is situated at the intersection of the symmetry axes and consequently coincides with the centroid of the section.

The coordinates of the principal pole in the principal central axes of the beam are expressed, according to equation (7.5), by the coordinates of

another arbitrarily chosen point B, which serves as the pole for the auxiliary sectorial area ω_B. In particular we may place B at the centroid of the section. Then equations (7.5) will have the simpler form

$$a_x = \frac{1}{J_x}\int_F \omega_B y\, dF,$$

$$a_y = -\frac{1}{J_y}\int_F \omega_B x\, dF.$$

We turn now to the determination of the sectorial origin of the principal sectorial coordinate ω. We find this point by using the condition of orthogonality $S_\omega = 0$.

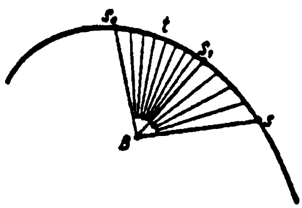


Figure 25

Note that the sectorial static moment, which is the integral of the product of the sectorial area ω with respect to the cross section area element dF, equals the difference between the static moments calculated separately for the parts of the section with positive and negative sectorial areas for an arbitrary choice of the point s_0 (but for a fixed position of the pole A). Because of this, we shall always be able to choose the origin of the sectorial areas in such a way that the sectorial static moment S_ω of the whole section vanishes.

Above we obtained equation (7.6) which related two sectorial areas having a common pole B but different sectorial area origins s_0 and s_1. This equation for the pole A of the principal sectorial coordinates will be:

$$\omega_A(s_1, s) = \omega_A(s_0, s) - \omega_A(s_0, s_1).$$

The quantity $\omega_A(s_0, s_1)$, which is equal to twice the area of the sector enclosed between the radius-vector As_0 and As_1 and the arc s_0s_1, is a function of the arc s_0s_1. Denoting the length of the arc by t (Figure 25), we introduce the following shortened notation:

$$\omega_A(s_0, s_1) = D(t). \tag{7.7}$$

Let s_1 be that point of origin for which the condition $S_\omega = 0$ is fulfilled. Imposing this condition on equation (7.7), we have:

$$\int_F \omega_A(s_0, s)\, dF - D(t)\, F = 0,$$

from which we find the following equation for $D(t)$:

$$D(t) = \frac{1}{F}\int_F \omega_A(s_0, s)\,dF.$$

In this equation sectorial area $D(t)$ is determind by the distance t between an arbitrarily chosen point s_0 (from which we measure the coordinate $\omega_A(s_0, s)$ under the integral sign) and the desired point s_1, for which the condition $S_\omega = 0$ is fulfilled. In the following we shall call this point the sectorial zero-point.

It is easy to show that the sectorial zero-point lies at the intersection of the symmetry axis with the cross section for hipped cross sections having a single symmetry axis.

There can be several or even an infinite number of sectorial zero-points. Thus, for a symmetrical I-section, any point on the web is a sectorial zero-point. We shall call the sectorial zero-point nearest to the principal pole the principal sectorial zero-point.

We shall dwell now on the determination of the principal sectorial coordinate, ω, and the corresponding principal sectorial moment, J_ω, for the case when the cross section is referred to an arbitrary system of coordinates Oxy.

We write equation (7.4) in a more compact form

$$\omega_A = \alpha_y x - \alpha_x y + \alpha + \omega_B, \tag{7.8}$$

where

$$\left.\begin{aligned} \alpha_x &= a_x - b_x, \\ \alpha_y &= a_y - b_y, \\ \alpha &= (a_x - b_x)\,y(s_0) - (a_y - b_y)\,x(s_0). \end{aligned}\right\} \tag{7.9}$$

Since the coordinates of the desired principal pole and the sectorial zero-points are unknown to us, the coefficients α_x, α_y and α are unknowns in the equation (7.8). The known quantities in this equation are the coordinates x and y and the sectorial coordinate ω_B, referred to an arbitrarily chosen pole B with an arbitrary origin. Assuming that ω_A is a principal sectorial coordinate which satisfies the conditions of orthogonality (7.1), we write these conditions term by term, substituting equation (7.8) for ω_A. We obtain the three equations

$$\begin{aligned} J_{\omega_A x} &= \int_F (\alpha_y x - \alpha_x y + \alpha + \omega_B)\,x\,dF = 0, \\ J_{\omega_A y} &= \int_F (\alpha_y x - \alpha_x y + \alpha + \omega_B)\,y\,dF = 0, \\ S_{\omega_A} &= \int_F (\alpha_y x - \alpha_x y + \alpha + \omega_B)\,dF = 0, \end{aligned}$$

or, expanded,

$$\left.\begin{aligned}
a_y\int_F x^2\,dF-a_x\int_F xy\,dF+a\int_F x\,dF+\int_F \omega_B x\,dF=0,\\
a_y\int_F yx\,dF-a_x\int_F y^2\,dF+a\int_F y\,dF+\int_F \omega_B y\,dF=0,\\
a_y\int_F x\,dF-a_x\int_F y\,dF+a\int_F dF+\int_F \omega_B dF=0.
\end{aligned}\right\}\tag{7.10}$$

Using (6.8)-(6.13), we may condense equations (7.10) as follows:

$$\left.\begin{aligned}
J_y a_y-J_{xy}a_x+S_y a=-J_{\omega_B x},\\
J_{xy}a_y-J_x a_x+S_x a=-J_{\omega_B y},\\
S_y a_y-S_x a_x+Fa=-S_{\omega_B}.
\end{aligned}\right\}\tag{7.11}$$

We obtain a system of three inhomogeneous algebraic equations in the three unknowns a_x, a_y and a. Since the matrix of the coefficients of this system is symmetric, it is possible to determine the unknowns a_x, a_y and a by using, for example, Gauss' shortened algorithm.

After determining a_x, a_y and a from equations (7.9) it is possible to construct the diagram of the principal sectorial coordinate ω_A and, at the same time, to obtain the position of the origin of the sectorial areas on the diagram.

Knowing the diagram of ω_A, it is easy to determine the principal sectorial moment of inertia J_{ω_A}. Indeed, substituting the ω_A according to (7.8) in the equality $J_{\omega_A}=\int_F \omega_A^2\,dF$, we obtain:

$$\begin{aligned}
J_{\omega_A}=a_y\left(J_y a_y-J_{xy}a_x+S_y a+J_{\omega_B x}\right)-a_x\left(J_{xy}a_y-J_x a_x+S_x a+J_{\omega_B y}\right)+\\
+a\left(S_y a_y-S_x a_x+Fa+S_{\omega_B}\right)+\\
+J_{\omega_B x}a_y-J_{\omega_B y}a_x+S_{\omega_B}a+J_{\omega_B}.
\end{aligned}\tag{7.12}$$

By virtue of (7.11) the expressions in the brackets must vanish. Therefore equation (7.12) assumes the simpler form:

$$J_{\omega_A}=a_y J_{\omega_B x}-a_x J_{\omega_B y}+aS_{\omega_B}+J_{\omega_B}.\tag{7.13}$$

Since we have determined a_x, a_y and a from the solution of the system (7.11) and for an arbitrary pole B and since we know $J_{\omega_B x}$, $J_{\omega_B y}$, S_{ω_B} and J_{ω_B}, we can now calculate J_{ω_A} from equation (7.13).

§ 8. Generalized cross-sectional forces. The bimoment and its physical meaning

Solving the differential equations (7.2) for a general load, or equations (7.3) for the case when $q_z = T_L = T_K = 0$, we determine the functions $\zeta(z)$, $\xi(z)$, $\eta(z)$ and $\theta(z)$ for given boundary conditions. Knowing these functions we are able to find the normal and tangential stresses, and also the bending moments which appear in the cross section of the beam from equations (5.6), (5.14), and (5.7). We had such equations for these stresses and moments only for the case of a beam under transverse load $(q_z = T_L = T_K = 0)$*

$$\left.\begin{aligned} \sigma &= E(\zeta' - \xi'' x - \eta'' y - \theta'' \omega), \\ \tau &= E\left[\xi''' \frac{S_y(s)}{\delta} + \eta''' \frac{S_x(s)}{\delta} + \theta''' \frac{S_\omega(s)}{\delta}\right], \\ H_k &= GJ_d \theta', \end{aligned}\right\} \tag{8.1}$$

where x, y and ω are the principal coordinates of the section points, $S_x(s)$, $S_y(s)$ are the static moments of the cut-off part of the cross section, and $S_\omega(s)$ is the sectorial static moment calculated for the same part of the section.

The author's theory of thin-walled beams is based on the "lumping" of a t w o - d i m e n s i o n a l c o n t i n u o u s e l a s t i c s y s t e m, i.e., a system whose equilibrium is described by differential equations with partial derivatives with respect to the coordinates z and s of the middle surface of the shell, into a d i s c r e t e - c o n t i n u o u s s y s t e m, i.e., a system possessing a finite number of degrees of freedom in the transverse directions and an infinite number of degrees of freedom in the direction of the generator of the shell.

Indeed, it was shown in §3 that an elementary (transverse) strip of the beam, enclosed between the sections $z = \text{const}$ and $z + dz = \text{const}$ has, according to the geometrical hypotheses adopted in our theory, seven degrees of freedom in space. The displacements from the plane $z = \text{const}$ (longitudinal displacements $u(z, s)$) are determined, according to (3.16), by the four generalized kinematical quantities: ζ, ξ', η' and θ'. The displacements in the plane $z = \text{const}$ (transverse displacements) $v(z, s)$ and $w(z, s)$ are determined, according to (3.9) and (3.10), by the three quantities ξ, η and θ.

Using these kinematical degrees of freedom, the internal normal and shear forces in a thin-walled beam, which arise on the internal surface elements of the cross section $z = \text{const}$, can be transformed into generalized longitudinal and transverse forces which refer to the whole cross section of the beam.

Starting from the notion of virtual work, we can determine the generalized longitudinal forces as the work of all the elementary longitudinal forces σdF in each of the admissible longitudinal generalized

* In this case, in virtue of the first equation (7.3), $\zeta'' = 0$.

displacements that we have examined, and put the work equal to unity for the kinematic model we have adopted. If we assume consecutively that in equation (3.16) $\zeta=1$, $\xi'=-1$, $\eta'=-1$ and $\theta'=-1$ (the remaining three variables vanish), we obtain the four elementary states of longitudinal displacement $u(z, s)$ in the section $z=$const: 1, x, y and ω. If we multiply the elementary longitudinal force, σdF, successively by each of the functions 1, x, y and ω and integrate over the whole area of the section, we obtain for the generalized longitudinal forces which correspond to these states the following equations:

$$\left.\begin{aligned} N &= \int_F \sigma 1\, dF, \\ M_x &= \int_F \sigma y\, dF, \\ M_y &= -\int_F \sigma x\, dF, \\ B &= \int_F \sigma \omega\, dF. \end{aligned}\right\} \qquad (8.2)$$

The first three equations determine the longitudinal force and the bending moment of the cross section of the beam. According to our sign convention (Figure 26), if a positive normal force σdF acts on an element of the section area in the first quadrant of the Oxy plane, a positive moment with respect to the Ox-axis and a negative moment with respect to the Oy-axis is produced. The fourth equation (8.2) determines a new statical quantity which has the dimensions $\text{kg}\cdot\text{cm}^2$ and which corresponds to the sectorial warping of the section ω. We denote this quantity by B and call it the b i m o m e n t. In contrast to the moment, the bimoment is a generalized balanced force system, i.e., a force system statically equivalent to zero.

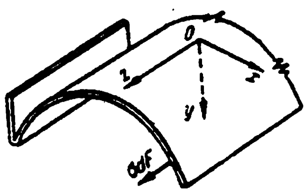


Figure 26

Substituting in (8.2) the first of equations (8.1) for σ and remembering that we are dealing with the principal generalized coordinates (1, x, y, ω) which satisfy the conditions of orthogonality, we obtain:

$$\left.\begin{aligned} N &= EF\zeta', \\ M_y &= EJ_y\xi'', \\ M_x &= -EJ_x\eta'', \\ B &= -EJ_\omega\theta''. \end{aligned}\right\}$$

Equations (8.3) establish a differential relation between the basic generalized quantities—the statical quantities N, M_y, M_x, B and the kinematical quantities ζ, ξ, η and θ —and characterize the variation of the stresses and strains in the beam through the variable z.

These equations can also be presented in the form:

$$\left.\begin{aligned}\zeta' &= \frac{N}{EF},\\ \xi'' &= \frac{M_y}{EJ_y},\\ \eta'' &= -\frac{M_x}{EJ_x},\\ \theta'' &= -\frac{B}{EJ_\omega}.\end{aligned}\right\} \tag{8.4}$$

Substituting expressions (8.4) in the equation (8.1) for the normal stresses σ, we obtain

$$\sigma = \frac{N}{F} - \frac{M_y}{J_y}x + \frac{M_x}{J_x}y + \frac{B}{J_\omega}\omega. \tag{8.5}$$

This equation expresses the normal stresses $\sigma(z, s)$ in terms of the generalized forces $N(z)$, $M_y(z)$, $M_x(z)$ and $B(z)$ and the principal generalized coordinates, 1, $x(s)$, $y(s)$ and $\omega(s)$. The first three terms of equation (8.5) coincide with the equations known from the theory of strength of materials and are based on the law of plane sections. The fourth term determines the normal stresses which arise because the sections do not remain plane under torsion, but are warped according to the law of sectorial areas. These stresses are proportional to the product of the bimoment B and the sectorial coordinate ω. The reciprocal of the principal sectorial moment of inertia J_ω serves here as the coefficient of proportionality.

We shall call the quantity EJ_ω, which figures in the last equations (8.3) and (8.4) and which characterizes the stiffness of the beam for those cases of torsion when normal stresses $\sigma_\omega = \frac{B}{J_\omega}\omega$ distributed according to the law of sectorial areas arise in the cross section, the *sectorial warping rigidity of the beam* (in analogy with the theory of bending).

Similarly we can express the functions, which characterize the variation of the tangential stresses τ through the variable z, by statical factors, to wit, forces. We determine these forces as the work of the elementary contour tractions $T\,ds = (\tau\delta)\,ds$ in each of the possible unit displacements $\xi = 1$, $\eta = 1$, $\theta = 1$ of the section contour in the plane $z = \text{const}$. From equations (3.9), (3.15) we can determine them in the following way,

$$\left.\begin{aligned}&Q_x = \int_F (\tau\delta)\,dx, \quad Q_y = \int_F (\tau\delta)\,dy,\\ &H_\omega = \int_F (\tau\delta)\,d\omega = \int (\tau\delta)\,h\,ds,\end{aligned}\right\} \tag{8.6}$$

where Q_x, Q_y are the transverse forces in the section $z=$const which act in the direction of the Ox and Oy axes, and H_ω is the flexural-torsional moment with respect to the shear center due to the axial stress forces $T=\tau\delta$ which act along the tangent to the arc of the section contour.

Substituting the stresses τ, determined by the second of equations (8.1) in (8.6), and taking into consideration that as a result of the orthogonality of the principal coordinates 1, x, y and ω the following relations are valid*

$$\int_F S_x\,dx=0,\quad \int_F S_x\,d\omega=0,\quad \int_F S_\omega\,dx=0,$$
$$\int_F S_y\,dy=0,\quad \int_F S_y\,d\omega=0,\quad \int_F S_\omega\,dx=0,$$

we obtain:

$$\left.\begin{aligned} Q_x&=-EJ_y\xi''', \\ Q_y&=-EJ_x\eta''', \\ H_\omega&=-EJ_\omega\theta'''. \end{aligned}\right\} \tag{8.7}$$

In equations (8.7) the transverse forces and the flexural-torsional moments H_ω are expressed in terms of the third derivatives of the deflections ξ, η of the lines of the principal poles and of the angle of rotation θ. The coefficients of these derivatives are the flexural rigidities EJ_x and EJ_y according to the law of plane sections, and the torsional rigidity EI_ω according to the law of sectorial areas. From equations (8.7) we find:

$$\left.\begin{aligned} \xi'''&=-\frac{1}{EJ_y}Q_x, \\ \eta'''&=-\frac{1}{EJ_x}Q_y, \\ \theta'''&=-\frac{1}{EJ_\omega}H_\omega. \end{aligned}\right\} \tag{8.8}$$

* Indeed, integrating by parts, we obtain

$$\int_F S_x\,dx=S_x x\Big|_F-\int_F x\,dS_x=S_x x\Big|_F-\int_F xy\,dF=0,$$

. . . .

$$\int_F S_\omega\,dx=S_\omega x\Big|_F-\int_F x\,dS_\omega=S_\omega x\Big|_F-\int_F x\omega\,dF=0,$$

but,

$$\int_F S_y\,dx=S_y x\Big|_F-\int_F x\,dS_y=S_y x\Big|_F-\int_F x^2\,dF=-J_y$$

and, analogously,

$$\int_F S_x\,dy=-J_x,\quad \int_F S_\omega\,d\omega=-J_\omega$$

By substituting the values found in (8.8) in equation (8.1) we obtain for the axial tangential stresses:

$$\tau = -\frac{1}{\delta}\left[\frac{Q_x}{J_y} S_y(s) + \frac{Q_y}{J_x} S_x(s) + \frac{H_\omega}{J_\omega} S_\omega(s)\right]. \tag{8.9}$$

By differentiating and eliminating the derivatives of the displacements, we obtain from equations (8.4) and (8.8) the relations:

$$Q_x = -M'_y, \quad Q_y = M'_x, \quad H_\omega = B'. \tag{8.10}$$

These relations, which are a generalization of Zhuravskii's theorem known from the theory of strength of materials, are valid only for the case of the action of transverse loads on the beam. As we remarked in the beginning of this section, equation (8.9) is valid only for the case of transverse loading. In the general case, when a longitudinal load is also present, applied as shearing tractions on the longitudinal edges, it is necessary to use instead of (8.9) the equation

$$\tau = \frac{1}{\delta}\left[T_K + (T_L - T_K)\frac{F(s)}{F} - \frac{Q_x}{J_y} S_y(s) - \frac{Q_y}{J_x} S_x(s) - \frac{H_\omega}{J_\omega} S_\omega(s)\right],$$

where T_K and T_L are the shear forces which act along the longitudinal edges of the beam and $F(s)$ is the area of the "cut-off" part of the section.

The third of equations (8.7) refers to the flexural-torsional moment H_ω about the principal pole, which is due to shear forces $\tau\delta$ which act along the tangent to the contour of the section. The third of equations (8.1) describes the torsional moment H_k, which appears due to the nonuniform distribution of the tangential stresses over the thickness of the wall. These stresses are proportional to the distance from the examined point to the profile line.

The sum of the moments H_ω and H_k gives the total torsional moment, which we shall denote by H:

$$H = H_\omega + H_k = -EJ_\omega\theta''' + GJ_d\theta'. \tag{8.11}$$

§ 9. The shear center*

1. We shall call the line lying parallel to the generator of the beam and that has the property that if the external transverse load (reactions included) passes through it the beam will be in the condition of central transverse bending, the line of the shear centers. By central transverse bending we mean a pure bending deformation of the beam, i.e., a deformation for which the cross sections which remain

* [The Russian text uses the term "center of flexure".]

plane suffer only translational displacements determined by the deflections $\xi=\xi(z)$, $\eta=\eta(z)$ (Figure 27a).

If the transverse load, including the support reactions, leads to a resultant even for one part of the beam which does not pass through the line of bending centers, then the beam will be in a condition of complicated response to bending and torsion. Complementary stresses of flexural torsion, determined by the law of sectorial areas, appear in the sections of the beam in addition to the bending stresses (Figure 27b).

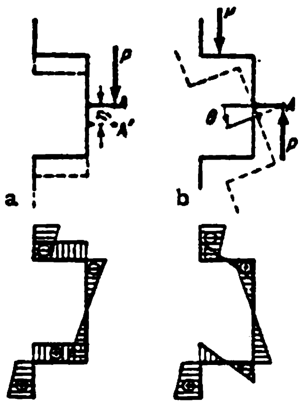


Figure 27

In § 7 we determined the coordinates of the principal pole A. We showed that for a beam of constant section the principal pole is simultaneously the line of the shear centers. For this case we write equation (8.9) for the tangential stresses as

$$\tau=-\frac{1}{\delta}\left[\frac{Q_x}{J_y}S_y(s)+\frac{Q_y}{J_x}S_x(s)+\right.$$
$$\left.+\frac{H_\omega}{J_\omega}S_\omega(s)\right]. \quad (8.9)$$

It is easy to verify that the first two terms of this equation determine the tangential stresses due to bending when the transverse load passes through the line of the principal poles. The third term determines the tangential stresses due to flexural torsion. They are proportional to the torsional moment H_ω and appear in the case when $H_\omega \neq 0$, i.e., when the transverse load (support reactions included) does not pass through the line of the principal poles. Indeed, the tangential stresses, determined by the first two terms of equation (8.9), contribute no torsional moment about the principal pole in central coordinate axes:

$$\int_F \tau\delta h\, ds=\int_F \tau\delta\, d\omega=-\frac{Q_x}{J_y}\int_F S_y\, d\omega-\frac{Q_y}{J_x}\int_F S_x\, d\omega=0.$$

The flexural-torsional moment H_ω is obtained from the tangential stresses which are determined only by the last term of equation (8.9):

$$\int_F \tau\delta\, d\omega=-\frac{H_\omega}{J_\omega}\int_F S_\omega\, d\omega=H_\omega. \quad (9.1)$$

Hence it follows that for $H_\omega\equiv 0$ the transverse force and, consequently, also the external transverse load which balances it in each section (i.e., for any value of z) passes through the line of the principal sectorial poles. This means that this line is also simultaneously the line of

shear centers. Therefore in the following, unless otherwise stated, the principal sectorial pole will be called the s h e a r c e n t e r , and the line of the principal sectorial poles will be called the l i n e o f s h e a r c e n t e r s .

It should be noted that it is also possible to reach our conclusions regarding the lines of the principal sectorial poles by another purely analytical method. To do this, we examine the last differential equation of equilibrium of the beam (7.3) for the angle of rotation $\theta(z)$:

$$EJ_{\omega}\theta^{IV} - GJ_{d}\theta'' = m. \tag{9.2}$$

When the beam is under transverse bending only, the integral of this equation $\theta(z)$ should vanish identically. This is obviously so providing, first, that the load distributed on the exterior passes through the line of principal sectorial poles, i.e., the equation is homogeneous and $m(z) \equiv 0$ and, second, that the boundary conditions for equation (9.2) are homogeneous*. The fulfillment of the last conditions leads in fact to no external concentrated torsional moments about the line of principal poles (i.e., the concentrated transverse load should also pass through this line) and to no concentrated longitudinal bimoments, since the influence of concentrated force factors is taken into account by the boundary conditions. If the transverse load satisfies these requirements, there will be no torsion. Therefore, the line of principal sectorial poles is also simultaneously the line of shear centers.

2. We shall denote the coordinates of the shear center by a_x, a_y, as determined by the general equations (7.5). Starting from the notions outlined above of the equilibrium of a beam under central transverse bending, we shall give here another statical method for determining the coordinates of the shear center. The shear center can be determined as the point of intersection of the line of action of the transverse forces Q_x and Q_y for two independent bending deformations in the principal planes.

As previously in § 7, let b_x and b_y be the coordinates of an arbitrarily chosen pole B and a_x and a_y be the coordinates of the shear center A.

For bending in the Oyz plane, the shear stresses $T = \tau\delta$ are determined by the equation:

$$\tau\delta = -\frac{Q_y}{J_x} S_x.$$

The torsional moment about the point B due to these forces, is given by

$$H_B = \int_F \tau\delta \, d\omega_B = -\frac{Q_y}{J_x} \int_F S_x \, d\omega_B = \frac{Q_y}{J_x} J_{\omega_B y}. \tag{9.3}$$

On the other hand, according to Varignon's theorem, this moment is equal to the moment of the resultant of the shear stresses, which passes through the shear center:

* This means that either of the conditions (2.8) and (2.10) of Chapter II is satisfied at the ends of the beam.

$$H_B = Q_y (a_x - b_x). \tag{9.4}$$

Equating the right-hand sides of equations (9.3) and (9.4), we find the coordinate a_x of the shear center:

$$a_x - b_x = \frac{J_{\omega_B y}}{J_x}.$$

Similarly we obtain from examination of the bending in the |Oxz plane:

$$a_y - b_y = -\frac{J_{\omega_B x}}{J_y}.$$

Both equations coincide with the equations (7.5) given earlier.

3. In the classical theory of bending, which is based on the law of plane sections, the concept of the beam axis is of fundamental importance. In the theory of strength of materials the line of centroids of the cross sections of the beam is tacitly assumed as such an axis. In problems of structural engineering, the bending element of any beam system (multiple span beams, frames) is also considered in the light of the elementary theory of bending and is identified with an elastic uniform model, namely, with an axis through the centroids of the beam sections. These fundamental concepts from the theory of the strength of materials and from the structural engineering of beam systems, which is a consequence of the law of plane sections and which is essentially based on Saint-Venant's principle, must be radically revaluated and generalized in the light of the general bimoment theory of flexural torsion that we have formulated. When the bending element (considered as a spatial system) of the structure has dimensions that classify it as a thin-walled beam, the line of shear centers assumes a fundamental importance. This line coincides with the line of centroids of the section for beams having two axes of symmetry in the cross section. For a beam with an arbitrary asymmetrical cross section or a cross section with one symmetry axis, the line of shear centers will not coincide with the line of centroids of the sections. In applying to such beams the usual theory of bending which is only valid for the case of pure flexure, we must take as the axis of the beam the line of shear centers instead of the line of centroids, and consider that the transverse loads, which cause the bending of the beam according to the law of plane sections, are applied at the points of the line of shear centers. If we refer the transverse load (distributed over the span or consisting of concentrated forces) to points of the line of centroids, the load will cause torsion in addition to flexure. The conventional theory of bending is not suitable for such a case.

We illustrate the foregoing discussion by an example of a beam with a channel cross section. For such a cross section the centroid O and the shear center A are situated on the symmetry axis on opposite sides of the web (see Figure 34). A transverse vertical load which passes through the line of centroids, causes not only bending but also torsion in this beam, accompanied by warping of the cross sections. When a transverse load acts on a channel beam it will undergo bending as prescribed by the

customary theory only for the case when this load is applied at points of the line of shear centers situated outside the section.

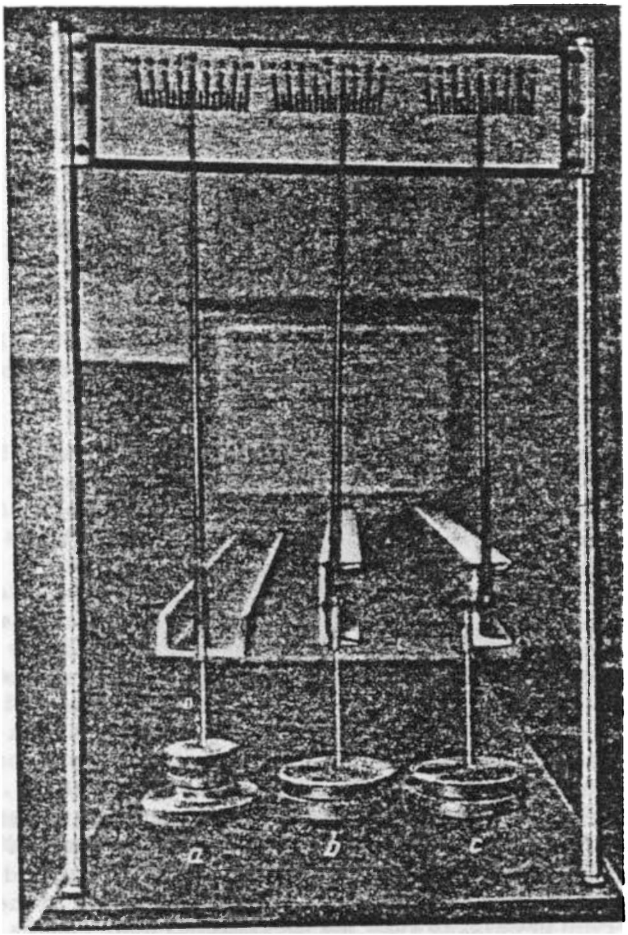


Figure 28

Figure 28 shows a model of a [cantilever] channel having one end encastered in a vertical wall and loaded at the free end by a concentrated force. At the free end, on the symmetry axis of the cross section, there is a ledge rigidly connected to the beam.

A force applied at the centroid of the section in the direction of the axis of symmetry of the cross section causes transverse bending of the beam (Figure 28a).

A force applied at the centroid of the section in a direction parallel to the wall causes torsion of the cantilever beam (Figure 28b) in addition to bending. The angle of rotation of the beam is determined by a pointer affixed to the free end of the beam.

A force applied at the shear center in a direction parallel to the web of the profile causes bending only (Figure 28c).

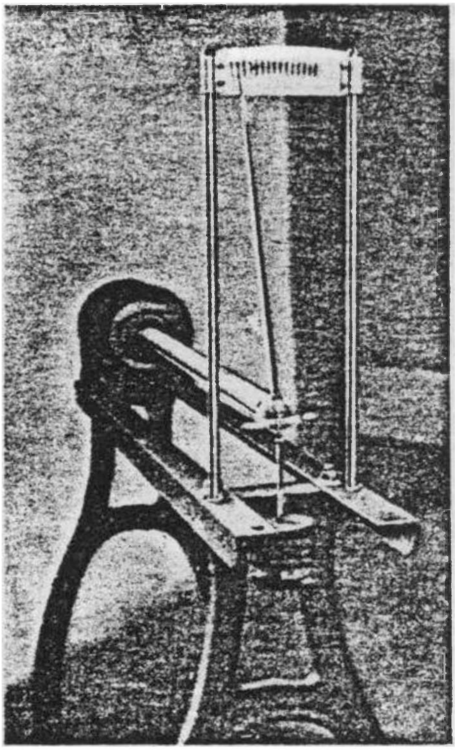


Figure 29

A model of a cantilever tube with a longitudinal slit is shown in Figure 29. The shear center of such a tube lies a distance of one diameter away from the centroid of the section on the farther side of the slit. A transverse load applied at the centroid normally to the plane of symmetry (which passes through the longitudinal slit) causes torsion of the beam in addition to bending.

Chapter II

CALCULATIONS FOR THIN-WALLED BEAMS OF OPEN CROSS SECTION

§ 1. Coordinates of the shear center and sectorial geometrical characteristics for certain cross sections

Acccrding to our theory it is necessary to calculate more than the geometrical characteristics of the cross section required by the law of plane sections when dealing with thin-walled beams of open cross section subjected to simultaneous bending and torsion. It is also necessary to calculate the geometrical characteristics required by the law of sectorial areas. We shall examine several examples of the calculation of the coordinates of the shear center and of sectorial characteristics. In the examples shown below we shall refer the section to the principal central axes.

1. I-Section with Unequal Flanges. We denote by d_1, d_2, d_3 and δ_1, δ_2, δ_3 (respectively) the width and thickness of the section elements. The axis of symmetry of the section is Oy (Figure 30). We use the general equations (I.7.5) in order to determine the coordinates of the shear center. It is unnecessary in the present case to determine the sectorial zero-point, since the cross section is symmetric about the web, and any point of the section lying on the web is a sectorial zero-point.

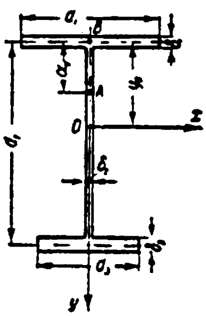


Figure 30

We place an auxiliary pole, B, of the sectorial area ω_B at the junction of the web and the upper flange of the I-beam. We take an arbitrary point of the web of the cross section as sectorial origin. The integrals of equations (I.7.5) are calculated by methods of structural mechanics, for which the diagrams of the geometrical quantities that occur in the integrals must be constructed (Figure 31).

The diagrams of the sectorial areas ω_B are shown in Figure 31a. The sectorial areas for the sections of the upper flange and the web are equal to zero. The lower flange has a skew-symmetrical diagram.

Figures 31b, c, show the diagrams of the distances x and y from the points of the section profile line to the principal axis.

We calculate the integrals $\int_F x\omega_B dF$ and $\int_F y\omega_B dF$ by the methods of structural mechanics, i. e., for each section the area of one diagram is multiplied by the ordinate of the other diagram situated under its centroid.

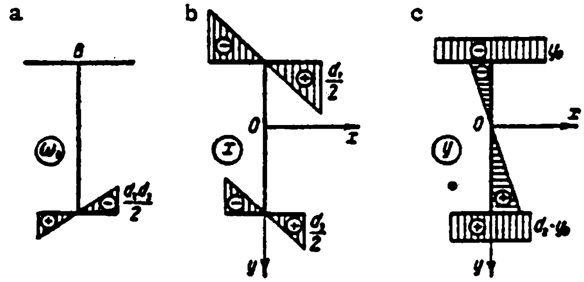


Figure 31

Thus, taking into account the thickness of the element, δ, we obtain

$$J_{\omega_B x}=\int_F \omega_B x\,dF=-\frac{d_3^3\delta_3}{12}d_2=-J_{3y}d_2,$$

$$J_{\omega_B y}=\int_F \omega_B y\,dF=0.$$

for the moment of inertia J_y, we have

$$J_y=\int_F x^2dF=\frac{d_1^3\delta_1}{12}+\frac{d_3^3\delta_3}{12}=J_{1y}+J_{3y},$$

where J_{1y}, J_{3y} are the moments of inertia of the flanges with respect to the y-axis.

The coordinates of the shear center, in accordance with equations (I.7.5) are therefore:

$$\left.\begin{aligned}a_x&=0,\\ a_y&=\frac{d_3^3\delta_3}{d_1^3\delta_1+d_3^3\delta_3}d_2=\frac{J_{3y}}{J_{1y}+J_{3y}}d_2.\end{aligned}\right\}\qquad(1.1)$$

The equation $a_x=0$ shows that the shear center lies on the axis of symmetry. The positive distance a_y from this center to the point B, on the Oy-axis, is determined by the second equation of (1.1). For the case of an I-section with equal flanges (having two axes of symmetry) the shear center coincides with the centroid of the section:

$$a_y=\frac{d_2}{2}.\qquad(1.2)$$

Equation (1.2) follows from (1.1) on substituting $d_1=d_3$ and $\delta_1=\delta_3$.

The equation for the coordinates of the shear center of a T-section are readily obtained from formulas (1.1). In this case it is necessary to assume that the area of one of the flanges is zero. Setting $d_2=0$, we obtain:

$$\alpha_x=0, \quad \alpha_y=0; \tag{1.3}$$

but when $d_1=0$, we obtain:

$$\alpha_x=0, \quad \alpha_y=d_2. \tag{1.4}$$

Equations (1.3) and (1.4) show that for a T-profile the shear center lies at the junction of web and flange. Indeed, as we have seen in § 4, Chapter I, the cross section of a beam consisting of a bundle of plates is not warped; consequently the bending moment H_ω vanishes, indicating that the resultant of the tangential stresses in the cross section (assuming that these stresses are uniformly distributed over the thickness of the plates) passes through the shear center. This point is the center of the stack, since the tangential stresses are parallel to the profile of the beam and, consequently, their resultant passes through the center.

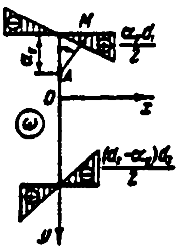


Figure 32

Having determined the shear center for an I-section with flanges of different widths, we can now construct the diagram for the principal sectorial areas ω by using the radius-vector AM. This diagram is shown in Figure 32. The point A and the sectorial area origin lie on the axis of symmetry Oy. The distribution over the section of the complementary normal stresses σ_ω due to flexural torsion is expressed by the diagram of the sectorial areas ω (Figure 32).

The equation for the sectorial moment of inertia is:

$$J_\omega=\int_F \omega^2 dF=\alpha_y^2 J_{1y}+(d_2-\alpha_y)^2 J_{2y}.$$

From equation (I.8.9) we have, for the tangential stresses τ_ω, due to the warping of the beam sections,

$$\tau_\omega=-\frac{H_\omega(z)S_\omega(s)}{J_\omega\delta(s)}. \tag{1.5}$$

It is evident from this that the complementary shear stresses τ_ω in the beam section vary according to the law of the sectorial static moment:

$$S_\omega(s)=\int_s \omega dF.$$

This moment is calculated for the "cut-off" section of the contour, i. e., for that section which is situated on one side of the examined point on the contour. Since the right-hand side of formula (1.5) contains the minus sign, and the moment of the complementary tangential stresses about the shear center is the torsional moment H_ω, it follows that the diagram of

the sectorial static moments, taken with an opposite sign, expresses the law of distribution of the tangential stresses τ_ω over the section. We also note that the sign of $S_\omega(s)$ depends on the direction of rotation of the section contour, which must be ascertained separately in each case according to the profile of the beam.

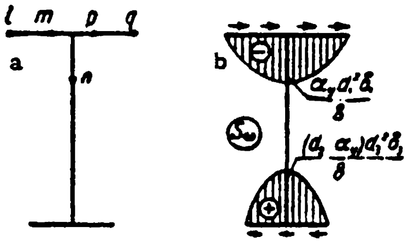


Figure 33

We return to our example, and agree that the coordinate s increases from left to right for the upper and for the lower flanges of the I-section. From this it is easy to construct the diagram S_ω. Thus, the sectorial static moment will be negative for the cut-off section of the profile lm (Figure 33a) since we move in the positive sense on the boundary of the contour l to the point m and the ω-diagram for the section lm is negative. The moment S_ω will also be negative for the cut-off section of the profile pq, since we move in the negative sense on the boundary of the contour q, to the point p, and the sectorial area for this section is positive. The diagram of the sectorial static moment is given in Figure 33b. The moments S_ω vary according to a parabolic law over the width of the shell, since the sectorial area for the upper and lower flanges is given by an odd linear function. For the web of the profile, the sectorial static moment vanishes, since the sectorial area for the points of the web is zero.

We mention that earlier (§ 5, Chapter I) we took the positive direction of the tangential stresses with respect to the positive cross section in the direction of the coordinate s, and considered the torsional moment with respect to the same area as positive in the clockwise sense. Taking this into consideration, it is easy to determine the direction of the complementary shear stresses in the cross section of the beam. The direction of these stresses in the flanges of the I-section is indicated by arrows in Figure 33b. The complementary shear stresses vanish at any point on the web of the H-beam, since for these points S_ω vanishes.

2. Trough Section (Channel). Let d and δ be, respectively, the depth and thickness of the web, and d_1 and δ_1 the width and thickness of the flanges of the trough section (Figure 34). The shear center A for such a section lies on the axis of symmetry Ox. The sectorial zero-point is at the intersection of the axis of symmetry and the profile of the beam. The distance a_x from the wall to the shear center A is determined by equation (I.7.5). The diagram of the sectorial areas ω_B with a pole B at the point of intersection of the web and the axis of symmetry Ox, is shown in Figure 35a; the diagram of the ordinates y is given in Figure 35b. Using the methods of structural mechanics for the calculation of the definite integral of the product of the function, given by the diagrams ω_B and y, and taking into consideration the thickness of the flange δ_1, we obtain:

$$J_{\omega_B y} = \int_F y\omega_B\, dF = -\frac{d^2 d_1^2 \delta_1}{4}. \qquad (1.6)$$

The moment of inertia J_x is calculated as the integral of the squares of the ordinates of the diagrams, y, over the area. F, of the section

$$J_x = \int_F y^2\, dF = \frac{d^3\delta}{12} + \frac{d^2 d_1 \delta_1}{2}. \tag{1.7}$$

Substituting (1.6) and (1.7) in the corresponding equation (I.7.5), we obtain the coordinates of the shear center:

$$a_x = \frac{J_{\omega_B y}}{J_x} = \frac{\int_F y\omega_B\, dF}{\int_F y^2\, dF} = -\frac{3d_1^2\delta_1}{d\delta + 6d_1\delta_1} = -\frac{d_1^2\delta_1}{2d_1\delta_1 + \frac{d\delta}{3}}.$$

The diagram of the principal sectorial areas ω, which is shown in Figure 36a, is skew-symmetrical with respect to the axis Ox. The point B, lying on this axis serves as the origin of the areas. The sectorial areas

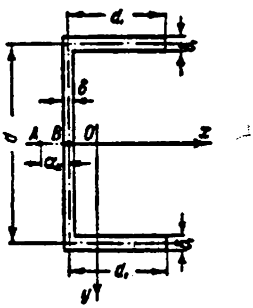


Figure 34

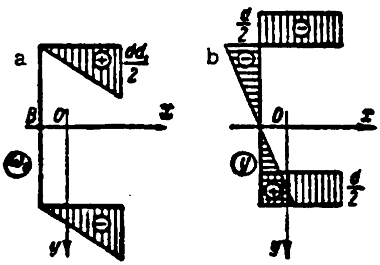


Figure 35

for the points of the web below the axis Ox, will be positive, since these areas are swept in the clockwise sense by the radius-vector AM. At the junctions of flanges and web, the sectorial areas attain their maximal values. This indicates that in torsion the maximum complementary stresses will appear in the channel corners.

The absolute value of the sectorial area for the flanges of the section decreases as we get further away from the web; at the point C, removed from the web a distance equal to that of the shear center A from the web, the sectorial area vanishes.

The sectorial moment of inertia J_ω is computed as the integral of the squares of the ordinates of the diagram ω over F. Using the methods of structural mechanics we obtain:

$$J_\omega = \int_F \omega^2 dF = \frac{1}{6}(d_1 - 3a_x)\, d^2 d_1^2 \delta_1 + a_x^2 J_x,$$

where J_x is the moment of inertia (1.7).

The diagram of the sectorial static moments S_ω, characterizing the distribution over the section of the shear stresses $T = \tau\delta$, due to torsion, is given in Figure 36b. We choose the clockwise sense around the contour as positive. The moments S_ω on each rectangular section of the contour vary according to a quadratic law and their absolute values are maximal at those points where the sectorial areas ω vanish. The directions of the torsional shear stresses in the cross section of the beam are indicated by arrows in Figure 36b.

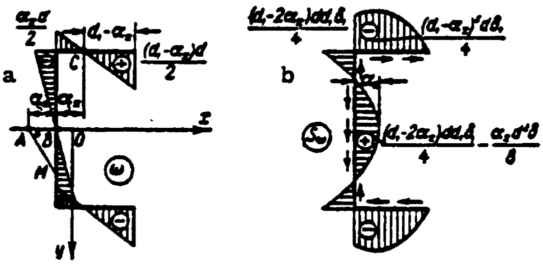


Figure 36

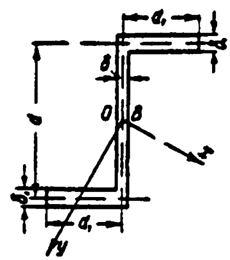


Figure 37

3. Z-Section. In order to determine the shear center in a Z-section with width and thickness equal to d and δ, respectively, for the web, and d_1 and δ_1, respectively, for the flanges, we choose an auxiliary pole B at the centroid of the section (Figure 37). An arbitrary point on the web of the profile is taken as the origin of the sectorial areas. The diagrams of the sectorial areas ω_B and of the coordinates x and y for the section contour are given in Figure 38. If we evaluate, with the help of these diagrams, the integrals in equation (I.7.5) and if we keep in mind the fact that the diagrams of the sectorial areas for the flanges are the same, whereas the diagrams of the coordinates differ only by their signs, we obtain:

$$a_x = 0; \quad a_y = 0. \tag{1.8}$$

Equation (1.8) shows that for a Z-section the shear center coincides with the centroid.

We now construct the diagram of the principal sectorial areas. The pole A of these areas is situated at the shear center, i. e., in this case at the centroid of the section. We have to construct the ω diagram so, that this diagram should be nullified not only in conjunction with the diagrams x and y of bending, but also with the diagram of a uniform extension of compression.

Since the diagram ω_B, shown in Figure 38a, is orthogonal to the diagram x and y (Figure 38b, c), we have now to superpose it on the diagram which corresponds to the stresses of an axial extension (compression), and to choose the ordinates of the diagram ω such that the sectorial static moment S_ω for the whole section is zero.

Let the flange point M_0, situated a distance t from the web, serve as the origin of the sectorial areas ω. The diagram of sectorial areas ω will then have the form shown in Figure 39a. The ordinates of this

diagram are linear in the parameter t, which determines the position of the origin M_0. We choose t such that the sectorial static moment S_ω for the whole section vanishes, i. e.,

$$S_\omega = \int_F \omega\, dF = 0. \tag{1.9}$$

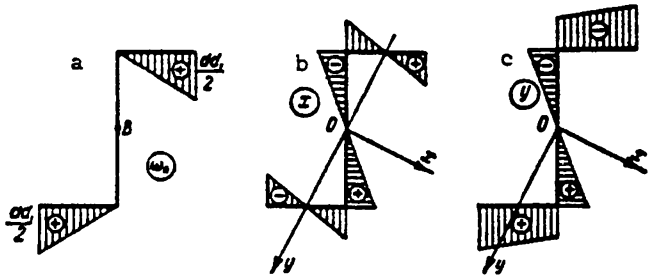


Figure 38

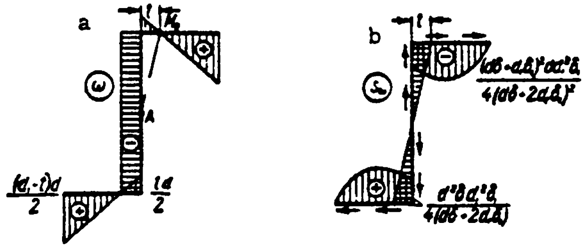


Figure 39

Developing expression (1.9) with the help of the diagram shown in Figure 39a, according to Vereshchagin's rule [?], we obtain:

$$\left(\frac{d_1}{2} - t\right) d d_1 \delta_1 - \frac{t d^2 \delta}{2} = 0,$$

whence

$$t = \frac{d_1^2 \delta_1}{d\delta + 2 d_1 \delta_1}. \tag{1.10}$$

The ω-diagram shown in Figure 39a, for t determined by equation (1.10) will be annulled by all the other terms of the general four-term equations (I.8.5). This means that of the four statical terms of formula (I.8.5) which vary over the section according to the law of the principal sectorial area ω, the normal stresses σ_ω give only the flexural-torsional bimoment B. The longitudinal force and the shear moments of these stresses vanish.

The sectorial area ω varies linearly over the width of the flange. On the web this area remains constant. This indicates that under flexural torsion of the Z-beam, the web undergoes a uniform extension or compression depending on the sign of the external torsional moment and on the conditions of support at the ends of the beam*.

For the sectorial moment of inertia we obtain:

$$J_\omega = \frac{d_1^3\delta_1}{12} d^2 \frac{d_1\delta_1 + 2d\delta}{d\delta + 2d_1\delta_1}.$$

The diagram of the sectorial static moment S_ω, characterizing the variation of the sectorial shear stresses, $T = \tau\delta$, in the cross section is given in Figure 39b. The sense of the motion on the section contour is from left to right. The S_ω-diagram on the flange sections is a parabola with maximum ordinates at the origin of sectorial areas. For the web, the sectorial static moment diagram has the form of a straight line, reaching the coordinate origin.

The resultants of the shear stresses, calculated on the segments of the cross section from the diagram of the sectorial static moments, is shown in the same figure. These resultants lead, for the whole section, to only one flexural-torsional moment H_ω.

4. Composite Asymmetrical Section. We examine a composite asymmetrical section, consisting of two No 12 channels (Figure 40a). Beams of such a section are used in industrial constructions such as truss girders.

We give the characteristics of the No 12 channel which we use in the calculations (GOST [Gosudarstvennyi obshchesoyuznyi standart—All-Union State Standard] 10017-39):

$$h = 12\text{cm}, \quad b = 5.3\text{cm}, \quad d = 0.55\text{cm}, \quad t = 0.9\text{cm},$$
$$F = 15.36\text{cm}^2, \quad J_x = 346.3\text{cm}^4, \quad J_y = 37.4\text{cm}^4, \quad z_0 = 1.62\text{cm}.$$

The centroid of the section is at the point O, which is situated 2.46 cm from the axis of the horizontal web and 3.54 cm from the axis of the vertical web. The principal axis Ox makes an angle $\beta = 29°\,53'$ with the horizontal axis (Figure 40b). The area of the section and the principal moments of inertia are respectively:

$$F = 30.72\text{cm}^2, \; J_x = 976.8\,\text{cm}^4, \quad J_y = 383.8\,\text{cm}^4. \qquad (1.11)$$

The axes of the section and length, thickness, and area of each portion are indicated in Figure 41a.

We shall take the straight axis of the web of the horizontal channel as the horizontal element of the section $2-B$ when determining the shear center and calculating the sectorial characteristics. The thickness of this element on the part $2-6$ is equal to the thickness of the channel web, and on the part $6-B$ is equal to the sum of the thicknesses of the web of the horizontal channel and of the flange of the vertical channel.

* The statement given here, which is based on Betti's theorem of reciprocity, is also valid inversely: a Z-beam subjected to an axial force, applied at a point of the web, also undergoes torsion, besides extension or compression. This will be discussed in greater detail in § 8.

By choosing an auxiliary pole at the point B which is the intersection of the web axes of the two channels, we obtain null diagrams of the sectorial areas for the sections $B-2$, $B-3$, and $B-4$, and triangular diagrams for the sections $1-2$ and $4-5$ (Figure 41b). The coordinates of the point B with respect to the principal axis will be

$$b_x = 4.29\,\text{cm}, \quad b_y = -0.38\,\text{cm}, \tag{1.12}$$

The diagrams of the coordinates of the points x and y of the profile line are given in Figure 42. Calculating the definite integrals of the product of ω_B and x and y and taking into consideration the thickness of the elements we obtain

$$J_{\omega_B x} = -1096.0\,\text{cm}^5, \quad J_{\omega_B y} = -38.3\,\text{cm}^5. \tag{1.13}$$

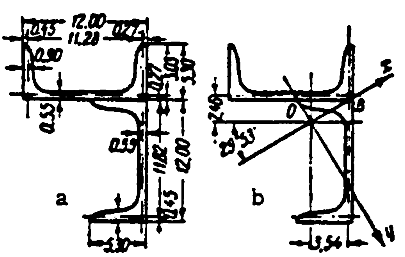


Figure 40

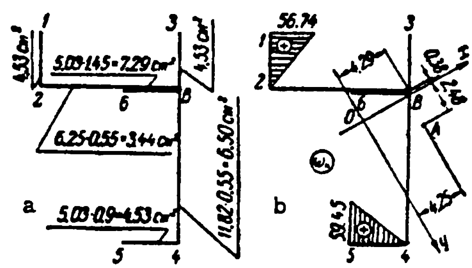


Figure 41

Introducing (1.11), (1.12), (1.13) in equation (I.7.5), we find the coordinates of the pole A:

$$a_x = 4.25\,\text{cm}, \quad a_y = 2.48\,\text{cm}.$$

The shear center A is 1.38 cm to the right of the vertical axis and 2.48 cm below the axis of the horizontal web (Figure 43a).

We determine the origin of the principal sectorial areas M_0. Let this point be on the web of the vertical channel at a distance t from the web axis of the horizontal channel. The diagram of the sectorial areas ω_A for a fixed distance t will be as shown in Figure 43a. Calculating with the help of this diagram the sectorial static moment S_ω for the whole section, we find:

$$S_\omega(t) = 42.53\,t - 102.72.$$

For an arbitrary t, this moment is a function of the position of the origin of the sectorial area.

Putting S_ω equal to zero we obtain:

$$t = 2.415\,\text{cm}.$$

For a given t the diagram of the principal sectorial areas ω will be as shown in Figure 43b.

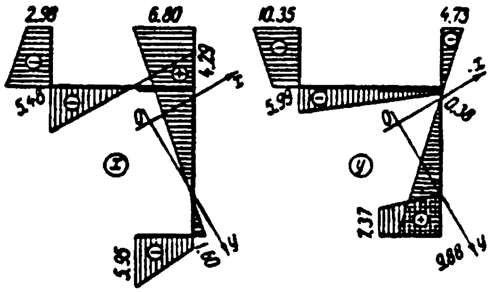


Figure 42

For the sectorial moment of inertia we obtain the value

$$J_{\omega} = 4829\,\text{cm}^6.$$

The diagram of the sectorial static moments S_{ω} is given in Figure 44.

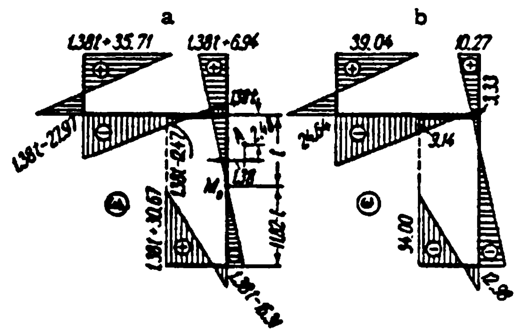


Figure 43

5. A Section in the Form of a Circular Arc. Let R be the radius of the profile of a circular section, δ the thickness of the section web, α half the central angle (Figure 45).

We place the pole B of the auxiliary sectorial area at the geometric center of the circular arc. We take the origin of the area ω_B at the point of intersection, M_0, of the axis of symmetry and the profile line. Denoting by φ the central angle corresponding to the point M that has the coordinate s on the contour, and letting the axis Oy coincide with the axis of symmetry of the section, we may write

$$\left.\begin{array}{c} s = R\varphi, \quad dF = R\delta d\varphi, \\ x = R\sin\varphi, \quad y = b_y - R\cos\varphi, \quad \omega_B = R^2\varphi. \end{array}\right\} \qquad (1.14)$$

According to (1.14), we obtain expressions for the sectorial products of inertia which enter into equation (I.7.5):

$$J_{\omega_B x} = \int_F x\omega_B dF = R^4\delta \int_{-\alpha}^{+\alpha} \varphi \sin \varphi d\varphi,$$
$$J_{\omega_B y} = \int_F y\omega_B dF = R^3\delta \int_{-\alpha}^{+\alpha} (b_y - R\cos\varphi)\,\varphi d\varphi.$$

In the expression for $J_{\omega_B x}$, the integral is an even function. This integral hence differs from zero. In the expression for $J_{\omega_B y}$ the integrand is an odd function. An integral of an odd function over an interval which is symmetrical with respect to the coordinate origin (as the area of the skew-symmetrical diagram) must vanish. Thus

$$\left.\begin{aligned} J_{\omega_B x} &= 2R^4\delta(\sin\alpha - \alpha\cos\alpha),\\ J_{\omega_B y} &= 0. \end{aligned}\right\} \tag{1.15}$$

Taking into consideration that $b_x = 0$, we obtain from equations (I.7.5) and (1.15),

$$\left.\begin{aligned} a_x &= a_x = 0,\\ a_y &= -\frac{J_{\omega_B x}}{J_y} = -\frac{2R^4\delta}{J_y}(\sin\alpha - \alpha\cos\alpha). \end{aligned}\right\} \tag{1.16}$$

The fact that $a_x = 0$ shows that the shear center is situated on the axis of symmetry. The distance, along the axis of symmetry, from the shear center to the geometric center of the circular arc (which coincides with the pole B), is determined by the second of equations (1.16).

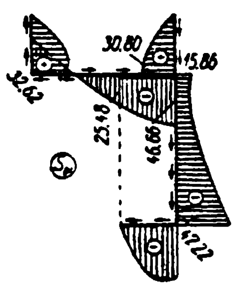


Figure 44

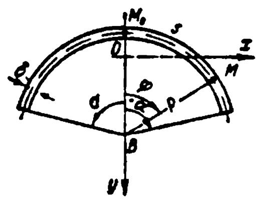


Figure 45

We calculate the moment of inertia J_y which enters into this equation. Neglecting the moment of inertia of the element $dF = \delta ds$ with respect to the tangent to the section contour arc as compared to the moment of inertia of the whole section, i. e., considering that the normal stresses σ are

uniformly distributed over the thickness of the wall, we may write

$$J_y = \int_F x^2\,dF.$$

Substituting in this equation the expressions for x and dF from equations (1.14) and integrating, we obtain:

$$J_y = R^3\delta \int_{-\alpha}^{+\alpha} \sin^2\varphi\,d\varphi = R^3\delta\,(\alpha - \sin\alpha\cos\alpha). \qquad (1.17)$$

The second of equations (1.16) now becomes:

$$a_y = -2R\,\frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}. \qquad (1.18)$$

Equation (1.18) determines the distance along the axis of symmetry from the shear center to the center of the circular arc as a function of the angle α (for a central angle 2α). The values of the coordinates a_y for various angles α are given below in Table 2.

Table 2

α	30°	60°	90°	120°	150°	180°
a_y	$-1.044\,R$	$-1.117\,R$	$-1.274\,R$	$-1.436\,R$	$-1.609\,R$	$-2\,R$

It is seen from this table that the shear center is situated outside the arc section.

For $\alpha = 180°$, i.e., in the case of a thin-walled tube slit along a generator, the shear center is at a distance $D = 2R$ from the geometrical center O, on the opposite side of the slit (Figure 46a)*.

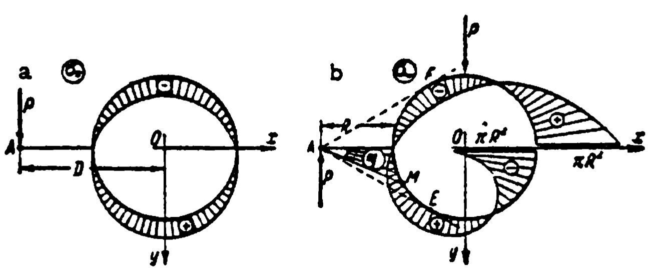


Figure 46

* In order to avoid misunderstanding we draw attention to the fact that in Figures 45, 48, 50, Oy represents the axis of symmetry while in Figures 46, 47, 49, this axis is represented by Ox.

If the ends of the tube are stiffened against bending and angular displacements and if we apply a concentrated transverse load to it in the middle of its span, the state of stress and strain of the tube will depend on the position of the load in the plane of the cross section. If the load passes through the shear center, the tube will be in a state of simple transverse bending (Figure 46a). If the load does not pass through the shear center, as in the case when it is loaded by its own weight (the load passes through the centroid of the section, Figure 47), the tube will be in a complex state of stress with simultaneous bending (Figure 46a) and torsion (Figure 46b). Complementary torsional stresses, determined by the law of sectorial areas, appear in addition to the bending stresses which obey the law of plane sections.

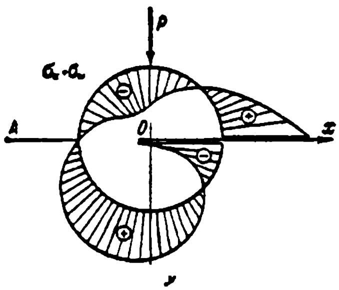


Figure 47

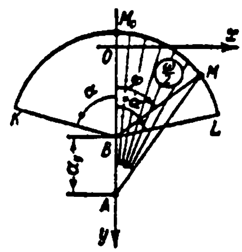


Figure 48

The diagrams of the ordinates y and of the sectorial areas ω, which characterize the distribution of the stresses σ_x and σ_{ω} in the cross section due to bending and torsion, are shown in Figures 46a and b. An example of the form of the diagram of the stress $\sigma_x + \sigma_{\omega}$ under the action of bending and torsion is given in Figure 47.

It is seen from the diagram of Figure 46b that there are three points in the cross section of the cut tube in which the torsional stresses σ_{ω} vanish. These points do not lie on a straight line, a fact which indicates that the section is warped. The sectorial stresses σ_{ω} reach their maximum absolute values at the points E and F (not taking into account the points which coincide with the slit) where the moving radius-vector, which sweeps the area ω, coincides with the tangent to the profile arc. At a point coinciding with the slit, the stresses σ_{ω} are discontinuous, owing to the free mobility of the lateral edges.

In the examined circular section one obtains a simple analytical expression for the diagram of the principal sectorial areas ω. From Figure 48 we have

$$\omega = R\,(a_y \sin\varphi + R\varphi).$$

In this equation a_y is taken with its true sign. For the sectorial moment of inertia we obtain the equation

$$J_{\omega} = \int_F \omega^2 dF = R^5\delta \int_{-\alpha}^{+\alpha} (a_y \sin\varphi + R\varphi)^2\, d\varphi =$$

$$= \frac{2}{3} R^5\delta \left[\alpha^3 - \frac{6(\sin\alpha - \alpha\cos\alpha)^2}{\alpha - \sin\alpha\cos\alpha}\right]; \qquad (1.19)$$

If $\alpha = \pi$ then $J_{\omega} = \frac{2}{3} R^5\delta\pi\,(\pi^2 - 6)$.

If the value of $\omega(\alpha)$ is known, it is easy to obtain an analytical expression for the sectorial static moment of the "cut-off" part of the section. We choose the clockwise sense of the contour as positive. Using Figure 48 for the cut-off part MK we obtain:

$$S_\omega = \int_F \omega dF = R^2\delta \int_{-\alpha}^{\varphi} (a_y \sin\varphi + R\varphi)\, d\varphi.$$

Setting in this equation $\alpha = \pi$, we obtain an expression S_ω for the slit tube,

$$S_\omega = R^3\delta \int_{-\pi}^{\varphi} (-2\sin\varphi + \varphi)\, d\varphi = R^3\delta\left(2\cos\varphi + \frac{\varphi^2}{2} - 2.93\right). \quad (1.20)$$

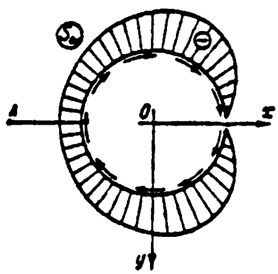


Figure 49

Figure 49 shows the diagram of the sectorial static moments computed by equation (1.20). It is seen from this diagram that in the case of torsion, the tangential stresses which are proportional to S_ω are maximal at two points of the cross section. These stresses lead statically to the flexural-torsional moment H_ω (§ 8, Chapter I) which together with the pure torsional moment counter-balances the external torsional couple.

Equations (I.7.5) for the calculations of the coordinates of the calculation of the sectorial moment of inertia are of a general character. In particular, the thickness of the wall of the cross section can vary according to any law.

When we examined the above profile, cut along a circular arc, we assumed the thickness of the wall to be constant. We can generalize the obtained results for the case, when the cross section of the beam-shell, cut along a circular arc, consists of portions of different thickness and has a single axis of symmetry (Figure 50a).

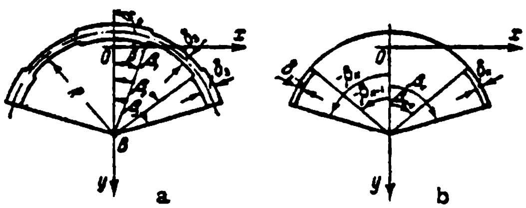


Figure 50

For this purpose we examine some k-th part whose thickness equals δ_k; this part is enclosed between radii which make angles β_{k-1} and β_k with the axis of symmetry in the positive quadrant. By symmetry, a similar part will lie on the other side of the axis of symmetry (Figure 50b). Now we formally denote the thickness of the part between the angles $-\beta_k$ and $+\beta_k$ by $+\delta_k$ while the part enclosed between $-\beta_{k-1}$ and $+\beta_{k-1}$, has a negative thickness $-\delta_k$ [these cancel on superposition to result in Figure 50b]. We obtain then from equation (1.15) for the k-th part:

$$J_{\omega_B x} = \int_F x\omega_B\, dF = R^4\delta_k\left(\int_{-\beta_k}^{+\beta_k} \varphi \sin\varphi\, d\varphi - \int_{-\beta_{k-1}}^{+\beta_{k-1}} \varphi \sin\varphi\, d\varphi\right) = \\ = 2R^4\delta_k[(\sin\beta_k - \beta_k\cos\beta_k) - (\sin\beta_{k-1} - \beta_{k-1}\cos\beta_{k-1})],$$

and from equation (1.17):

$$J_y = \int_F x^2 dF = R^3\delta_k[(\beta_k - \sin\beta_k\cos\beta_k) - (\beta_{k-1} - \sin\beta_{k-1}\cos\beta_{k-1})].$$

Adding the obtained values with k running from 1 to n, where n is the number of the parts which have a different thickness, we obtain an equation for the coordinates of the shear center:

$$a_y = -\frac{J_{\omega_B x}}{J_y} = -2R\frac{\sum\limits_{k=1}^{n}[(\sin\beta_k - \beta_k\cos\beta_k) - (\sin\beta_{k-1} - \beta_{k-1}\cos\beta_{k-1})]\,\delta_k}{\sum\limits_{k=1}^{n}[(\beta_k - \sin\beta_k\cos\beta_k) - (\beta_{k-1} - \sin\beta_{k-1}\cos\beta_{k-1})]\,\delta_k}. \quad (1.21)$$

Analogously we obtain an equation for the sectorial moment of inertia:

$$J_\omega = 4R^5\left\{\frac{1}{6}\sum_{k=1}^{n}(\beta_k^3 - \beta_{k-1}^3)\,\delta_k - \right. \\ \left. - \frac{\left\{\sum\limits_{k=1}^{n}[(\sin\beta_k - \beta_k\cos\beta_k) - (\sin\beta_{k-1} - \beta_{k-1}\cos\beta_{k-1})]\,\delta_k\right\}^2}{\sum\limits_{k=1}^{n}[(\beta_k - \sin\beta_k\cos\beta_k) - (\beta_{k-1} - \sin\beta_{k-1}\cos\beta_{k-1})]\,\delta_k}\right\}. \quad (1.22)$$

6. Section Reinforced by Longitudinal Ribs (Stringers). We shall examine a thin-walled beam, reinforced by longitudinal ribs. The cross section of such a beam will show the areas of the sections of the longitudinal ribs as concentrated at different points (Figure 51). If we apply to such a beam the hypothesis of inflexibility of the section we may generalize the above theory of the sectorial areas for a section with wall thickness varying according to any discontinuous law. For this purpose it is sufficient to regard the definite integrals discussed above, referred to various sectorial characteristics of the section, as Stieltjes integrals. This means that we must replace these integrals by sums of products of values of the integrands and the corresponding elementary areas, where finite

parts of the working cross section are concentrated at isolated points of the section. In such a generalization the sectorial products of inertia $J_{\omega x}$ and $J_{\omega y}$, entering into the equation (I.7.5) become

$$\left.\begin{aligned} J_{\omega_B x} &= \int_F x\omega_B\,dF = \int_L x\omega_B\,\delta\,ds + \sum_k x_k\,\omega_{Bk} f_k,\\ J_{\omega_B y} &= \int_F y\omega_B\,dF = \int_L y\omega_B\,\delta\,ds + \sum_k y_k\,\omega_{Bk} f_k,\end{aligned}\right\} \tag{1.23}$$

where $\delta = \delta(s)$ is the thickness of the section wall, varying in the general case; f_k is the area of the section of the longitudinal rib with a centroid at the point K; ω_{Bk} is the sectorial area for the point K; this area is determined by the pole B and by the profile line M_0M and does not depend on the area of the section of the element of the thin-walled beam.

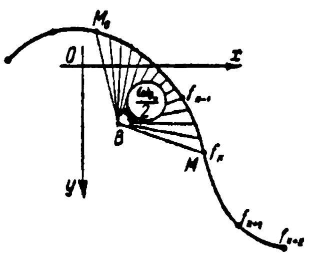


Figure 51

The integrals on the right-hand side of equations (1.23), are taken with respect to the variable s and over the whole contour of the section. The sums are taken over all the areas of the sections which are concentrated at points of the contour. For the axial moments of inertia J_x and J_y we obtain analogous expressions:

$$\left.\begin{aligned} J_x &= \int_L y^2\delta\,ds + \sum_k y_k^2 f_k,\\ J_y &= \int_L x^2\delta\,ds + \sum_k x_k^2 f_k.\end{aligned}\right\} \tag{1.24}$$

Applying equations (1.23) and (1.24), equations (I.7.5) become:

$$\begin{aligned} a_x &= \frac{\int_L y\omega_B\delta\,ds + \sum_k y_k\omega_{Bk} f_k}{\int_L y^2\delta\,ds + \sum_k y_k^2 f_k},\\ a_y &= -\frac{\int_L x\omega_B\delta\,ds + \sum_k x_k\omega_{Bk} f_k}{\int_L x^2\delta\,ds + \sum_k x_k^2 f_k}.\end{aligned} \tag{1.25}$$

For sectorial characteristics S_ω and J_ω we obtain the expressions:

$$\left.\begin{aligned} S_\omega &= \int_{L(s)} \omega\delta\,ds + \sum_{k(s)} \omega_k f_k,\\ J_\omega &= \int_L \omega^2\delta\,ds + \sum_k \omega_k^2 f_k.\end{aligned}\right\} \tag{1.26}$$

The first of equations (1.26) is determined by the sectorial static moment for the contour point M. The integral and the sums on the right-hand side of this equation are taken over the "cut-off" part of the section

only; $S_\omega = 0$ for the whole section. The sectorial moment of inertia is determined by the second of equations (1.26). In this equation the integration with respect to s and the summation in the concentrated areas f_k, are taken over the whole cross section.

By using equations (1.25) and (1.26) we can calculate the coordinates of the shear center and the sectorial characteristics of the sections for rolled sections taking into consideration the variation of the thickness of the wall and the increase in thickness at the joints of the different parts of the section.

By using these equations it is possible to design a section consisting only of concentrated areas. In such a case it is obviously necessary to consider the thickness of the wall δ as equal to zero. We obtain

$$\left.\begin{aligned} a_x &= \frac{\sum\limits_k y_k \omega_{Bk} f_k}{\sum\limits_k y_k^2 f_k}, \quad a_y = -\frac{\sum\limits_k x_k \omega_{Bk} f_k}{\sum\limits_k x_k^2 f_k}; \\ S_\omega &= \sum_{k(s)} \omega_k f_k, \qquad J_\omega = \sum_k \omega_k^2 f_k. \end{aligned}\right\} \tag{1.27}$$

Equations (1.27) refer to beams in which the normal forces in the cross sections are sustained only by the longitudinal elements. Such beams are found, for example, in girders of open railroad bridges, three-dimensional airplane structures, reinforced ribs, etc. The solid walls of a girder in the calculation above should be considered as solid plates subjected only to shear forces in the cross sections. The normal forces are sustained by the longitudinal members alone.

As an example we shall examine a bridge girder along the roadway (bridge floor) and reinforced by transverse frames. In order to determine the shear center by equations (1.27) we choose the pole at the point B and assume $f_1 = f_2 = f_3 = f_4$. In that case we obtain $a_y = \frac{d_2}{2}$ (Figure 52a). As we see, the shear center is situated in this case, too, outside the section. Under torsion the longitudinal extensions are distributed over the section according to the law of sectorial areas. The diagram of these areas is given in Figure 52c. By multiplying the extension by the rigidity EJ of the longitudinal members, we obtain the normal stresses in the webs of the three-dimensional girder during torsion. These stresses should be added to the flexural stresses which appear in the members of the girder.

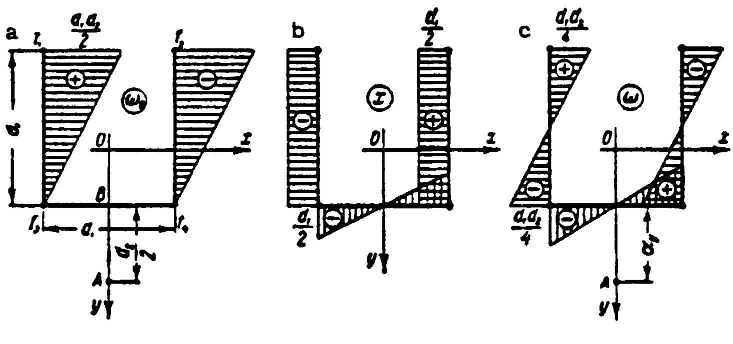


Figure 52

Thus, by applying to the girders the hypothesis of inflexibility of the section contour, we can view these girders as solid three-dimensional systems, taking into consideration the additional stresses which appear in them due to torsion.

7. A Cylindrical Shell, Stiffened Longitudinally by Plates and Beams. Figure 53a shows the cross section of a system, consisting of a cylindrical shell, two plates, and two rolled sections, each of which consists of a thin-walled beam.

If, by its dimensions, this structure is to be classified with long cylindrical shells or if it is reinforced in the transverse direction by ribs (frames), then it is possible to apply the theory of thin-walled beams effectively, considering if as a three-dimensional system. The distribution of the longitudinal normal stresses under bending is given by the law of plane sections. This law is applied to all the section elements, including the part of the section, which consists of the longitudinal stringers, the plates, and the thin-walled beams, acting together with the shell and forming with this shell a single body of rigid cross section. Analogously, in the case of flexural torsion, the distribution of the longitudinal normal stresses over the cross section is given by the law of sectorial areas. This law, which characterizes the warping of the section, remains valid for all the elements composing it.

Figure 53b shows the diagram of ordinates $x = x(s)$, which expresses the law of plane sections of a beam-shell bent away from the plane of symmetry. Figure 53c gives the diagram of the ordinates $\omega = \omega(s)$ which expresses the law of the sectorial areas under flexural torsion with respect to the shear center A. The position of the center A is determined by our general equations (I.7.5). The definite integrals entering into these equations are taken over all the elements of the section. The main characteristics of the section of this complex profile are calculated by equations (6.9)-(6.13), in which the integration is also performed over all the parts of the complex profile, including the intermediate plates and the rolled thin-walled beams.

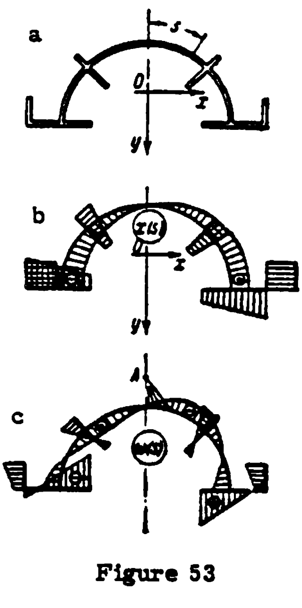


Figure 53

§ 2. Torsion of a beam subjected to a transverse load

1. It was established in § 9, Chapter I, that thin-walled beams of rigid cross section undergo not only bending but also torsion when subjected to a transverse load which does not pass through the shear center. Here we shall omit all that refers to the bending of beams based on the hypothesis of plane sections (the first three equations of (I.7.3)) and concentrate on the study of torsion in a thin-walled beam, which in the general case is produced by a transverse load. The last of equations (I.7.3) gives the following differential equation for the torsional angle $\theta(z)$, in case only a transverse load is acting,

$$EJ_\omega \theta^{IV} - GJ_d \theta^{II} - m = 0. \tag{2.1}$$

We mention that m denotes the free term which represents the external torsional couple per unit length of the shell in the case of purely transverse loading. If q_x and q_y are the magnitudes of the transverse loads acting in the direction of the principal axis, and e_x and e_y are the arms of these loads with respect to the shear center (Figure 54), then the torsional couple m is determined by the equation

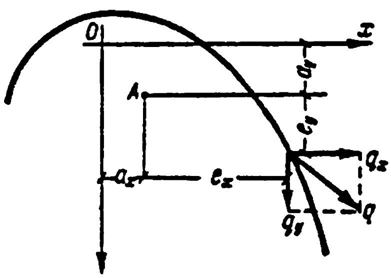


Figure 54

$$m = q_y e_x - q_x e_y.$$

Dividing equation (2.1) by the coefficient of θ^{IV} and introducing the notation

$$k^2 = \frac{GJ_d}{EJ_\omega} l^2 \tag{2.2}$$

(l is the span length of the beam-shell along the generator),

$$f(z) = \frac{1}{EJ_\omega} m.$$

Equation (2.1) is simplified:

$$\theta^{IV} - \frac{k^2}{l^2}\theta^{II} - f(z) = 0. \tag{2.3}$$

2. The differential equation (2.3) is linear and inhomogeneous with constant coefficients (in the case when $m \neq 0$).

The general integral of the differential equation (2.3) can be written in the form

$$\theta(z) = C_1 + C_2 z + C_3 \operatorname{sh}\frac{k}{l} z + C_4 \operatorname{ch}\frac{k}{l} z + \bar{\theta}(z). \tag{2.4}$$

Here, C_1, C_2, C_3 and C_4 are arbitrary constants, obtained from the integration of the homogeneous differential equation, i.e., of the equation resulting from equation (2.3) by dropping the free term $f(z)$; $\bar{\theta}(z)$ is any particular solution of the inhomogeneous equation (2.3); k is a dimensionless characteristic number, determined by equation (2.2)*:

$$k = l\sqrt{\frac{GJ_d}{EJ_\omega}}.$$

As we see, this number depends on the ratio of the pure torsional rigidity GJ_d, understood in the sense of Saint-Venant's theory, and the sectorial rigidity EJ_ω (i.e., the torsional rigidity of the beam which is associated with the appearance of normal stresses in the cross sections) and proportional to the span-length, l, of the beam.

Having obtained the general integral for the torsional angle (2.4), we can also obtain the general integrals for the warping $\theta' = \frac{d\theta}{dz}$ and for the internal force factors of the cross section of the beam.

The bimoment, B, obtained from the normal torsional stresses is such a factor (instead of the bimoment, it is possible to take the normal

* We recapitulate the general procedure of solution of the homogeneous equation

$$\theta^{IV} - \frac{k^2}{l^2}\theta^{II} = 0. \tag{a}$$

We look for a solution of this equation in the form

$$\theta = e^{rz}, \tag{b}$$

where r is an unknown factor to be determined. Substituting (b) in equation (a), we obtain

$$r^4 e^{rz} - \frac{k^2}{l^2} r^2 e^{rz} = 0.$$

Dropping the common factor e^{rz}, we get the characteristic equation for determining the value of r

$$r^4 - \frac{k^2}{l^2} r^2 = 0 \quad \text{or} \quad r^2\left(r^2 - \frac{k^2}{l^2}\right) = 0. \tag{c}$$

Solving equation (c), we find the following four roots:

$$r_1 = r_2 = 0 \text{ (a double root)}, \; r_3 = \frac{k}{l}, \; r_4 = -\frac{k}{l}.$$

Consequently, the general integral of the homogeneous equation (a) can be written in the form

$$\theta = C_1 + C_2 z + \bar{C}_3 e^{\frac{k}{l}z} + \bar{C}_4 e^{-\frac{k}{l}z}.$$

Passing from exponential functions to hyperbolic functions through the formulas

$$e^{\frac{k}{l}z} = \operatorname{ch}\frac{k}{l}z + \operatorname{sh}\frac{k}{l}z, \quad e^{-\frac{k}{l}z} = \operatorname{ch}\frac{k}{l}z - \operatorname{sh}\frac{k}{l}z$$

and writing for $(\bar{C}_3 + \bar{C}_4)$ and $(\bar{C}_3 - \bar{C}_4)$ the new constants $C_3 = \bar{C}_3 + \bar{C}_4$ and $C_4 = \bar{C}_3 - \bar{C}_4$, we obtain the desired expression

$$\theta = C_1 + C_2 z + C_3 \operatorname{sh}\frac{k}{l}z + C_4 \operatorname{ch}\frac{k}{l}z.$$

stresses themselves which are related to the bimoment by the equation $\sigma = \frac{B}{J_\omega}\omega$). Another force factor is the total torsional moment H, which consists of the sum of the flexural-torsional moment H_ω due to the axial stresses and of the torsional moment H_k due to the nonuniform distribution of the tangential stresses over the thickness of the wall. For these quantities we obtained equations (I.8.3) and (I.8.11):

$$\left.\begin{aligned} B &= -EJ_\omega\theta'', \\ H &= -EJ_\omega\theta''' + GJ_d\theta'. \end{aligned}\right\} \tag{2.5}$$

Having found (2.4), the first, second, and third order derivatives of the function $\theta(z)$ with respect to the variable z and introducing them into equations (2.5), we obtain the general integrals for B and H. Finally, for the four basic design quantities (two kinematical: the torsional angle θ and the warping θ'; and two statical: the bimoment B and the total torsional moment H) we obtain the following equations:

$$\left.\begin{aligned} \theta(z) &= C_1 + C_2 z + C_3 \operatorname{sh}\frac{k}{l}z + C_4 \operatorname{ch}\frac{k}{l}z + \bar{\theta}(z), \\ \theta'(z) &= C_2 + C_3\frac{k}{l}\operatorname{ch}\frac{k}{l}z + C_4\frac{k}{l}\operatorname{sh}\frac{k}{l}z + \bar{\theta}'(z), \\ B(z) &= -GJ_d\left[C_3\operatorname{sh}\frac{k}{l}z + C_4\operatorname{ch}\frac{k}{l}z + \frac{l^2}{k^2}\bar{\theta}''(z)\right], \\ H(z) &= GJ_d\left[C_2 + \bar{\theta}'(z) - \frac{l^2}{k^2}\bar{\theta}'''(z)\right]. \end{aligned}\right\} \tag{2.6}$$

3. General methods for finding particular integrals of an inhomogeneous equation are discussed in the theory of differential equations. Thus, for example, knowing the general integral of a homogeneous equation, it is possible to find the particular integral of the inhomogeneous equation by the method of variation of constants. This method is generally applicable to linear equations with constant and variable coefficients.

There are simpler and faster methods for obtaining the particular integrals for various particular forms of loads. We shall not examine these cases here, since in the following we intend to solve the differential equation of torsion by using the method of initial parameters. The determination of the particular integrals by this method differs from the classical methods mentioned above. We shall explain the application of the method of initial parameters to our problem in the following section. We shall now give several examples of obtaining particular integrals by the usual theory for some common cases of loading.

a) Concentrated torsional moment. Such a case exists in the absence of surface tractions. A concentrated torsional moment is obtained due to the action of a concentrated transverse load which does not pass through the shear center. The load per unit section, due to the torsional moment m, is equal to zero. The particular integral of the inhomogeneous equation (2.3) is also zero:

$$\bar{\theta}(z) = 0.$$

Equations (2.6) assume the following form:

$$\left.\begin{aligned} \theta(z) &= C_1 + C_2 z + C_3 \operatorname{sh}\frac{k}{l}z + C_4 \operatorname{ch}\frac{k}{l}z, \\ \theta'(z) &= C_2 + C_3\frac{k}{l}\operatorname{ch}\frac{k}{l}z + C_4\frac{k}{l}\operatorname{sh}\frac{k}{l}z, \\ B(z) &= -GJ_d\left(C_3 \operatorname{sh}\frac{k}{l}z + C_4 \operatorname{ch}\frac{k}{l}z\right), \\ H(z) &= GJ_d C_2. \end{aligned}\right\} \qquad (2.7)$$

It is seen from the fourth equation (2.7) that the total torsional moment due to internal forces on the unloaded part does not depend on the variable z. This also follows from the condition of equilibrium for that part of the beam, which is enclosed between bounding sectional planes, to which the torsional moments due to the concentrates forces are restricted.

Equation (2.7) allowed also the determination of the stresses and strains of the beam due to an external load, which is applied at the end ap- appearing as normal or tangential traction, distributed over the section according to the law of the sectorial areas.

b) Evenly distributed torsional moment. If a transverse load, having a constant magnitude and constant direction and applied along a line parallel to the generator, acts on a beam, the external torsional couple per unit section, m, due to such loading will be constant (independent of z).

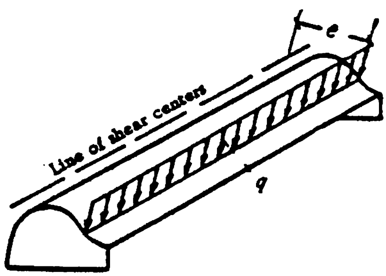


Figure 55

Denoting the magnitude of the transverse load by q and the moment-arm of this load about the shear center by e, we obtain for the magnitude of the torsional moment m the expression (Figure 55):

$$m = qe.$$

The differential equation (2.3) for this case has the following form:

$$\theta^{IV} - \frac{k^2}{l^2}\theta'' - \frac{qe}{EJ_\omega} = 0.$$

A particular solution of this inhomogeneous equation is

$$\bar{\theta} = -\frac{qe}{2EJ_\omega}\frac{l^2}{k^2}z^2.$$

The general integrals (2.6) for the strains and stresses due to torsion will be

$$\begin{aligned}
\theta(z) &= C_1 + C_2 z + C_3 \operatorname{sh}\frac{k}{l}z + C_4 \operatorname{ch}\frac{k}{l}z - \frac{qe}{2EJ_\omega}\frac{l^2}{k^2}z^2,\\
\theta'(z) &= C_2 + C_3\frac{k}{l}\operatorname{ch}\frac{k}{l}z + C_4\frac{k}{l}\operatorname{sh}\frac{k}{l}z - \frac{qe}{EJ_\omega}\frac{l^2}{k^2}z,\\
B(z) &= -GJ_d\left(C_3 \operatorname{sh}\frac{k}{l}z + C_4 \operatorname{ch}\frac{k}{l}z - \frac{qe}{EJ_\omega}\frac{l^4}{k^4}\right),\\
H(z) &= GJ_d\left(C_2 - \frac{qe}{EJ_\omega}\frac{l^2}{k^2}z\right).
\end{aligned}$$

c) Torsional moment for a trapezoidal distribution. The magnitude of the torsional moment m can be expressed by the equation:

$$m = e\left(q_0 + \frac{q_1}{l}z\right),$$

where the function $q_0 + \frac{q_1}{l}z$ describes a trapezoidal load applied along a line parallel to the generator and having an arm e about the shear center. The general integrals (2.6) will be in this case

$$\begin{aligned}
\theta(z) &= C_1 + C_2 z + C_3 \operatorname{sh}\frac{k}{l}z + C_4 \operatorname{ch}\frac{k}{l}z - \frac{e}{6EJ_\omega}\left(3q_0 z^2 + \frac{q_1}{l}z^3\right)\frac{l^2}{k^2},\\
\theta'(z) &= C_2 + C_3\frac{k}{l}\operatorname{ch}\frac{k}{l}z + C_4\frac{k}{l}\operatorname{sh}\frac{k}{l}z - \frac{e}{EJ_\omega}\left(q_0 z + \frac{1}{2}\frac{q_1}{l}z^2\right)\frac{l^2}{k^2},\\
B(z) &= -GJ_d\left[C_3 \operatorname{sh}\frac{k}{l}z + C_4 \operatorname{ch}\frac{k}{l}z - \frac{e}{EJ_\omega}\left(q_0 + \frac{q_1}{l}z\right)\frac{l^4}{k^4}\right],\\
H(z) &= GJ_d\left[C_2 - \frac{e}{EJ_\omega}\left(q_0 z + \frac{1}{2}\frac{q_1}{l}z^2\right)\frac{l^2}{k^2} + \frac{q_1 e}{EJ_\omega}\frac{l^3}{k^4}\right].
\end{aligned}$$

4. Having found by this or other methods the general integrals of the four basic design quantities as functions of four arbitrary integration constants, we can proceed to determine these constants. This can be done by imposing the boundary conditions at the ends of the beam. Since there are four arbitrary conditions, we must set two conditions at each end of the beam.

The principal forms of the boundary conditions which we are likely to encounter can be divided into the following types.

a) Clamped end. This means that at the support the section is fixed so as to prevent displacements in the plane of the section (no torsional angle θ) and away from this plane (no warping ξ). These are purely kinematical boundary conditions; consequently, they can be written thus:

$$\theta = 0, \quad \theta' = 0 \qquad (2.8)$$

for $z = 0$ or $z = l$, depending on which end this condition refers to.

b) Hinged end. We call a support a hinge support, when the torsional angle vanishes for the supported section (the section is fixed

so as to prevent rotation about the z axis) and the supported section can freely deform out of its plane (no sectorial longitudinal forces in the section). The boundary conditions for a hinge support at the end of the beam will be:

$$\theta = 0, \quad B = 0. \tag{2.9}$$

Here, one condition is kinematical, the other statical.

c) Free end. By this we mean that there are no statical factors at the end (absence of longitudinal sectorial forces and general torsional moment). Consequently, at the free end of the beam we have purely statical conditions:

$$B = 0, \quad H = 0. \tag{2.10}$$

Since the beam has two ends, the boundary conditions can appear in various combinations; thus, for example, conditions (2.8) as well as conditions (2.9) can be satisfied at both ends of a single beam; conditions (2.10) cannot be satisfied at both ends of the same beam [for an obvious reason]; conditions (2.8) can be satisfied at one end and (2.9) or (2.10) at the other.

By enumerating the combinations we have not, of course, exhausted all the forms of boundary conditions which may be encountered in practice; torsional moments of given magnitude, bimoments or longitudinal normal forces, etc, may be applied at the ends. Some of these forms of boundary conditions will be examined below.

When the boundary conditions have been applied and the integration constants determined, we have essentially completed the basic part of the design; the remainder consists of calculating forces or kinematical factors.

§ 3. Application of the method of initial parameters to the design of beams subject to torsion

1. The general solution of the homogeneous differential equation of torsion

$$\theta^{IV} - \frac{k^2}{l^2}\theta'' = 0 \tag{3.1}$$

is given by equations (2.6). In these equations the functions 1, z, $\mathrm{sh}\frac{k}{l}z$, $\mathrm{ch}\frac{k}{l}z$ are particular linearly independent solutions of equation (3.1), and C_1, C_2, C_3, C_4 are arbitrary constants. These constants are determined from the boundary conditions in each particular case. By using the method of initial parameters, we can easily find these constants and considerably simplify the design of beams subject to torsion.

We place the origin of the coordinate z in some arbitrary section of the beam. We shall postulate that all the geometrical and statical factors involved in the description of torsion of the beam by the law of sectorial

areas have definite values prescribed for this initial section. Denoting these factors by θ_0, θ'_0, B_0 and H_0 respectively, we assume that the beam is subjected to the action of the initial factors only, i.e., we take the external forces to vanish along the beam. Putting $z=0$ for this section, we obtain from the general equations (2.6) the following system of equations for the arbitrary constants C_1, C_2, C_3 and C_4 in terms of the initial parameters θ_0, θ'_0, B_0 and H_0:

$$\theta_0 = C_1 + C_4, \qquad \theta'_0 = C_2 + \frac{k}{l} C_3,$$
$$B_0 = -C_4 GJ_d, \qquad H_0 = C_2 GJ_d.$$

Solving this system of equations, we obtain

$$\left.\begin{aligned} C_1 &= \theta_0 + \frac{1}{GJ_d} B_0, \\ C_2 &= \frac{1}{GJ_d} H_0, \\ C_3 &= \frac{l}{k}\theta'_0 - \frac{l}{k}\frac{1}{GJ_d} H_0, \\ C_4 &= -\frac{1}{GJ_d} B_0. \end{aligned}\right\} \tag{3.2}$$

Substituting the expressions for C_1, C_2, C_3 and C_4 found from equation (2.6), we obtain the general equations of the method of initial parameters for all four design quantities:

$$\left.\begin{aligned} \theta &= \theta_0 + \frac{l}{k}\theta'_0 \operatorname{sh}\frac{k}{l}z - \frac{1}{GJ_d} B_0\left(\operatorname{ch}\frac{k}{l}z - 1\right) + \\ &\qquad + \frac{1}{GJ_d} H_0\left(z - \frac{l}{k}\operatorname{sh}\frac{k}{l}z\right), \\ \theta' &= \theta'_0 \operatorname{ch}\frac{k}{l}z - \frac{k}{l}\frac{1}{GJ_d} B_0 \operatorname{sh}\frac{k}{l}z + \frac{1}{GJ_d}\left(1 - \operatorname{ch}\frac{k}{l}z\right) H_0, \\ B &= -\frac{l}{k} GJ_d\theta'_0 \operatorname{sh}\frac{k}{l}z + B_0 \operatorname{ch}\frac{k}{l}z + \frac{l}{k} H_0 \operatorname{sh}\frac{k}{l}z, \\ H &= H_0. \end{aligned}\right\} \tag{3.3}$$

These equations define a set of linear transformations of the flexural-torsional factors of the initial section θ_0, θ'_0, B_0 and H_0 into the flexural-torsional factors $\theta(z)$, $\theta'(z)$, $B(z)$ and $H(z)$, z being the coordinate of the section. The coefficients of this transformation do not depend only on the coordinate z, as determined by the relative position of the two sections (the initial and the examined), but also on the quantities GJ_d, EJ_ω and $k = l\sqrt{\frac{GJ_d}{EJ_\omega}}$ which characterize the elastic properties of the beam under torsion. It is convenient to present the linear transformation expressed by equations (3.3) in tabular form (see Table 3).

The upper row and in the extreme left column of this table give the flexural-torsional factors which refer to two different sections of the

beam*. The coefficients of the transformation (3.3) are shown at the intersections of the columns and rows. These coefficients form a matrix, which we shall call the matrix of the initial parameters.

Table 3

	θ_0	θ_0'	$\frac{1}{GJ_d}B_0$	$\frac{1}{GJ_d}H_0$
$\theta(z)$	1	$\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}z$	$1-\mathrm{ch}\,\frac{k}{l}z$	$z-\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}z$
$\theta'(z)$	0	$\mathrm{ch}\,\frac{k}{l}z$	$-\frac{k}{l}\,\mathrm{sh}\,\frac{k}{l}z$	$1-\mathrm{ch}\,\frac{k}{l}z$
$\frac{1}{GJ_d}B(z)$	0	$-\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}z$	$\mathrm{ch}\,\frac{k}{l}z$	$\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}z$
$\frac{1}{GJ_d}H(z)$	0	0	0	1

2. It follows from the method by which the matrix of the initial parameters was obtained that this matrix can be written in a more general form, namely, the initial section need not be $z=0$, but an arbitrary section $z=t$ $(l>t>0)$. We should then use the argument $(z-t)$ instead of z in the coefficients of the matrix. The matrix can be written as follows:

Table 4

	θ_t	θ_t'	$\frac{1}{GJ_d}B_t$	$\frac{1}{GJ_d}H_t$
$\theta(z)$	1	$\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}(z-t)$	$1-\mathrm{ch}\,\frac{k}{l}(z-t)$	$(z-t)-$ $-\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}(z-t)$
$\theta'(z)$	0	$\mathrm{ch}\,\frac{k}{l}(z-t)$	$-\frac{k}{l}\,\mathrm{sh}\,\frac{k}{l}(z-t)$	$1-\mathrm{ch}\,\frac{k}{l}(z-t)$
$\frac{1}{GJ_d}B(z)$	0	$-\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}(z-t)$	$\mathrm{ch}\,\frac{k}{l}(z-t)$	$\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}(z-t)$
$\frac{1}{GJ_d}H(z)$	0	0	0	1

* We put $\frac{1}{GJ_d}B$ and $\frac{1}{GJ_d}H$ instead of B and H, to which they are proportional, in order to keep the coefficients in the table uniform.

We now prove certain important properties of the matrix of the initial parameters.

a) The determinant of the matrix of the initial parameters is equal to unity for any value of z.

Indeed, expanding this determinant, we obtain

$$\Delta = \text{ch}^2 \frac{k}{l}(z-t) - \text{sh}^2 \frac{k}{l}(z-t) = 1\ ^{*}).$$

b) Since the coefficients of the matrix, for given elastic characteristics and for a given span length, depend on the reduced coordinate $(z-t)$, which determines the relative position of the two sections of the beam $t=\text{const}$ and $z=\text{const}$, it is possible to interpret the expressions of Table 4 differently, depending on which of the two sections is considered variable.

Taking the initial section $t=\text{const}$ as given and allowing z to take on different values, we may use the formulas of Table 4 to draw the graphs of the variation of the factors $\theta(z)$, $\theta'(z)$, $B(z)$ and $H(z)$ due to the variation of each of the four initial factors θ_t, θ'_t, B_t and H_t separately.

Definitely fixing now the section $z=\text{const}$, for which the values of $\theta(z)$, $\theta'(z)$, $B(z)$ and $H(z)$ are sought, and letting the section $t=\text{const}$ assume different positions along the beam, we obtain from the formulas of Table 4 the equations of the influence line (influent) for each of the four factors $\theta(z)$, $\theta'(z)$, $B(z)$, $H(z)$ separately. Therefore the coefficients of this matrix, which depends on the argument $(z-t)$, will in this case be the influence functions.

Since the independent variable in the case of the influence line is the same relative coordinate and the equations of the influence line coincide with the equations of the diagrams, the graphs of the variation of the flexural-torsional factors are simultaneously also the influence lines of these factors. This is the property of reciprocity of diagrams and influence lines (influents).

c) The linear transformation represented by the matrix of Table 4 has also the property of reversibility, i.e., with this matrix it is possible to carry out not only the direct transformation of the initial factors θ_t, θ'_t, $\frac{1}{GJ_d}B_t$ and $\frac{1}{GJ_d}H_t$ of the initial section $t=\text{const}$, into the factors $\theta(z)$, $\theta'(z)$, $\frac{1}{GJ_d}B(z)$ and $\frac{1}{GJ_d}H(z)$ of the section having the coordinate z, but also the inverse transformation of the factors θ_z, θ'_z, $\frac{1}{GJ_d}B_z$ and $\frac{1}{GJ_d}H_z$ back into the factors $\theta(t)$, $\theta'(t)$, $\frac{1}{GJ_d}B(t)$ and $\frac{1}{GJ_d}H(t)$. For this purpose it is necessary to interchange the sections $z=\text{const}$ and $t=\text{const}$, i.e., to consider the section $z=\text{const}$ as initially given and the section $t=\text{const}$ as variable. Naturally, the argument $(z-t)$ changes sign. When the sign

* The determinant of the matrix of the initial parameters is well known in the theory of linear differential equations as the Wronskian. The system of fundamental solutions of a homogeneous differential equation for which the Wronskian equals unity at the initial value of the argument is called, in the theory of linear differential equations, the normal fundamental system. Our solutions, consisting of the linearly independent functions l, $\frac{l}{k}\,\text{sh}\,\frac{k}{l}(z-t)$, $1-\text{ch}\,\frac{k}{l}(z-t)$, $(z-t)-\frac{l}{k}\,\text{sh}\frac{k}{l}(z-t)$, is thus the normal fundamental system of equation (3.1).

of the argument changes, the signs of the even influence functions do not, but those of the odd influence functions do. Thus, the matrix of the inverse transformation, whose coefficients are obtained from the coefficients of the matrix of Table 4 by changing the signs of the odd functions, can be given in the form of Table 5.

Table 5

	θ_z	θ'_z	$\frac{1}{GJ_d}B_z$	$\frac{1}{GJ_d}H_z$
$\theta(t)$	1	$-\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}(z-t)$	$1-\mathrm{ch}\,\frac{k}{l}(z-t)$	$-(z-t)+$ $+\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}(z-t)$
$\theta'(t)$	0	$\mathrm{ch}\,\frac{k}{l}(z-t)$	$\frac{k}{l}\,\mathrm{sh}\,\frac{k}{l}(z-t)$	$1-\mathrm{ch}\,\frac{k}{l}(z-t)$
$\frac{1}{GJ_d}B(t)$	0	$\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}(z-t)$	$\mathrm{ch}\,\frac{k}{l}(z-t)$	$-\frac{l}{k}\,\mathrm{sh}\,\frac{k}{l}(z-t)$
$\frac{1}{GJ_d}H(t)$	0	0	0	1

The inverse linear transformation (Table 5) represents essentially the solution of the system of four equations of Table 3 if we consider left-hand sides θ_z, θ'_z, $\frac{1}{GJ_d}B_z$ and $\frac{1}{GJ_d}H_z$ to be given and the initial parameters θ_t, θ'_t, $\frac{1}{GJ_d}B_t$ and $\frac{1}{GJ_d}H_t$ to be the quantities sought.

The determinant of the inverse transformation matrix is similarly equal to unity, since reversing the sign of all the odd functions does not change the value of the determinant*.

3. We shall proceed now to the integration of the homogeneous equation (3.1) resulting from various concentrated force-factors arbitrarily distributed along the beam. We first write the matrix of the initial parameters of Table 3 in a more compact form denoting the coefficients of this matrix by the letter K with two indexes which give the row and the column in which this coefficient is found in the table. The matrix of the initial parameters in this symbolic notation is given in Table 6.

The formulas as written in Table 6, and in expanded form for the

* This matrix possesses a number of interesting properties. These properties were examined in detail by the author /65/, /66/, for a matrix of a more general type, referring to a differential equation with two generalized elastic characteristics r^2 and s^4:

$$\theta^{IV} - 2r^2\theta^{II} + s^4\theta = 0,$$

This equation is derived in § 5 of Chapter III.

case of restrained torsion as given in Table 3, permit the determination of the flexural-torsional factors $\theta(z)$, $\theta'(z)$, $\frac{1}{GJ_d}(B)z$ and $\frac{1}{GJ_d}H(z)$ for any section $z = \text{const}$ if these factors for the initial section $z = 0$ are known. Considering the coefficients $K(z)$ of Table 6 to be influence functions, and using the principle of superposition which follows from the linearity of this transformation, we can determine from this table, for any section of

Table 6

	θ_0	θ_0'	$\frac{1}{GJ_d}B_0$	$\frac{1}{GJ_d}H_0$
$\theta(z)$	$K_{\theta\theta}(z)$	$K_{\theta\theta'}(z)$	$K_{\theta B}(z)$	$K_{\theta H}(z)$
$\theta'(z)$	$K_{\theta'\theta}(z)$	$K_{\theta'\theta'}(z)$	$K_{\theta' B}(z)$	$K_{\theta' H}(z)$
$\frac{1}{GJ_d}B(z)$	$K_{B\theta}(z)$	$K_{B\theta'}(z)$	$K_{BB}(z)$	$K_{BH}(z)$
$\frac{1}{GJ_d}H(z)$	$K_{H\theta}(z)$	$K_{H\theta'}(z)$	$K_{HB}(z)$	$K_{HH}(z)$

the beam, the flexural-torsional factors $\theta(z)$, $\theta'(z)$, $\frac{1}{GJ_d}B(z)$ and $\frac{1}{GJ_d}H(z)$ resulting from concentrated kinematical and statical factors applied at arbitrary sections. Thus, for example, if in addition to the initial parameters θ_0, θ_0', $\frac{1}{GJ_d}B_0$, $\frac{1}{GJ_d}H_0$, the concentrated factors θ_t, θ_t', $\frac{1}{GJ_d}B_t$ and $\frac{1}{GJ_d}H_t$, applied at the section $z = t$, also act on the beam, the flexural-torsional factors θ, θ', $\frac{1}{GJ_d}B$ and $\frac{1}{GJ_d}H$ for sections with abscissas $z \geqslant t$ will be composed of the factors $\theta(z)$, $\theta'(z)$, $\frac{1}{GJ_d}B(z)$ and $\frac{1}{GJ_d}H(z)$, determined only by the initial parameters θ_0, θ_0', $\frac{1}{GJ_d}B_0$ and $\frac{1}{GJ_d}H_0$, and of the factors $\theta(z-t)$, $\theta'(z-t)$, $\frac{1}{GJ_d}B(z-t)$ and $\frac{1}{GJ_d}H(z-t)$ which are the result of the action of the concentrated factors θ_t, θ_t', $\frac{1}{GJ_d}B_t$ and $\frac{1}{GJ_d}H_t$ only. The influence coefficients are the same in both cases but in the first case they are computed for the argument z, while in the second for the argument $(z-t)$. Consequently, for the part of the beam lying beyond the section $t = \text{const}$ or on this section (i. e., for $z \geqslant t$), the formulas given in Table 7 will hold good for the calculation of the basic quantities.

For those parts of the beam on the near side of the section $t = \text{const}$ (for $z < t$) we should use the formulas of Table 6 to calculate the basic quantities. In practical calculations by the method of initial parameters,

Table 7

	θ_0	θ'_0	$\frac{1}{GJ_d}B_0$	$\frac{1}{GJ_d}H_0$	θ_t	θ'_t	$\frac{1}{GJ_d}B_t$	$\frac{1}{GJ_d}H_t$
$\theta(z)$	$K_{\theta\theta}(z)$	$K_{\theta\theta'}(z)$	$K_{\theta B}(z)$	$K_{\theta H}(z)$	$K_{\theta\theta}(z-t)$	$K_{\theta\theta'}(z-t)$	$K_{\theta B}(z-t)$	$K_{\theta H}(z-t)$
$\theta'(z)$	$K_{\theta'\theta}(z)$	$K_{\theta'\theta'}(z)$	$K_{\theta' B}(z)$	$K_{\theta' H}(z)$	$K_{\theta'\theta}(z-t)$	$K_{\theta'\theta'}(z-t)$	$K_{\theta' B}(z-t)$	$K_{\theta' H}(z-t)$
$\frac{1}{GJ_d}B(z)$	$K_{B\theta}(z)$	$K_{B\theta'}(z)$	$K_{BB}(z)$	$K_{BH}(z)$	$K_{B\theta}(z-t)$	$K_{B\theta'}(z-t)$	$K_{BB}(z-t)$	$K_{BH}(z-t)$
$\frac{1}{GJ_d}H(z)$	$K_{H\theta}(z)$	$K_{H\theta'}(z)$	$K_{HB}(z)$	$K_{HH}(z)$	$K_{H\theta}(z-t)$	$K_{H\theta'}(z-t)$	$K_{HB}(z-t)$	$K_{HH}(z-t)$

the formulas of Table 7 are fundamental. In the following we shall consider the values of the concentrated factors θ_t, θ'_t, $\frac{\Gamma}{GJ_d}B_t$ and $\frac{1}{GJ_d}H_t$ and the section at which they are applied ($z=t$) as given in these formulas. In most cases static loads are encountered; kinematical loads are much rarer. In Table 7, therefore, it is possible to retain only B_t and H_t assuming $\theta_t=\theta'_t=0$.

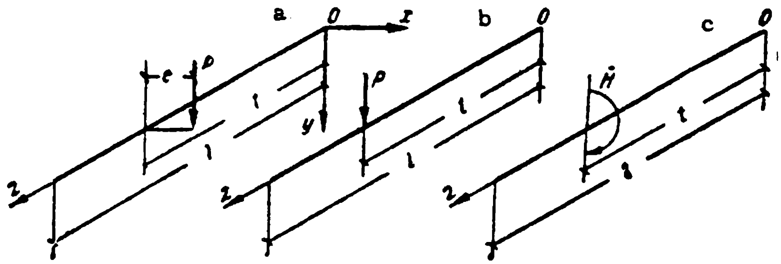


Figure 56

4. By using the law of superposition we can easily generalize the formulas of Table 7 for more complicated cases of action of loads applied at various sections of the beam in the form of factors either concentrated or continuously distributed over a certain part of the beam.

We first wish to remark on the sign of the initial parameter H_t due to a concentrated external torsional moment $\bar{H}$ applied at the section $z=t$ of the beam. We shall show that for a positive value of $\bar{H}$, the initial parameter H_t is determined by the equation $H_t=-\bar{H}$.

Let an external concentrated torsional moment $\bar{H}$ act in the section $z=t$ (Figure 56c). This moment is due, in this case, to the external transverse force P, applied at the section $z=t$ at a distance e from the line of the shear centers (Figure 56a) and is determined by the equation $\bar{H}=Pe$. According to Figure 56c, the torsional moment $\bar{H}$ is positive, since it acts clockwise on the side of the values $z-t>0$.

In the section $z=0$ the initial value of the torsional moment, being an initial parameter, is equal to H_0. The positive sense for this moment on an area with a negative normal will be anticlockwise, if we look on this area from the side where $z-t>0$ (§ 5, Chapter I). Advancing along the axis Oz we find that the function $H(z)$ is diminished in traversing the section $z=t$. Indeed, for the part of the beam $z-t>0$ the torsional moment $\bar{H}$ on an area with a negative normal causes a clockwise rotation if we look at this area from the side $z-t>0$ and, therefore, its value, like the value of the initial factor H_t in the section $z-t=0$ will be negative, i. e., $H_t=-\bar{H}$.

An analogous observation can be made with respect to B_t.

We shall examine, as an example, a beam loaded on the part from $z=t_1$ to $z=t_2$ by $q(z)$, applied at a distance e off the line of shear centers, and which consequently gives rise to a torsional moment distribution $\bar{H}(z)=q(z)e$ (Figure 57) which may conform to any function of the coordinate z.

On the part of the beam from $z=0$ to $z=t_1$ there is no load and, consequently, all the basic functions are expressed in terms of the initial parameters only through the formulas of Table 6, or in expanded form by the formulas of Table 3.

Summing the "concentrated" moments $\overline{H}(t)\,dt$ over the whole region from $z=t_1$ to z $(t_1 \leqslant z < t_2)$ or, in other words, integrating the last column of Table 7 with respect to the variable t between the limits $t=t_1$ and $t=z$, we obtain an equation which accounts for the influence of the distributed load only, i.e., of the torsional moment $\overline{H}(z)$. These equations will have the form:

$$\left.\begin{aligned}
\theta(z) &= -\frac{1}{GJ_d}\int\limits_{t=t_1}^{t=z}\overline{H}(t)\,K_{\theta H}(z-t)\,dt,\\
\theta'(z) &= -\frac{1}{GJ_d}\int\limits_{t=t_1}^{t=z}\overline{H}(t)\,K_{\theta' H}(z-t)\,dt,\\
\frac{1}{GJ_d}B(z) &= -\frac{1}{GJ_d}\int\limits_{t=t_1}^{t=z}\overline{H}(t)\,K_{BH}(z-t)\,dt,\\
\frac{1}{GJ_d}H(z) &= -\frac{1}{GJ_d}\int\limits_{t=t_1}^{t=z}\overline{H}(t)\,K_{HH}(z-t)\,dt.
\end{aligned}\right\} \qquad (3.4)$$

Combining equations (3.4) with the formulas of Table 6, which depend only on the initial parameters, we obtain equations for computing the basic design quantities for the second part of the beam $(t_2 > z \geqslant t_1)$.

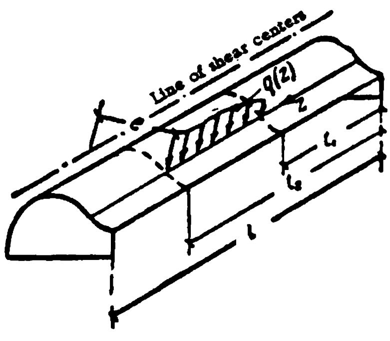


Figure 57

The third part of the beam from $z=t_2$ to $z=l$ does not bear any load. The equations for the third part $(l \geqslant z \geqslant t_2)$ differ from the analogous equations for the second part only in that the integrals will have the upper limit $t=z$ instead of $t=t_2$, since the load is distributed over the entire section (from t_1 to t_2).

5. We shall now calculate the integrals in the above-mentioned equations for a specific case of loading and present these equations in expanded form. Let a uniformly-distributed torsional moment $H(t)=qe$ be applied to the part of the beam $t_2 \geqslant z \geqslant t_1$, q and the eccentricity e being constant Putting qe in front of the integral sign and expanding the influence coeffi-

cients in the integrands, we obtain, by the corresponding equations of Table 3:

$$\frac{1}{GJ_d}\int_{t=t_1}^{t=z}\overline{H}(t)\,K_{\theta H}(z-t)\,dt=\frac{qe}{GJ_d}\int_{t=t_1}^{t=z}\left[(z-t)-\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t)\right]dt=$$

$$=\frac{qe}{GJ_d}\left[-\frac{(z-t)^2}{2}+\frac{l^2}{k^2}\operatorname{ch}\frac{k}{l}(z-t)\right]_{t=t_1}^{t=z}=$$

$$=\frac{qe}{GJ_d}\left\{\frac{(z-t_1)^2}{2}+\frac{l^2}{k^2}\left[1-\operatorname{ch}\frac{k}{l}(z-t_1)\right]\right\},$$

$$\frac{1}{GJ_d}\int_{t=t_1}^{t=z}\overline{H}(t)\,K_{\theta' H}(z-t)\,dt=\frac{qe}{GJ_d}\int_{t=t_1}^{t=z}\left[1-\operatorname{ch}\frac{k}{l}(z-t)\right]dt=$$

$$=\frac{qe}{GJ_d}\left[t+\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t)\right]_{t=t_1}^{t=z}=$$

$$=\frac{qe}{GJ_d}\left[(z-t_1)-\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t_1)\right],$$

$$\frac{1}{GJ_d}\int_{t=t_1}^{t=z}\overline{H}(t)\,K_{BH}(z-t)\,dt=\frac{qe}{GJ_d}\int_{t=t_1}^{t=z}\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t)\,dt=$$

$$=\frac{qe}{GJ_d}\left[-\frac{l^2}{k^2}\operatorname{ch}\frac{k}{l}(z-t)\right]_{t=t}^{t=z}=$$

$$=\frac{qe}{GJ_d}\frac{l^2}{k^2}\left[-1+\operatorname{ch}\frac{k}{l}(z-t_1)\right],$$

$$\frac{1}{GJ_d}\int_{t=t_1}^{t=z}\overline{H}(t)\,K_{HH}(z-t)\,dt=\frac{qe}{GJ_d}\int_{t=t_1}^{t=z}dt=\frac{qe}{GJ_d}\Big[t\Big]_{t=t_1}^{t=z}=$$

$$=\frac{qe}{GJ_d}(z-t_1).$$

Analogously, we calculate the integrals for the third part of the beam ($z\geqslant t_2$):

$$\frac{1}{GJ_d}\int_{t=t_1}^{t=t_2}\overline{H}(t)\,K_{\theta H}(z-t)\,dt=$$

$$=\frac{qe}{GJ_d}\left\{-\frac{(z-t_2)^2}{2}+\frac{(z-t_1)^2}{2}+\frac{l^2}{k^2}\left[\operatorname{ch}\frac{k}{l}(z-t_2)-\operatorname{ch}\frac{k}{l}(z-t_1)\right]\right\},$$

$$\frac{1}{GJ_d}\int_{t=t_1}^{t=t_2}\overline{H}(t)\,K_{\theta' H}(z-t)\,dt=$$

$$=\frac{qe}{GJ_d}\left\{t_2-t_1+\frac{l}{k}\left[\operatorname{sh}\frac{k}{l}(z-t_2)-\operatorname{sh}\frac{k}{l}(z-t_1)\right]\right\},$$

$$\frac{1}{GJ_d}\int_{t=t_1}^{t=t_2}\overline{H}(t)\,K_{BH}(z-t)\,dt=\frac{qe}{GJ_d}\frac{l^2}{k^2}\left[-\operatorname{ch}\frac{k}{l}(z-t_2)+\operatorname{ch}\frac{k}{l}(z-t_1)\right],$$

$$\frac{1}{GJ_d}\int_{t=t_1}^{t=t_2}\overline{H}(t)\,K_{HH}(z-t)\,dt=\frac{qe}{GJ_d}(t_2-t_1).$$

Table 8

	θ_0	θ_0'	$\frac{1}{GJ_d}B_0$	$\frac{1}{GJ_d}H_0$
$\theta(z)$	1	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$z-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$
$\theta'(z)$	0	$\operatorname{ch}\frac{k}{l}z$	$-\frac{k}{l}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$
$\frac{1}{GJ_d}B(z)$	0	$-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$\operatorname{ch}\frac{k}{l}z$	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$
$\frac{1}{GJ_d}H(z)$	0	0	0	1

The equations for the basic design quantities for the indicated type of load are given in expanded form in Table 8 (for the part $z < t_1$), in Table 9 (for the part $t_2 > z \geqslant t_1$), and in Table 10 (for the part $z \geqslant t_2$).

Tables 8, 9, and 10 may give the wrong impression that the loads applied to any part of the beam influence only the parts which lie beyond it but not those that precede; thus, for instance, in the example considered the load is applied to the second part and the equations for the second and third parts contain load terms (i.e. definite integrals of the load) while in the equations for the first part such terms do not appear.

Table 9

	θ_0	θ_0'	$\frac{1}{GJ_d}B_0$	$\frac{1}{GJ_d}H_0$	$\frac{1}{GJ_d}qe$
$\theta(z)$	1	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$z-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$-\frac{(z-t_1)^2}{2}-\frac{l^2}{k^2}\times$ $\times\left[1-\operatorname{ch}\frac{k}{l}(z-t_1)\right]$
$\theta'(z)$	0	$\operatorname{ch}\frac{k}{l}z$	$-\frac{k}{l}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$-(z-t_1)+$ $+\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t_1)$
$\frac{1}{GJ_d}B(z)$	0	$-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$\operatorname{ch}\frac{k}{l}z$	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$-\frac{l^2}{k^2}\left[-1+\right.$ $\left.+\operatorname{ch}\frac{k}{l}(z-t_1)\right]$
$\frac{1}{GJ_d}H(z)$	0	0	0	1	$-(z-t_1)$

Table 10

	θ_0	θ_0'	$\frac{1}{GJ_d}B_0$	$\frac{1}{GJ_d}H_0$	$\frac{1}{GJ_d}q_e$
$\theta(z)$	1	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$z-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$-\frac{(z-t_1)^2}{2}+\frac{(z-t_2)^2}{2}-\frac{l^2}{k^2}\left[\operatorname{ch}\frac{k}{l}(z-t_2)-\operatorname{ch}\frac{k}{l}(z-t_1)\right]$
$\theta'(z)$	0	$\operatorname{ch}\frac{k}{l}z$	$-\frac{k}{l}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$-(t_2-t_1)-\frac{l}{k}\left[\operatorname{sh}\frac{k}{l}(z-t_2)-\operatorname{sh}\frac{k}{l}(z-t_1)\right]$
$\frac{1}{GJ_d}B(z)$	0	$-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$\operatorname{ch}\frac{k}{l}(z)$	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$-\frac{l^2}{k^2}\left[-\operatorname{ch}\frac{k}{l}(z-t_2)+\operatorname{ch}\frac{k}{l}(z-t_1)\right]$
$\frac{1}{GJ_d}H(z)$	0	0	0	1	$-(t_2-t_1)$

This, however, is not the case. The error becomes apparent if we remember that the initial parameters θ_0, θ'_0, B_0 and H_0 are arbitrary integration constants of the differential equation (3.1) and have to be determined from the boundary conditions at the ends of the beam $z=0$ and $z=l$. Two conditions at the end $z=0$ immediately give definite values (usually zero) for two of the initial parameters, and the two other parameters are determined from the corresponding conditions at the end $z=l$ This end belongs to the last part of the beam, that on which all the loads act. Thus, by means of the two parameters determined from the equations of the last part, the influence of all the loads is also transmitted to the preceding parts, including the first, which does not directly contain definite integrals of the load.

§ 4. Beams subjected to terminal torsional moments

We apply the method of initial parameters to the design of torsional members which are to withstand a given external load. We begin with the simpler cases of external load, namely, when the state of stress and strain throughout the beam is described by the same equations; such loads include, for example, those which are applied only at the ends of the beam. We examine in particular a beam subjected to the action of mutually balancing torsional moments, applied at the ends (Figure 58).

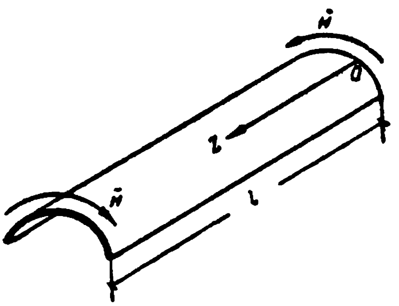


Figure 58

The strains and stresses of the beam will depend on the rigidity of the end supports. We examine here two cases of boundary conditions.

1. Beam Ends Free of Longitudinal Sectorial Forces. In such a case the bimoments, B, should vanish at both ends of the beam. Two further conditions are that the torsional moments at the ends of the beam should equal a given external torsional moment $\bar{H}$. However, from Table 3 we see that $H(z)=H_0$ and therefore it is sufficient to equate $H(z)$ to the given $\bar{H}$ only at one end; therefore, we have here only one condition. Let this condition refer to the end $z=l$. We take as the fourth boundary condition the vanishing of twist at the end $z=0$.

Thus, the boundary conditions in this case will be:

$$\text{for } z=0 \quad B=0 \text{ and } \theta=0;$$
$$\text{for } z=l \quad B=0 \text{ and } H=\bar{H}.$$

From these conditions we obtain:

$$B_0=0, \quad \theta_0=0 \text{ and } H_0=\bar{H}. \tag{4.1}$$

Thus, the three initial parameters are already determined; we determine the fourth initial parameter, θ_0', from the last condition: $B(l) = 0$. Using (4.1) we obtain:

$$B(l) = -\frac{l}{k} GJ_d \theta_0' \operatorname{sh} k + \frac{l}{k} \overline{H} \operatorname{sh} k = 0.$$

Hence

$$\theta_0' = \frac{\overline{H}}{GJ_d}. \tag{4.2}$$

Substituting in the matrix of Table 3 the found values (4.1), (4.2) of the initial parameters, we obtain the following equations for the basic design quantities:

$$\theta(z) = \frac{\overline{H}}{GJ_d} z, \quad \theta'(z) = \frac{\overline{H}}{GJ_d}, \quad B(z) = 0, \quad H(z) = \overline{H}. \tag{4.3}$$

For the torsion angle $\theta(z)$ we obtain an equation identical to the known equation from the theory of pure torsion of thin-walled beams.

The flexural-torsional bimoment $B(z)$ vanishes everywhere (for any z) for the chosen boundary conditions. This indicates that in a beam with free ends the normal torsional stresses vanish. Only tangential stresses appear in the beam, which form closed trajectories in the cross section and which give rise to a torsional moment $\overline{H}$.

The twist $\theta'(z)$ is proportional to the torsional moment $\overline{H}$ and inversely proportional to the torsional rigidity GJ_d in the case of pure torsion. As was mentioned earlier, the quantity J_d for thin-walled open profiles (I.5.8) is calculated from the equation

$$J_d = \frac{\alpha}{3} \sum d\delta^3,$$

where α is a coefficient depending on the shape of the section. Table 11, below, gives the values of the coefficient α for several types of rolled-steel sections which were obtained experimentally.

Table 11

Rolled-steel sections	Coefficient α	
	Diverse specimens	Average
Angle	0.86–1.10	0.99
Channel	0.98–1.25	1.12
Tee	0.92–1.25	1.15
I-beam	1.16–1.44	1.31
Wide-flanged beam . .	1.21–1.47	1.29
Zee	1.13–1.20	1.16

2. The Supported Sections Remain Plane in Torsion. With these conditions at both ends of the beam, the warping θ' vanishes everywhere.

Assuming, as before, that the initial section does not rotate, we may present the boundary conditions in the following form:

$$\text{for } z=0 \quad \theta'=0, \quad \theta=0;$$
$$\text{for } z=l \quad \theta'=0, \quad H=\overline{H}.$$

These conditions determine immediately the three initial parameters

$$\theta_0=0, \quad \theta_0'=0, \quad H_0=\overline{H}. \tag{4.4}$$

We find the fourth initial parameter from the conditions $\theta'(l)=0$. We take $\theta'(l)$ from Table 3; from (4.4) we obtain:

$$\theta'(l)=-\frac{k}{l}\frac{B_0}{GJ_d}\operatorname{sh}k+\frac{\overline{H}}{GJ_d}(1-\operatorname{ch}k)=0,$$

whence

$$B_0=\frac{l}{k}\frac{1-\operatorname{ch}k}{\operatorname{sh}k}. \tag{4.5}$$

Inserting in Table 3 the found values (4.4) and (4.5) of the initial parameters, we obtain the following equations for the basic design quantities:

$$\left.\begin{aligned}
\theta(z)&=\frac{\overline{H}}{GJ_d}\left[-\frac{l}{k}\frac{1-\operatorname{ch}k}{\operatorname{sh}k}\left(\operatorname{ch}\frac{k}{l}z-1\right)+z-\frac{l}{k}\operatorname{sh}\frac{k}{l}z\right],\\
\theta'(z)&=\frac{\overline{H}}{GJ_d}\left(-\frac{1-\operatorname{ch}k}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}z+1-\operatorname{ch}\frac{k}{l}z\right),\\
B(z)&=\overline{H}\frac{l}{k}\left(\frac{1-\operatorname{ch}k}{\operatorname{sh}k}\operatorname{ch}\frac{k}{l}z+\operatorname{sh}\frac{k}{l}z\right),\\
H(z)&=\overline{H}.
\end{aligned}\right\} \tag{4.6}$$

Since the conditions of support at the ends of the beam are symmetrical and the external load is antisymmetrical, these equations can be simplified if we choose the origin of the coordinate z in the middle of the beam span (Figure 59) and if we assume that the torsion angle in the initial section is equal to zero. In this case the torsion angle, θ, will be an odd function of z. Consequently, the bimoment B which is proportional to the second derivative of this angle will also be an odd function of z which vanishes for $z=0$. This occurs because double differentiation leaves the parity of the function unchanged. It is also possible to show directly that in the middle section of the beam the bimoment vanishes if we assume $z=\frac{l}{2}$ in the third of equations (4.6).

Thus, the boundary conditions will be

$$\text{for } z=0 \quad \theta=0, \quad B=0;$$
$$\text{for } z=\frac{l}{2} \quad \theta'=0, \quad H=\overline{H}.$$

The initial parameter θ'_0 will be

$$\theta'_0 = \frac{\operatorname{ch}\frac{k}{2} - 1}{\operatorname{ch}\frac{k}{2}} \frac{\bar{H}}{GJ_d}$$

and we obtain the following expressions for the basic design quantities:

$$\left.\begin{aligned} \theta(z) &= \frac{\bar{H}}{GJ_d}\left(z - \frac{l}{k}\frac{\operatorname{sh}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}\right), \quad \theta'(z) = \frac{\bar{H}}{GJ_d}\left(1 - \frac{\operatorname{ch}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}\right), \\ B(z) &= \bar{H}\frac{l}{k}\frac{\operatorname{sh}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}, \quad H(z) = \bar{H}. \end{aligned}\right\} \tag{4.7}$$

It follows from equations (4.7) that in a beam reinforced at both ends by rigid transverse plates and subjected to torsion, there will be not only tangential torsional stresses like in pure torsion (i. e.. torsion without elongation of the separate fibers in the direction Oz), but also sectorial normal and tangential stresses, associated with the bending of separate elementary longitudinal strips in their planes when their ends are rigidly fixed.

From (I.8.5), (I.8.9), (I.8.11), and (4.7), the equations for the sectorial stresses σ_ω and τ_ω will have the form

$$\left.\begin{aligned} \sigma_\omega &= \frac{B}{J_\omega}\omega = \frac{l}{k}\frac{\bar{H}}{J_\omega}\omega(s)\frac{\operatorname{sh}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}, \\ \tau_\omega &= -\frac{H_\omega}{J_\omega}\frac{S_\omega(s)}{\delta(s)} = \frac{\bar{H}}{J_\omega}\frac{S_\omega(s)}{\delta(s)}\frac{\operatorname{ch}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}. \end{aligned}\right\} \tag{4.8}$$

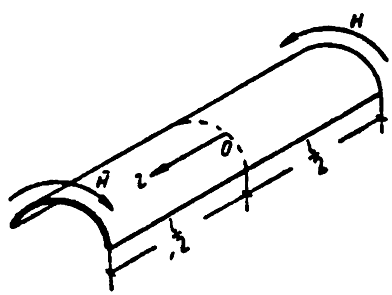


Figure 59

It is seen from these equations that the normal stresses vary with respect to the middle cross section (origin of the coordinate z) like an odd function $\left(\operatorname{sh}\frac{k}{l}z\right)$, and the tangential stresses vary like an even

function $\left(\operatorname{ch}\frac{k}{l}z\right)$; both reach their maximum values at the terminal sections. Assuming $z=\frac{l}{2}$ in (4.8), we obtain the following equations for these maximum values:

$$\max \sigma_\omega = \frac{l}{k}\frac{\overline{H}}{J_\omega}\omega(s)\operatorname{th}\frac{k}{2},$$
$$\max \tau_\omega = \frac{\overline{H}}{J_\omega}\frac{S_\omega(s)}{\delta(s)}.$$

§ 5. A beam under transverse load which does not pass through the shear center

We shall examine a thin-walled beam of open section with a span l, having its ends supported in some definite way.

Let a concentrated transverse force, P, at a distance e from the shear center, act on the beam at some point $z=t$ (Figure 56a). In this case the beam will be in a condition of combined flexure and torsion. As a matter of fact, the load P, if transferred parallel to itself, to the shear center (Figure 56b) will give rise to stresses, determined by the usual elementary theory of bending. The concentrated external torsional moment $\overline{H}=Pe$ obtained by transferring the force P to the shear center (Figure 56c), gives rise to complementary sectorial stresses σ_ω and τ_ω. Omitting the calculation of the beam for a load, which causes flexure according to the law of plane sections, we shall examine here the calculation of the beam for the action of a concentrated torsional moment $\overline{H}=Pe$.

Since only an external concentrated force-factor $\overline{H}$ acts on the section $z=t$ we have $H_t=-\overline{H}=-Pe$ (§ 3, subsection 4), and the expanded form of the equations of Table 7 (§ 3) becomes:

Table 12

	θ_0	θ'_0	$\frac{1}{GJ_d}B_0$	$\frac{1}{GJ_d}H_0$	$\frac{1}{GJ_d}\overline{H}_t=-\frac{1}{GJ_d}Pe$
$\theta(z)$	1	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$z-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$(z-t)-\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t)$
$\theta'(z)$	0	$\operatorname{ch}\frac{k}{l}z$	$-\frac{k}{l}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}(z-t)$
$\frac{1}{GJ_d}B(z)$	0	$-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$\operatorname{ch}\frac{k}{l}z$	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t)$
$\frac{1}{GJ_d}H(z)$	0	0	0	1	1

By these formulas the quantities $\theta(z)$, $\theta'(z)$, $B(z)$, $H(z)$ are calculated only for those sections of the beam which are situated to the left of the section $z=t$ (for $z>t$) where the torsional moment $\overline{H}=Pe$ is applied.

For sections situated near the section $z=t$ (for $z<t$), it is necessary to retain only the terms containing the initial parameters θ_0, θ_0', B_0 and H_0 in the formulas of Table 12, and discard those containing the external load $\overline{H}=Pe$.

We pass now to the determination of the initial parameters, and we examine some particular cases of boundary conditions.

1. Beam Hinged at Both Ends. Here we apply conditions (2.9) to both ends. Therefore, the boundary conditions are written as follows:

$$\left.\begin{array}{lll} \text{for } z=0 & \theta=0, & B=0; \\ \text{for } z=l & \theta=0, & B=0. \end{array}\right\} \tag{5.1}$$

The first two conditions directly determine the two initial parameters: $\theta_0=0$ and $B_0=0$. The two other initial parameters θ_0' and H_0 are found from the last two conditions (5.1). Writing, with the help of Table 12, the expresses for $\theta(l)$ and $B(l)$ and equating them to zero we obtain:

$$\left.\begin{array}{l} \theta_0'\dfrac{l}{k}\,\mathrm{sh}\,k+\dfrac{H_0}{GJ_d}\left(l-\dfrac{l}{k}\,\mathrm{sh}\,k\right)=-\dfrac{Pe}{GJ_d}\left[\dfrac{l}{k}\,\mathrm{sh}\dfrac{k}{l}(l-t)-(l-t)\right], \\ -\theta_0'\dfrac{l}{k}\,\mathrm{sh}\,k+\dfrac{H_0}{GJ_d}\dfrac{l}{k}\,\mathrm{sh}\,k=\dfrac{Pe}{GJ_d}\dfrac{l}{k}\,\mathrm{sh}\dfrac{k}{l}(l-t). \end{array}\right\} \tag{5.2}$$

The solution of equations (5.2) gives the following values for θ_0' and H_0:

$$\left.\begin{array}{l} \theta_0'=\dfrac{Pe}{GJ_d l\,\mathrm{sh}\,k}\left[(l-t)\,\mathrm{sh}\,k-l\,\mathrm{sh}\dfrac{k}{l}(l-t)\right], \\ H_0=Pe\dfrac{l-t}{l}. \end{array}\right\} \tag{5.3}$$

Substituting the values found for the initial parameters (5.3) in Table 12 and keeping in mind that the analytic expressions for the flexural-torsional factors will be different for each part, we obtain finally: for the part $0\leqslant z<t$

$$\left.\begin{array}{l} \theta(z)=\dfrac{Pe}{GJ_d}\left[\dfrac{l-t}{l}z-\dfrac{l\,\mathrm{sh}\dfrac{k}{l}(l-t)}{k\,\mathrm{sh}\,k}\,\mathrm{sh}\dfrac{k}{l}z\right], \\ \theta'(z)=\dfrac{Pe}{GJ_d}\left[\dfrac{l-t}{l}-\dfrac{\mathrm{sh}\dfrac{k}{l}(l-t)}{\mathrm{sh}\,k}\,\mathrm{ch}\dfrac{k}{l}z\right], \\ B(z)=Pe\dfrac{l}{k}\dfrac{\mathrm{sh}\dfrac{k}{l}(l-t)}{\mathrm{sh}\,k}\,\mathrm{sh}\dfrac{k}{l}z, \\ H(z)=Pe\dfrac{l-t}{l}. \end{array}\right| \tag{5.4}$$

for the part $t \leq z \leq l$

$$\left.\begin{aligned}
\theta(z) &= \frac{Pe}{GJ_d}\left[\frac{t}{l}(l-z) - \frac{l}{k}\frac{\operatorname{sh}\frac{k}{l}t}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}(l-z)\right],\\
\theta'(z) &= \frac{Pe}{GJ_d}\left[-\frac{t}{l} + \frac{\operatorname{sh}\frac{k}{l}t}{\operatorname{sh}k}\operatorname{ch}\frac{k}{l}(l-z)\right],\\
B(z) &= Pe\frac{l}{k}\frac{\operatorname{sh}\frac{k}{l}t}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}(l-z),\\
H(z) &= -Pe\frac{t}{l}.
\end{aligned}\right\} \tag{5.5}$$

Equations (5.4) and (5.5) have a general character and permit the determination of the kinematical and statical factors $\theta(z)$, $\theta'(z)$, $B(z)$ and $H(z)$ for any section and for any position of the concentrated force P in the span. As was already said in § 3, we can construct the diagrams of $\theta(z)$, $\theta'(z)$, $B(z)$ and $H(z)$ for an external concentrated torsional moment Pe applied at a definite section $t=$const, by holding in these equations the section $t=$const and letting z vary. If we consider in these formulas z constant and the distance t to the section as varying over all values in the interval $0 < t < l$, equations (5.4) and (5.5) are those of the influence line of the displacements $\theta(z)$, $\theta'(z)$ for the forces $B(z)$ and $H(z)$ on the section $z=$const due to a concentrated torsional moment Pe moving along the beam.

Given the equations (5.4) and (5.5), we can easily determine the complementary normal and tangential stresses due to torsion, by the equations:

$$\left.\begin{aligned}
\sigma_\omega &= \frac{B(z)}{J_\omega}\omega(s),\\
\tau_\omega &= -\frac{H_\omega(z)S_\omega(s)}{J_\omega\delta(s)},
\end{aligned}\right\} \tag{5.6}$$

where, according to (I.8.11), $H_\omega(z) = H(z) - GJ_d\theta'(z)$.

In equations (5.6) the statical factors $B(z)$ and $H_\omega(z)$ characterize the variation of the stresses σ_ω and τ_ω along the beam (for a given position of the load), and the geometric factors $\omega(s)$, $S_\omega(s)$ and $\delta(s)$ characterize these stresses over the section contour.

Considering (5.4) and (5.5) as the equations of the influence line, i.e., taking z constant and t varying, in view of the law of independent action of forces we can easily calculate the displacements and forces in a beam twisted by a system of external concentrated torsional moments P_1e_1, P_2e_2, ..., applied at the points t_1, t_2, ..., etc.

It is also possible to determine the quantities $\theta(z)$, θ (z), $B(z)$ and $H(z)$, and then from (5.6), find the values of σ_ω and τ_ω under a continuous transverse load.

If the magnitude and position of a transverse load varies over the length of the beam, but its direction remains constant, the magnitude of

the applied torsional moment is given by

$$m(t)=q(t)\,e(t), \tag{5.7}$$

where $q(t)$ is the specific load, $e(t)$ its distance in the section t from the shear center.

Regarding the transverse load $q(t)$ as continuous, i.e., distributed over the whole length of the beam, and $m(t)\,dt$ a concentrated torsional moment (over the length dt) analogous to the moment denoted by Pe in (5.4) and (5.5), we obtain equations for the basic design quantities by integrating the corresponding quantities over the whole length of the beam. Thus, for a continuous load,

$$\left.\begin{aligned}
\theta(z)&=\int_0^z \theta(z,t)\,m(t)\,dt+\int_z^l \theta(z,t)\,m(t)\,dt,\\
\theta'(z)&=\int_0^z \theta'(z,t)\,m(t)\,dt+\int_z^l \theta'(z,t)\,m(t)\,dt,\\
B(z)&=\int_0^z B(z,t)\,m(t)\,dt+\int_z^l B(z,t)\,m(t)\,dt,\\
H(z)&=\int_0^z H(z,t)\,m(t)\,dt+\int_z^l H(z,t)\,m(t)\,dt.
\end{aligned}\right\} \tag{5.8}$$

The section z (which is where we determine the values of the relevant parameters) divides the interval from 0 to l (the length of the beam) into two parts. The functions $\theta(z,t)$, $\theta'(z,t)$, $B(z,t)$ and $H(z,t)$ in equations (5.8) are the influence functions, determined either by (5.4) or by (5.5), depending on which side of the examined section z lies the part for which the influence of the load $m(t)\,dt$ is evaluated. Obviously, the influence functions of the first terms of equations (5.8) should be calculated by equations (5.5) since here $t<z$, and the influence functions of the second terms (5.8) should be calculated by equations (5.4) since for this part $t>z$.

Taking $m(t)$ as constant in equations (5.8), which is true, for example, in the case $q=\text{const}$ and $e=\text{const}$, and integrating, we obtain equations for $\theta(z)$, $\theta'(z)$, $B(z)$, $H(z)$ due to the influence of a uniformly distributed external torsional moment in an arbitrary section z.

These equations have the following form:

$$\left.\begin{aligned}
\theta(z)&=\frac{l^2}{k^2}\frac{m}{GJ_d}\left[\frac{k^2}{2l^2}z(l-z)+\frac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}-1\right],\\
\theta'(z)&=\frac{l}{k}\frac{m}{GJ_d}\left[\frac{k}{l}\left(\frac{l}{2}-z\right)-\frac{\operatorname{sh}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}\right],\\
B(z)&=\frac{l^2}{k^2}m\left[1-\frac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}\right],\\
H(z)&=m\left(\frac{l}{2}-z\right).
\end{aligned}\right\} \tag{5.9}$$

For the stresses σ_ω and τ_ω we obtain the following equations:

$$\left.\begin{aligned}\sigma_\omega &= \frac{l^2}{k^2}\frac{m}{J_\omega}\omega(s)\left[1-\frac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}\right],\\ \tau_\omega &= \frac{l}{k}\frac{m}{J_\omega\delta(s)}S_\omega(s)\frac{\operatorname{sh}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}.\end{aligned}\right\} \tag{5.10}$$

It is seen from equations (5.10) that in the case of a uniform load q acting at a constant distance e from the shear center, the greatest normal stresses σ_ω are obtained in the mid-section (half-length) while the tangential stresses τ_ω are maximal at the supported sections

$$\left.\begin{aligned}\max\sigma_\omega &= \frac{l^2}{k^2}\frac{m}{J_\omega}\omega(s)\left(1-\frac{1}{\operatorname{ch}\frac{k}{2}}\right).\\ \max\tau_\omega &= \frac{l}{k}\frac{m}{J_\omega\delta(s)}S_\omega(s)\operatorname{th}\frac{k}{2}.\end{aligned}\right\} \tag{5.11}$$

We could have obtained equations (5.9) differently, starting from the matrix of Table 9 where, based on the considerations developed in § 3, the particular integrals for a uniform load q, acting at a constant distance e from the shear center from $z=t_1$ to $z=t_2$ are given in the last column. In the given case, where q acts over the whole length of the beam, it is possible to assume $t_1=0$ in Table 9 and to consider the formulas of this table to be valid for the whole interval from 0 to l.

Applying the boundary conditions (5.1) we obtain the values of the initial parameters:

$$\theta_0=0,\quad \theta_0'=\frac{qe}{GJ_d}\left(\frac{l}{2}+\frac{l}{k}\frac{1-\operatorname{ch}k}{\operatorname{sh}k}\right),$$
$$B_0=0,\quad H_0=qe\frac{l}{2}.$$

Introducing in Table 9 these values of the initial parameters modified to suit the present case, we arrive at equations (5.9).

As a numerical example we examine a truss girder, shown in Figure 40. For this girder we obtained in § 1:

$$J_x=976.8\text{ cm}^4,\quad J_y=383.8\text{ cm}^4,\quad J_\omega=4829\text{ cm}^6.$$

We calculate the torsional rigidity J_d for $\alpha=1$ by equation (I.5.8),

$$J_d=6.17\text{ cm}^4.$$

For the elastic constants of the material we have:

$$E = 2\,100\,000\ \text{kg/cm}^2,$$
$$G = 0.4E.$$

The length of the girder will be $l = 3.00$ m. For the elastic constant k we obtain by equation (2.2)

$$k = 6.782.$$

We determine the stresses σ_ω due to a uniformly distributed transverse load $q = 0.9\,\frac{\text{ton}}{\text{m}}$ which is applied with an eccentricity $e = -7.11$ cm with respect to the shear center (Figure 60). Such a load is due to the weight of a wall built on the girder. This load causes torsional stresses in addition to flexural stresses. The flexural stresses are computed by the usual equations of the theory of strength of materials. Maximal torsional stresses are obtained from the first of equations (5.11)

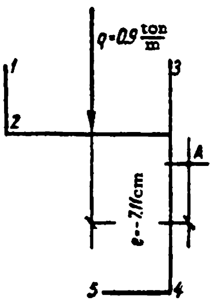


Figure 60

$$\max \sigma_\omega = -24.43\,\omega(s).$$

The diagram of the sectorial areas $\omega(s)$ is shown in Figure 43b. Given this diagram, it is easy to compute the stresses σ_ω for any point of the contour, by the above equation.

The results of the calculations for the girder (in kg/cm^2) are given in Table 13.

Table 13

No of points	σ_x	σ_y	$\sigma_x + \sigma_y$	σ_ω	$\sigma_x + \sigma_y + \sigma_\omega$
1	+391.5	−930.1	−538.6	−953.8	−1492.4
2	+719.9	−538.3	+181.6	+602.0	+783.6
3	−893.4	−425.1	−1318.5	−250.9	−1569.4
4	+210.2	+887.9	+1098.1	+317.1	+1415.2
5	+781.7	+662.3	+1444.0	−830.7	+613.3

Figure 61a shows the diagram of maximal flexural stresses $\sigma_{\text{flex}} = \sigma_x + \sigma_y$. Figure 61b shows the diagram of the torsional stresses σ_ω. The diagram of total stress $\sigma = \sigma_x + \sigma_y + \sigma_\omega$ for the middle section is shown in Figure 61c.

It is seen from the comparison of the diagrams $\sigma_{\text{flex}} = \sigma_x + \sigma_y$ and $\sigma = \sigma_x + \sigma_y + \sigma_\omega$ shown in Figure 61a and c that the error in calculating the girder by the usual bending theory as compared with the calculation allowing for torsion can be very large. This disparity of the results of calculations according to the theory of plane sections and those obtained

from the theory presented here is explained by the fact that we consider the given girder as a separate element, having no supports to restrain torsion. Actually, a brick wall resting on the girder will resist torsion owing to its monolithic nature. Due to this resistance, the weight of the wall will be transmitted to the girder with a smaller eccentricity. This reduced eccentricity will lower the complementary sectorial stress σ_ω for a constant, load, whereas the flexural stresses will not depend on the eccentricity. The stress diagram $\sigma=\sigma_x+\sigma_y+\sigma_\omega$ with the elastic compliance of the brick wall taken in, will occupy an intermediate position between the diagrams shown in Figure 61a and c.

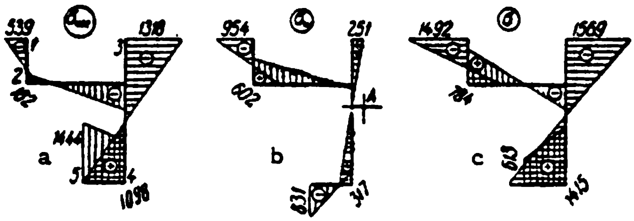


Figure 61

In order to get an idea of the influence of the eccentricity on the value of σ_ω, we bring in Table 14 below the result of the calculation of σ_ω for three eccentricities: $e=-7.11$ cm (corresponding to the load examined above), $e=-1.38$ cm (corresponding to the same load, but applied at the point 3), and $e=-12.84$ cm (the same load applied at the point 1) (Figure 60).

Table 14

No of points	σ_ω kg/cm^2		
	$e=-1.38$	$e=-7.11$	$e=-12.84$
1	-185.1	-953.8	-1722.5
2	$+116.8$	$+602.0$	$+1087.0$
3	-48.7	-250.9	-453.1
4	$+61.5$	$+317.1$	$+572.5$
5	-161.2	-830.7	-1500.2

As shown, the flexural stresses σ_x and σ_y do not change in magnitude when the eccentricity is changed. Their magnitudes remain at their previous values (Table 13).

2. A Beam With Rigidly Encastered Ends. In this case we have at both ends of the beams boundary conditions of the form (2.8):

$$\begin{aligned} &\text{for } z=0 \quad \theta=0, \quad \theta'=0; \\ &\text{for } z=l \quad \theta=0, \quad \theta'=0. \end{aligned} \tag{5.12}$$

The first two conditions directly determine two initial parameters;

the remaining two initial parameters B_0 and H_0 are found from the last two conditions by the formulas of Table 12.

We obtain two equations:

$$\frac{B_0}{GJ_d}(1-\operatorname{ch}k)+\frac{H_0 l}{GJ_d}\left(1-\frac{1}{k}\right)\operatorname{sh}k+\frac{Pe}{GJ_d}\left[\frac{l}{k}\operatorname{sh}\frac{k}{l}(l-t)-(l-t)\right]=0.$$

$$-\frac{B_0}{GJ_d}\frac{k}{l}\operatorname{sh}k+\frac{H_0}{GJ_d}(1-\operatorname{ch}k)+\frac{Pe}{GJ_d}\left[\operatorname{ch}\frac{k}{l}(l-t)-1\right]=0.$$

From these equations we find:

$$\left.\begin{aligned}
B_0&=-Pe\frac{t+(l-t)\operatorname{ch}k+\frac{l}{k}\operatorname{sh}\frac{k}{l}t-\frac{l}{k}\operatorname{sh}k+\frac{l}{k}\operatorname{sh}\frac{k}{l}(l-t)-t\operatorname{ch}\frac{k}{l}(l-t)}{k\operatorname{sh}k+2(1-\operatorname{ch}k)},\\
H_0&=Pe\frac{1+\frac{k}{l}(l-t)\operatorname{sh}k-\operatorname{ch}k+\operatorname{ch}\frac{k}{l}t-\operatorname{ch}\frac{k}{l}(l-t)}{k\operatorname{sh}k+2(1-\operatorname{ch}k)}.
\end{aligned}\right\} \tag{5.13}$$

By introducing the values of the initial parameters in Table 12, we obtain the equations for the basic design quantities for these boundary conditions. The equations are:

on the part $0\leqslant z<t$:

$$\left.\begin{aligned}
\theta(z)&=\frac{B_0}{GJ_d}\left(1-\operatorname{ch}\frac{k}{l}z\right)+\frac{H_0}{GJ_d}\left(z-\frac{l}{k}\operatorname{sh}\frac{k}{l}z\right),\\
\theta'(z)&=-\frac{B_0}{GJ_d}\frac{k}{l}\operatorname{sh}\frac{k}{l}z+\frac{H_0}{GJ_d}\left(1-\operatorname{ch}\frac{k}{l}z\right),\\
B(z)&=B_0\operatorname{ch}\frac{k}{l}z+H_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z,\\
H(z)&=H_0.
\end{aligned}\right\} \tag{5.14}$$

on the part $t\leqslant z\leqslant l$

$$\left.\begin{aligned}
\theta(z)&=\frac{B_0}{GJ_d}\left(1-\operatorname{ch}\frac{k}{l}z\right)+\frac{H_0}{GJ_d}\left(z-\frac{l}{k}\operatorname{sh}\frac{k}{l}z\right)+\\
&\quad+\frac{Pe}{GJ_d}\left[(z-t)-\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t)\right],\\
\theta'(z)&=-\frac{B_0}{GJ_d}\frac{k}{l}\operatorname{sh}\frac{k}{l}z+\frac{H_0}{GJ_d}\left(1-\operatorname{ch}\frac{k}{l}z\right)+\\
&\quad+\frac{Pe}{GJ_d}\left[1-\operatorname{ch}\frac{k}{l}(z-t)\right],\\
B(z)&=B_0\operatorname{ch}\frac{k}{l}z+H_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z+Pe\frac{l}{k}\operatorname{sh}\frac{k}{l}(z-t),\\
H(z)&=H_0+Pe.
\end{aligned}\right\} \tag{5.15}$$

where B_0 and H_0 are determined by equations (5.13).

Considering expressions (5.14) and (5.15) as influence functions, we can, in analogy with the previous case, determine the flexure-torsion factors due to any continuous load from equations (5.8). We can do that

specifically for a uniformly-distributed torsional moment $m = qe = \text{const}$ which acts over the whole length of the beam.

In the latter case it is even simpler to start from Table 9, § 3, assuming there $t_1 = 0$. We determine the initial parameters from these conditions by applying the boundary conditions (5.12).

$$\theta_0 = \theta'_0 = 0,$$

$$B_0 = -m\frac{l^2}{k^2}\,\frac{k\,\mathrm{ch}\frac{k}{2} - 2\,\mathrm{sh}\frac{k}{2}}{2\,\mathrm{sh}\frac{k}{2}},$$

$$H_0 = m\frac{l}{2}.$$

By introducing the values of the initial parameters in the expressions of Table 9, we obtain the following equations for the basic design quantities for the case of a uniformly distributed torsional moment, acting over the whole length of the beam,

$$\left.\begin{aligned}
\theta(z) &= \frac{m}{EJ_\omega}\frac{l^2}{k^2}\left[\frac{z(l-z)}{2} - \frac{l^2\,\mathrm{sh}\frac{kz}{2l}\,\mathrm{sh}\frac{k}{2l}(l-z)}{k\,\mathrm{sh}\frac{k}{2}}\right],\\
\theta'(z) &= \frac{m}{EJ_\omega}\frac{l^2}{k^2}\left[\frac{l}{2} - z + \frac{l}{2}\,\frac{\mathrm{sh}\frac{k}{l}\left(z-\frac{l}{2}\right)}{\mathrm{sh}\frac{k}{2}}\right],\\
B(z) &= m\frac{l^2}{k^2}\left[1 - \frac{k}{2}\,\frac{\mathrm{ch}\frac{k}{l}\left(z-\frac{l}{2}\right)}{\mathrm{sh}\frac{k}{2}}\right],\\
H(z) &= m\left(\frac{l}{2} - z\right).
\end{aligned}\right\} \tag{5.16}$$

From (5.16) we obtain for the sectorial stresses σ_ω and τ_ω the expressions

$$\left.\begin{aligned}
\sigma_\omega &= \frac{m}{J_\omega}\frac{l^2}{k^2}\left[1 - \frac{k}{2}\,\frac{\mathrm{ch}\frac{k}{l}\left(z-\frac{l}{2}\right)}{\mathrm{sh}\frac{k}{2}}\right]\omega(s),\\
\tau_\omega &= \frac{m}{2J_\omega}\,\frac{l}{\mathrm{sh}\frac{k}{2}}\,\frac{S_\omega(s)}{\delta(s)}\,\mathrm{sh}\frac{k}{l}\left(z-\frac{l}{2}\right).
\end{aligned}\right\} \tag{5.17}$$

Expressions (5.17) are maximal at the supported sections of the beam, i.e., for $z = 0$ and $z = l$:

$$\max\sigma_\omega = \frac{m}{J_\omega}\frac{l^2}{k^2}\left(1 - \frac{k}{2}\,\mathrm{cth}\frac{k}{2}\right)\omega(s),$$

$$\max\tau_\omega = \pm\frac{ml}{2J_\omega}\,\frac{S_\omega(s)}{\delta(s)}.$$

3. A Beam with One End Encastered and the Other Hinged. In this case conditions (2.8) should be satisfied at one end of the beam, and conditions (2.9) at the other. Let us assume specifically that conditions (2.8) refer to the end $z=0$, and conditions (2.9) (hinge support) to the end $z=l$.

Conditions (2.8) at the end $z=0$ then immediately determine the two initial parameters:

$$\theta_0=0; \quad \theta_0'=0.$$

We determine the other two initial parameters B_0 and H_0 from the two equations which arise from the conditions (2.9) for the end $z=l$. These equations, which we obtain with the help of Table 12, are:

$$\left.\begin{aligned} B_0(1-\operatorname{ch}k)+H_0\left(l-\frac{l}{k}\operatorname{sh}k\right)+Pe\left[\frac{l}{k}\operatorname{sh}\frac{k}{l}(l-t)-(l-t)\right]=0,\\ B_0\operatorname{ch}k+H_0\frac{l}{k}\operatorname{sh}k+Pe\frac{l}{k}\operatorname{sh}\frac{k}{l}(l-t)=0. \end{aligned}\right\} \tag{5.18}$$

Solving the system (5.18), we find:

$$\left.\begin{aligned} B_0&=Pe\frac{\frac{l}{k}(l-t)\operatorname{sh}k-\frac{l^2}{k}\operatorname{sh}\frac{k}{l}(l-t)}{\frac{l}{k}\operatorname{sh}k-l\operatorname{ch}k},\\ H_0&=Pe\frac{\frac{l}{k}\operatorname{sh}\frac{k}{l}(l-t)-(l-t)\operatorname{ch}k}{\frac{l}{k}\operatorname{sh}k-l\operatorname{ch}k}. \end{aligned}\right\} \tag{5.19}$$

The equations for the basic design quantities will have the same form as in the previous case: (5.14) for the part $0\leqslant z<t$ and (5.15) for the part $t\leqslant z\leqslant l$, only that now we have to substitute their values for B_0 and H_0 as determined by equations (5.19).

In the case of a uniformly-distributed torsional moment $m=qe=\text{const}$ acting over the whole length of the beam, the equations for the basic design quantities will be:

$$\left.\begin{aligned} \theta(z)&=\frac{m}{GJ_d}\left\{-\left(\frac{l^2}{k^2}+\frac{z^2}{2}\right)+\frac{1}{\operatorname{sh}k-k\operatorname{ch}k}\left[l^2\operatorname{sh}\frac{k}{2}\left(\operatorname{ch}\frac{k}{2}-\right.\right.\right.\\ &\quad\left.-\frac{2}{k}\operatorname{sh}\frac{k}{2}\right)-\frac{l}{2k}[2+(k^2-2)\operatorname{ch}k]z+\\ &\quad\left.\left.+\frac{l^2}{k^2}\left(\operatorname{sh}\frac{k}{l}z-k\operatorname{ch}\frac{k}{l}z\right)+\left(\frac{l^2}{k^2}-\frac{l^2}{2}\right)\operatorname{sh}\frac{k}{l}(l-z)\right]\right\},\\ \theta'(z)&=\frac{m}{GJ_d}\left\{-z+\frac{1}{\operatorname{sh}k-k\operatorname{ch}k}\left[-\frac{l}{2k}[2+(k^2-2)\operatorname{ch}k]+\right.\right.\\ &\quad\left.\left.+\frac{l}{k}\left(\operatorname{ch}\frac{k}{l}z-k\operatorname{sh}\frac{k}{l}z\right)+\frac{l\left(\frac{k^2}{2}-1\right)}{k}\operatorname{ch}\frac{k}{l}(l-z)\right]\right\},\\ B(z)&=m\frac{l^2}{k^2}\left[1-\frac{\left(1-\frac{k^2}{2}\right)\operatorname{sh}\frac{k}{l}(l-z)+\operatorname{sh}\frac{k}{l}z-k\operatorname{ch}\frac{k}{l}z}{\operatorname{sh}k-k\operatorname{ch}k}\right],\\ H(z)&=-m\left\{z+\frac{\frac{l}{2k}[2+(k^2-2)\operatorname{ch}k]}{\operatorname{sh}k-k\operatorname{ch}k}\right\}. \end{aligned}\right\} \tag{5.20}$$

From equations (5.6) it is easy to calculate the normal and tangential stresses due to torsion: σ_ω and τ_ω.

4. A Beam with One End Encastered and the Other Free. Considering as in the previous case, that the initial section is the one encastered and that the statical factors vanish at the other end, it is possible to write the boundary conditions for this case as follows:

$$\left.\begin{array}{lll} \text{for } z=0 & \theta=0, & \theta'=0; \\ \text{for } z=l & B=0, & H=0. \end{array}\right\} \qquad (5.21)$$

The equations for determining the remaining initial parameters B_0 and H_0 which are derived with the help of Table 12 will be:

$$B_0 \operatorname{ch} k + H_0 \frac{l}{k} \operatorname{sh} k = Pe \frac{l}{k} \operatorname{sh} \frac{k}{l}(l-t),$$
$$H_0 = Pe.$$

Whence we find:

$$\left.\begin{array}{l} H_0 = Pe, \\ B_0 = Pe \dfrac{l}{k} \dfrac{\operatorname{sh} \frac{k}{l}(l-t) - \operatorname{sh} k}{\operatorname{ch} k}. \end{array}\right\} \qquad (5.22)$$

As in the two previous cases, the equations for the determination of the basic design quantities will be given by the expressions (5.14) and (5.15), in which we have introduced the values of B_0 and H_0 (5.22).

The equations for the basic design quantities in the case of a uniformly distributed torsional moment $m=qe=$const acting over the whole length of the beam will have the following form for such boundary conditions:

$$\left.\begin{array}{l} \theta(z) = -\dfrac{m}{GJ_d \operatorname{ch} k}\Big[-\dfrac{l^2}{k^2} - \dfrac{l^2}{k}\operatorname{sh} k + z\left(l - \dfrac{z}{2}\right)\operatorname{ch} k + \\ \qquad\qquad + \dfrac{l^2}{k^2}\operatorname{ch}\dfrac{k}{l}z + \dfrac{l^2}{k}\operatorname{sh}\dfrac{k}{l}(l-z)\Big], \\ \theta'(z) = -\dfrac{m}{GJ_d \operatorname{ch} k}\Big[(l-z)\operatorname{ch} k + \dfrac{l}{k}\operatorname{sh}\dfrac{k}{l}z - l\operatorname{ch}\dfrac{k}{l}(l-z)\Big], \\ B(z) = -\dfrac{m}{\operatorname{ch} k}\dfrac{l^2}{k^2}\Big[\operatorname{ch} k - \operatorname{ch}\dfrac{k}{l}z - k\operatorname{sh}\dfrac{k}{l}(l-z)\Big], \\ H(z) = -m(l-z). \end{array}\right\} \qquad (5.23)$$

From equations (5.6) the normal and tangential stresses due to flexure plus torsion σ_ω and τ_ω, are easily calculated.

We have examined four types of beams which differ in their boundary conditions. For all these types the boundary conditions were given explicitly. Of the four initial flexure-torsion factors two vanished.

The given solution can also be easily extended to a more general case, where the boundary conditions are given in the form of linear relations between the kinematic and static factors. We encounter such boundary conditions in the case where the beam is *elastically* encastered

at the ends. A beam of intermediate span, in a continuous girder, which is elastically joined to the neighboring elements of this girder, can serve as an example. The statical factors which act in the supported sections of this beam will be proportional to the corresponding kinematic factors. The coefficients of proportionality are determined from the condition of equality of the longitudinal sectorial deformations and torsion angles at the supported sections.

5. The Torsion of a Beam under the Action of a Bending Moment. We examined the problem of torsion of a beam, whose ends are under different boundary conditions, and which is subjected to a transverse load not passing through the shear center.

We will show that, given the solution for the case of a concentrated force acting on the beam, it is easy to obtain from this solution the basic equations for the design of beams twisted by a concentrated bending moment (consequently, it is also possible to derive them for a distribution of moments).

Let a concentrated bending moment M act on the beam at the point $z=t$ in a plane parallel to the beam axis and at a distance e from the shear center (Figure 62a).

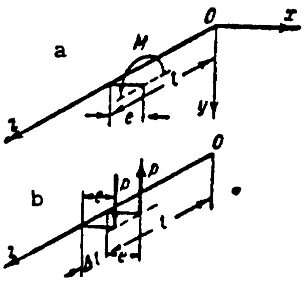


Figure 62

Since the plane of action of this moment does not pass through the shear center, torsional stresses appear in the beam, in addition to flexural stresses. In this case, when the plane of action of the moment M does not pass through the line of shear centers, we determine the complementary deformations and stresses which appear in the beam according to the law of sectorial areas, through the formulas of Table 12, which are derived for the case of a concentrated transverse force.

We collectively denote any one of the four factors $\theta(z)$, $\theta'(z)$, $\frac{1}{GJ_d}B(z)$ and $\frac{1}{GJ_d}H(z)$ by $\Phi(z,t)$ keeping in mind that this factor is a function of the two variables z and t. Furthermore, we assume a unit load $Pe=1$, and indicate this by affixing the index P to the function Φ. We can then write any of the formulas of Table 12 symbolically, as follows:

$$\Phi_p = \Phi_p(z,t)$$

We now replace the external bending moment M, applied at the point $z=t$, by a transverse couple consisting of forces P with an arm Δt (Figure 62b):

$$P = \frac{M}{\Delta t}. \tag{5.24}$$

Calculating the value of the factor Φ_p for the action of two equal and opposite transverse concentrated forces P, applied at the sections

t and $t+\Delta t$, we may write on the strength of (5.24):

$$\Phi_M=[\Phi_p(z, t+\Delta t)-\Phi_p(z, t)]Pe=\frac{\Phi_p(z, t+\Delta t)-\Phi_p(z, t)}{\Delta t}Me.$$

At the limit, as $\Delta t \to 0$ we obtain

$$\Phi_M=\frac{\partial\Phi_p}{\partial t}Me. \tag{5.25}$$

In this way the influence function for a flexure-torsion factor Φ_M due to a bending moment M, acting in a plane parallel to the beam axis at a distance e from the shear center, is found to be proportional to the derivative of Φ_p with respect to t, where Φ_p stands for the influence function for an analogous factor for the case of a concentrated external torsional moment $Pe = 1$ acting on the beam. The quantity Me serves as the coefficient of proportionality.

Thus, for example, for a beam hinged at both ends subjected to a concentrated bending moment M in the plane $z=t$ at an eccentricity e (off the shear center), we obtain from (5.4), (5.5), and (5.25) the following expressions for the principal flexure-torsion factors:

for the part $0 \leqslant z < t$

$$\theta(z)=\frac{Me}{GJ_d}\left[-\frac{z}{l}+\frac{\operatorname{ch}\frac{k}{l}(l-t)}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}z\right],$$

$$\theta'(z)=\frac{Me}{GJ_d}\left[-\frac{1}{l}+\frac{k}{l}\frac{\operatorname{ch}\frac{k}{l}(l-t)}{\operatorname{sh}k}\operatorname{ch}\frac{k}{l}z\right],$$

$$B(z)=-Me\frac{\operatorname{ch}\frac{k}{l}(l-t)}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}z,$$

$$H(z)=-\frac{Me}{l};$$

for the part $t \leqslant z \leqslant l$

$$\theta(z)=\frac{Me}{GJ_d}\left[\frac{l-z}{l}-\frac{\operatorname{ch}\frac{k}{l}t}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}(l-z)\right],$$

$$\theta'(z)=\frac{Me}{GJ_d}\left[-\frac{1}{l}+\frac{k}{l}\frac{\operatorname{ch}\frac{k}{l}t}{\operatorname{sh}k}\operatorname{ch}\frac{k}{l}(l-z)\right],$$

$$B(z)=Me\frac{\operatorname{ch}\frac{k}{l}t}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}(l-z),$$

$$H(z)=-\frac{Me}{l}.$$

Analogously, it is also possible to find the equations for other cases of boundary conditions.

It follows from equations (5.25) that for $e = 0$, Φ_M also vanishes. This means that if the external bending moment acts in a plane passing through the shear center, all the additional secotrial factors related to

the beam torsion vanish. The beam will, in this case, be in the state of central transverse flexure. In the general case, when the plane of action of the external bending moment does not pass through the shear center, complementary factors associated with torsion will appear in the beam, in addition to the principal flexural strains.

§ 6. Beam torsion and the determination of the bimoments under terminally applied longitudinal force

1. The departure from the law of plane sections occurs, as we shall see below, not only in the case of bending due to a transverse load which does not pass through the shear center, but also in the case of longitudinal forces (compressive, tensile, or shearing) acting at the ends or somewhere in the middle surface, and in the case of the longitudinal force applied outside the section and transmitted to it through a rigid arm fixed to some point of the beam contour.

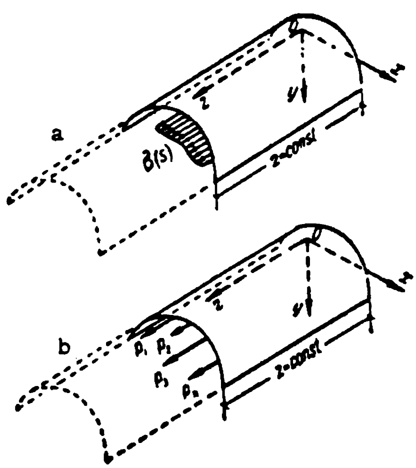


Figure 63

We mention first that the law of sectorial areas affords to replace a given external longitudinal load by four force factors equivalent to it, viz., a longitudinal normal force $\overline{N}$, bending moments $\overline{M}_x$ and $\overline{M}_y$ and a bimoment $\overline{B}$. Each of these iorce factors can be represented as a generalized force, by considering the work done by a given external load on the section z = const through the corresponding longitudinal displacement which is determined by one of the four components of the general four-term equation (I.3.16).

If the external longitudinal load in the section z = const is given in the form of normal stresses $\bar{\sigma}(s)$ (Figure 63a), then by (I.8.2) we have for the statically equivalent force factors,

$$\left.\begin{aligned}\overline{N}&=\int_F \bar{\sigma}\, dF,\\ \overline{M}_x&=\int_F \bar{\sigma} y\, dF,\\ \overline{M}_y&=-\int_F \bar{\sigma} x\, dF,\\ \overline{B}&=\int_F \bar{\sigma}\omega\, dF.\end{aligned}\right\} \qquad (6.1)$$

When the longitudinal load consists of a set of concentrated forces $P_k (k = 1, 2, 3, \ldots, n)$ applied at various points of the contour of the cross section (Figure 63b), equations (6.1) for the determination of the four force factors of the section z = const become

$$\left.\begin{aligned} \overline{N} &= \sum_k P_k, \\ \overline{M}_x &= \sum_k P_k y_k, \\ \overline{M}_y &= -\sum_k P_k x_k, \\ \overline{B} &= \sum_k P_k \omega_k. \end{aligned}\right\} \quad (6.2)$$

Here x_k and y_k denote the coordinates of the point of application of the force P_k, and ω_k the sectorial area at this point. The summation is carried out for all forces P_k $(k = 1, 2, 3, \ldots, n)$ applied to the contour of the section z = const.

If the load consists of a sole longitudinal force P, applied at the contour point $K(x_k, y_k, \omega_k)$ equations (6.2) will have the simpler form:

$$\left.\begin{aligned} \overline{N} &= P, \\ \overline{M}_x &= P y_k, \\ \overline{M}_y &= -P x_k, \\ \overline{B} &= P \omega_k. \end{aligned}\right\} \quad (6.3)$$

We should do well to note here that the sign of the bimoment depends on:

1) the sign of the outward normal to the section z = const at the point of application of the force P,

2) the direction of the force P, and

3) the sign of the corresponding sectorial area ω.

The external normal and the force P are taken as positive if their direction coincides with the positive direction of the axis Oz. The sectorial area ω is taken as positive if, when looking at the section z = const from the side of the positive outward normal, it is swept by a radius-vector moving clockwise.

If the sectorial area coordinate ω corresponding to the force P turns out to be positive, the bimoment will be positive for the cases (a) and (c) of the four modes of application of the force P, shown in Figure 64, and negative for the cases (b) and (d), in accordance with the adopted rule of signs.

The given equations permit the calculation of the static factors at the section z = const for a given diagram of the normal stresses $\overline{\sigma}(s)$ or for a given diagram of the longitudinal normal forces. Since the values of these factors are known, it is easy to determine the stresses at any point of the beam. We recall equation (I.8 5) for the normal stresses:

$$\sigma = \frac{N}{F} - \frac{M_y}{J_y} x + \frac{M_x}{J_x} y + \frac{B}{J_\omega} \omega.$$

In this equation the first three terms express the stresses which obey the law of plane sections. The static factors, which characterize the variation of these stresses over the length of the beam under the action of a longitudinal load only, do not depend on the variable z and are determined by the equations

$$N=\overline{N};\quad M_x=\overline{M}_x;\quad M_y=\overline{M}_y.$$

The last term in (I.8.5) gives the stresses σ_ω, which appear as a result of the torsion of the beam. The flexural-torsional bimoment B, which characterizes the variation of the sectorial normal stresses with respect to the variable z, as opposed to the static factors N, M_x, and M_y does not remain constant along the beam.

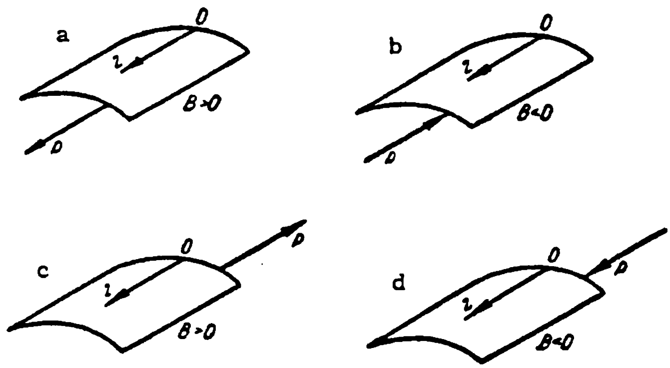


Figure 64

We based our theory of thin-walled beams on the hypothesis of invariability of the beam cross section. This hypothesis implies that any transverse load in the plane of the cross section can be replaced by a statically equivalent load in the same plane, since the cross section of the beam behaves like an absolutely rigid body.

The case of a longitudinal load applied at points of the cross section is a different affair. In fact, the cross section has four and not three degrees of freedom with respect to longitudinal displacements. The longitudinal displacements of the section do not obey the law of plane sections. The sections of the beam are warped. Each degree of freedom has a generalized longitudinal force corresponding to it. These generalized longitudinal forces, referred to the principal coordinates, are the tensile force N, the bending moments M_x and M_y and the bimoment B.

The generalized longitudinal force called the bimoment is actually a special system of longitudinal forces, applied at points of the cross section, i.e., a system of forces, which is statically equivalent to zero. For a beam of given profile, the bimoment in any section has a completely determined value depending on the external load and on the boundary conditions. Since the replacement of one longitudinal load applied at points of the cross section of the beam by another such load statically equivalent to it

changes the value of the bimoment in this section, it is, generally speaking, impossible to effect such a replacement. We can exchange one longitudinal load for another in the cross section of the beam only in such a way that the values of the four generalized longitudinal forces in this section remain unchanged. In particular, we can replace on longitudinal load in the cross section of a beam by another, statically equivalent to the first, if both loads are taken at the boundaries of a single bundle of plates, which constitute the given beam. It is easy to show that in this case the value of the bimoment in the examined cross section does not change since at the boundaries of a single bundle of plates, as we have shown earlier, the longitudinal displacements obey the law of plane sections.

We shall further show that it is necessary to distinguish clearly to what couple, longitudinal or transversal, the bending moment is due. We examine a thin-walled beam with an open cross section, subjected to the action of a concentrated bending moment at the end of the beam. The diagram of the principal sectorial areas for such a section is shown in Figure 36a. Let us assume that the moment bends the beam in a longitudinal plane passing through the shear center A and the zero sectorial point S_1. If this moment consists in a transverse couple (Figure 65a) there will obviously be no torsion, since the transverse forces lie in the plane which passes through the shear center. If this moment consists in a longitudinal couple applied at the points B and S_1 (Figure 65b), the bimoment $\overline{B}$ in the initial section is determined by the equation

$$\overline{B}=P\omega_B$$

and the beam will undergo torsion. Let, on the contrary, the plane of action of a concentrated bending moment pass through the zero sectorial points S_1 and S_2 (Figure 65c, d). In this case the beam will undergo torsion if the bending moment consists in a transverse couple, since the transverse forces do not pass through the shear center. If the bending moment consists in a longitudinal couple, applied at the points S_1 and S_2, there will be no torsion, since S_1 and S_2 are the zero sectorial points.

It is also necessary to remember, that if the bending moment consists in a transverse couple, the forces composing this couple can be transferred along their line of action to any point of the cross section of the beam, since the profile of the beam is considered rigid. If the bending moment consists in a longitudinal couple, the torsion of the beam will depend on the points of application of the component forces: it is impossible to arbitrarily replace one longitudinal couple by another statically equivalent to it. As we mentioned above, it is generally possible to replace an arbitrary longitudinal load by another, with the proviso that in the examined cross section the generalized longitudinal forces retain their previous values.

As before, we shall omit the cases of extension (compression) and bending according to the law of plane sections since there are sufficiently familiar from the theory of strength of materials, and shall deal only with the derivation of the equations for the determination of the complementary (sectorial) strains and stresses, which appear as a result of torsion in a beam under the action of an external longitudinal load.

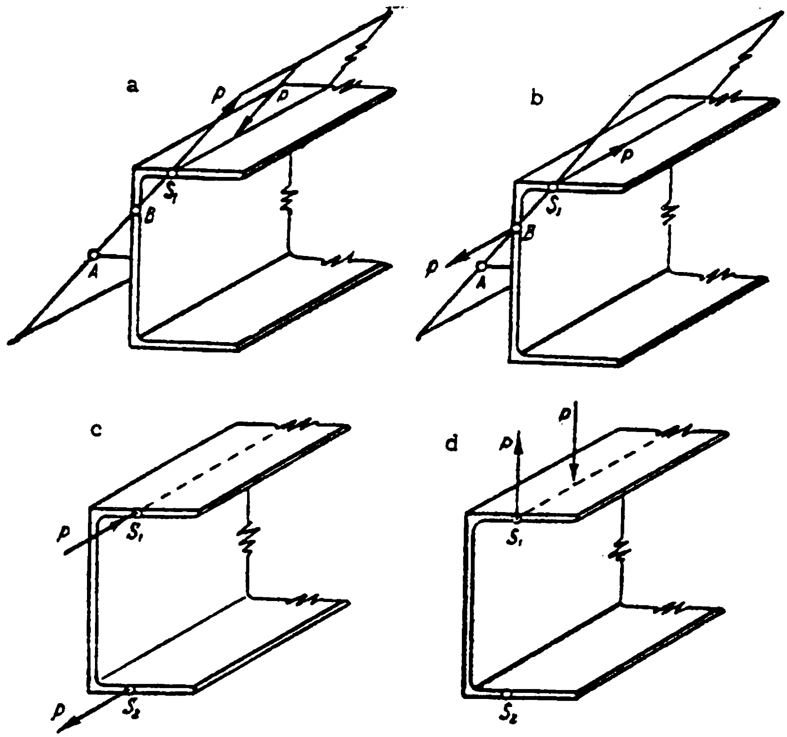


Figure 65

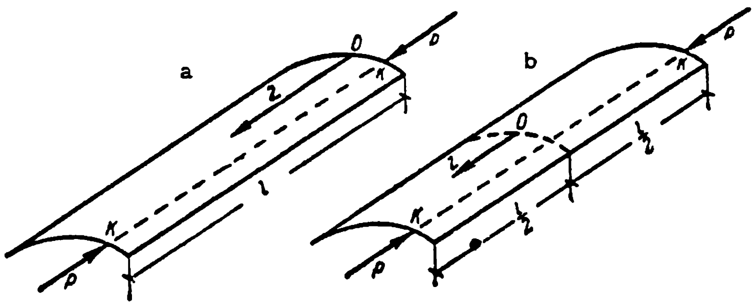


Figure 66

2. We shall examine a case where only flexure-torsion bimoments B are applied at the ends of the beam. We assume that the beam is subjected to the action of compressive forces P applied at its ends and directed along the line $K—K$ (Figure 66a). The bimoment B is determined in this case by the last of equations (6.3), which, according to our sign convention, will be (for both ends):

$$\overline{B} = -P\omega_k.$$

We shall assume that the initial section $z=0$ is constrained so as to

prevent torsion. The boundary conditions for the given loading can then be written:

$$\text{for } z=0 \quad \theta=0, \quad \overline{B}=-P\omega_k;$$

$$\text{for } z=l \quad H=0, \quad \overline{B}=-P\omega_k.$$

The initial parameters for these boundary conditions will be

$$\begin{aligned} &\theta_0=0,\\ &\theta_0'=-\frac{P\omega_k}{GJ_d}\frac{k}{l}\operatorname{th}\frac{k}{2},\\ &B_0=-P\omega_k,\\ &H_0=0, \end{aligned}$$

and the basic design quantities will be calculated from the equations

$$\left.\begin{aligned} &\theta(z)=\frac{P\omega_k}{GJ_d}\left(\operatorname{ch}\frac{k}{l}z-\operatorname{th}\frac{k}{2}\operatorname{sh}\frac{k}{l}z-1\right),\\ &\theta'(z)=\frac{P\omega_k}{GJ_d}\frac{k}{l}\left(\operatorname{sh}\frac{k}{l}z-\operatorname{th}\frac{k}{2}\operatorname{ch}\frac{k}{l}z\right),\\ &B(z)=-P\omega_k\left(\operatorname{ch}\frac{k}{l}z-\operatorname{th}\frac{k}{2}\operatorname{sh}\frac{k}{l}z\right),\\ &H(z)=0. \end{aligned}\right\} \tag{6.4}$$

Equations (6.4) are simplified if we place the origin of the coordinate z in the cross section at the middle of the beam (Figure 66b) and if we again assume that in this case the torsion angle vanishes at the end.

The boundary conditions in this case can be written

$$\text{for } z=0 \quad H=0, \quad \theta'=0;$$

$$\text{for } z=\frac{l}{2} \quad B=-P\omega_k, \quad \theta=0.$$

Equations (6.4), for these conditions, will be

$$\left.\begin{aligned} &\theta(z)=\frac{P\omega_k}{GJ_d}\frac{\operatorname{ch}\frac{k}{l}z-\operatorname{ch}\frac{k}{2}}{\operatorname{ch}\frac{k}{2}},\\ &\theta'(z)=\frac{P\omega_k}{GJ_d}\frac{k}{l}\frac{\operatorname{sh}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}},\\ &B(z)=-P\omega_k\frac{\operatorname{ch}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}},\\ &H(z)=0. \end{aligned}\right\} \tag{6.5}$$

For the normal stresses, which appear only under the action of a flexural-torsional bimoment we now obtain the following equation

$$\sigma_\omega=-\frac{P\omega_k}{J_\omega}\frac{\operatorname{ch}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}\omega(s). \tag{6.6}$$

These stresses, as seen from equation (6.6), depend not only on the position s of the point on the profile line of the contour but also on the position of the point on the generator (the coordinate z). The smallest stresses are obtained in the middle cross section for $z=0$. As we move farther away from this section on one side or the other, the stresses increase as $\operatorname{ch}\frac{k}{l}z$; for $z=\pm\frac{l}{2}$, i.e., in the extreme sections, the stresses reach the maximum values

$$\max \sigma_{\omega}=-\frac{P\omega_k}{J}\omega(s). \tag{6.7}$$

Equation (6.7) determines the sectorial diagram of stresses, which is statically equivalent to the given flexural-torsional bimoment $\bar{B}$.

We are thus in a position to deduce a very important result, viz.,

A beam is twisted not only, as in the case of pure torsion, by a transverse bending force that does not pass through the shear center, but also when subjected to terminal longitudinal forces only.

It will be shown below that this conclusion refers also to longitudinal forces and moments applied at any point of the middle surface of the beam, as well as outside it, when the load is passed on to the beam through a rigid strut.

The normal stresses which occur in the corss section of the beam under an arbitrary force does not as a rule obey the law of plane sections. In the case of longitudinal forces this law is valid only provided these forces are distributed over the section so that the flexural-torsional bimoment B due to them, as determined by the last of equations (6.1), (6.2), or (6.3), vanishes. Thus, for example, if the longitudinal load consists only of the force P, the flexural-torsional bimoment vanishes only if the force is applied at a contour point where the sectorial area $\omega(s)$ vanishes.

In the cross sections of the beam, in the general case of a longitudinal force (when $\bar{B}\neq 0$) tangential stresses $\tau(z, s)$ and torsional moments H_k also appear in addition to the normal stresses. The tangential axial shearing stresses $\tau(z, s)$ are determined by the general equation (I.8.9), whence

$$\tau_{\omega}(z, s)=-\frac{H_{\omega}(z)\,S_{\omega}(s)}{J_{\omega}\delta(s)}. \tag{6.8}$$

Here, $H_{\omega}(z)$ is the flexural-torsional moment, equal, according to the third of equations (I.8.10), to the derivative of the bimoment $B(z)$ with respect to z,

$$H_{\omega}(z)=\frac{dB(z)}{dz}=-P\omega_k\frac{k}{l}\,\frac{\operatorname{sh}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}. \tag{6.9}$$

Substituting (6.9) in (6.8), we obtain:

$$\tau_{\omega}(z, s)=\frac{P\omega_k}{J_{\omega}}\,\frac{k}{l}\,\frac{\operatorname{sh}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}\,\frac{S_{\omega}(s)}{\delta(s)}. \tag{6.10}$$

It is seen from equation (6.10) that the axial tangential stresses over the length of the beam vary as the odd function $\operatorname{sh}\frac{k}{l}z$. For $z=0$, i.e., in the middle cross section, these stresses are equal to zero. In the cross section the tangential stresses are proportional to the sectorial statical moment $S_{\omega}(s)$ calculated for the cut-off part of the section, and are inversely proportional to the thickness of the beam wall $\delta(s)$. The tangential stresses τ, multiplied by the thickness of the wall $\delta(s)$ give the shear forces $T=\tau\delta$ per unit arc-length of the section contour. These forces, T, give rise to a couple which acts in the plane of the cross section and is equal to the flexural-torsional moment

$$H_{\omega}(z)=\int\limits_{L}\tau\delta d\omega.$$

In addition to the flexural-torsional moments $H_{\omega}(z)$ in the cross sections of the beam, torsional moments H_k also appear, which can be calculated by equation (I.5.7). For these moments we obtain on the basis of equations (I.5.7) and (6.5)

$$H_k(z)=GJ_d\theta'(z)=P\omega_k\frac{k}{l}\frac{\operatorname{sh}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}. \tag{6.11}$$

It is easy to show that the general torsional moment $H(z)$, equal, according to equation (I.8.11), to the sum of the moments $H_{\omega}(z)$ and $H_k(z)$ vanishes when only a longitudinal force acts,

$$H(z)=H_{\omega}(z)+H_k(z)=0. \tag{6.12}$$

Equation (6.12), obtained from (6.9) and (6.11), expresses one of the equilibrium conditions of the part of the beam which lies to one side of the section.

3. As an example, we examine the asymmetrical section, shown in Figure 67. The principal axes are denoted by x and y. The auxiliary central axes are denoted by ξ and η; the angle between the central and the principal axes is $\varphi=21°42'$.

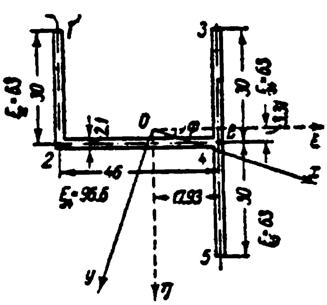


Figure 67

The area of the section and the principal moments of inertia have the following values:

$$F=285.6\text{cm}^2,$$
$$J_x=43\,040\text{cm}^4,$$
$$J_y=120\,300\text{cm}^4.$$

In order to determine the coordinates of the shear center we choose point 4 as an auxiliary pole B. This point lies at the intersection of the axis of the horizontal element with the axis of the right vertical wall. We construct the diagram of the secotrial areas ω_B with respect to this pole (Figure 68a). Calculating the integrals of the products of diagram ω_B and x and also of ω_B and y

(Figure 68b and c) throughout the cross section, we obtain

$$J_{\omega x} = \int_F \omega_B x dF = -1\,402\,000 \text{cm}^5,$$
$$J_{\omega y} = \int_F \omega_B y dF = -223\,000 \text{cm}^5.$$

From equations (I.7.5) for the coordinates of the shear center, evaluated relative to the pole B along the principal axes, we obtain

$$a_x = \frac{J_{\omega y}}{J_x} = -5.18 \text{cm}, \quad a_y = -\frac{J_{\omega x}}{J_y} = 11.66 \text{cm}.$$

Since in the principal axes the auxiliary pole B has the coordinates (Figure 68)

$$b_x = 17.88 \text{cm}, \quad b_y = -3.55 \text{cm},$$

the coordinates of the shear center A along the principal axes will be

$$a_x = 12.70 \text{cm}, \quad a_y = 8.11 \text{cm}.$$

With these data it is easy to compute the distance from the shear center to the axes of the horizontal and the vertical elements on the right of the cross section. They will equal 8.93 cm and 9.13 cm respectively (Figure 69a).

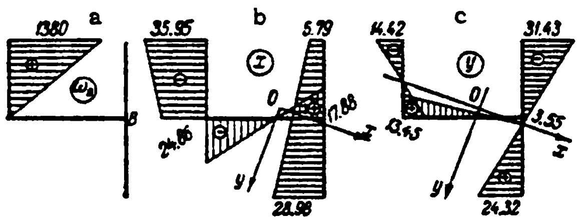


Figure 68

Placing the pole in the bending center, A, the distance of the origin of the sectorial areas M_0 from the left vertical axis being t (Figure 69a), we construct the diagram of the sectorial areas and from the condition $S_\omega = \int_F \omega dF = 0$ we find $t = 41.73$ cm.

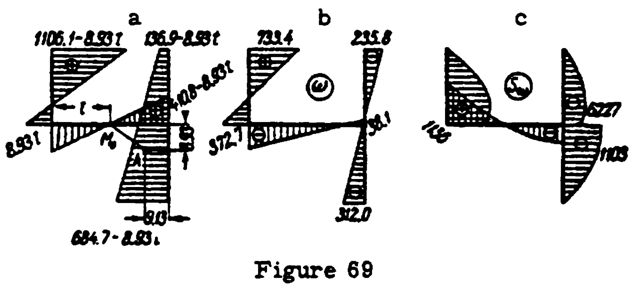


Figure 69

Figure 69b gives the diagram of the principal sectorial area ω, which is constructed after determining the position of the sectorial origin M_φ. Figure 69c gives the diagram of the sectorial statical moments S_ω. From the diagram of ω we calculate the sectorial moment of inertia. The torsional rigidty is determined by equation (I.5.8); for $\alpha = 1$, we find

$$J_d = \sum \frac{d\delta^3}{3} = 419.8\text{cm}^4.$$

It is seen from the diagram of the sectorial areas (Figure 69b) that the distribution of the stresses σ over the section is linear only in the particular case when the longitudinal force P is applied at one of the three points of the profile line where the sectorial area ω vanishes. For any other position of the force P complementary (sectorial) stresses associated with the warping of the section appear in the cross sections. These stresses, for the boundary conditions given in subsection 2 of the present section for a beam with the origin of z in the middle cross section, are determined by equation (6.6) for an arbitrary point of the cross section at a distance z from the middle cross section. We now write this equation in the form

$$\sigma_\omega = -\frac{P\omega_k}{J_\omega}\omega(s)\,\varphi(k, z),$$

where

$$\varphi(k, z) = \frac{\operatorname{ch}\frac{k}{l}z}{\operatorname{ch}\frac{k}{2}}. \tag{6.13}$$

It is plain from (6.13) that the function $\varphi(k, z)$ depends on the position of the point along the beam and on the elastic constant k which can be calculated from equation (2.2). In the present case, since we examine half of the beam (the coordinate z is taken from the middle section), we should understand l to be half the length of the beam when we calculate k.

The function $\varphi(k, z)$ characterizes the variation of the sectorial deformations along the beam. Figure 70 shows the graphs of the function $\varphi(k, \alpha)$, plotted for various lengths of the beam with a Poisson ratio $\mu = \frac{G}{E} = 0.4$. In these graphs the abscissa is the reduced distance $\alpha = \frac{z}{l/2}$ expressed as a part of half the length of the beam and measured from the middle cross section. The ordinate gives $\varphi(k, \alpha)$.

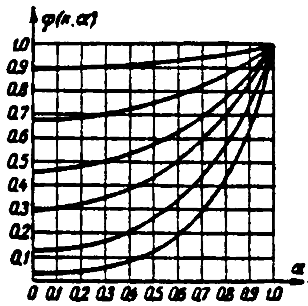


Figure 70

The graphs show that the sectorial deformations of the section and consequently also the complementary stresses σ_ω decrease as we move from the ends of the beam toward the middle. The rate of decrease for the given dimensions of the beam cross section and a given Poisson ratio μ depends on the length of the beam l. For longer beams the sectorial normal stresses are much smaller in the middle part. For very long beams these stresses have a local character, and are consequently in agreement with Saint-Venant's principle. For $k \to \infty$, i. e., for an infinitely long beam, the sectorial normal stresses

are everywhere equal to zero, excluding the ends of the beam for $\alpha = \pm 1$ or $z = \pm \frac{l}{2}$ where the function φ equals unity. Indeed, expressing the hyperbolic cosine by the well-known Euler formula, we may write the function φ in the form

$$\varphi = \frac{\operatorname{ch} \frac{k}{l} z}{\operatorname{ch} \frac{k}{2}} = \frac{\operatorname{ch} \frac{k}{2} \alpha}{\operatorname{ch} \frac{k}{2}} = \frac{e^{\frac{k}{2}\alpha} + e^{-\frac{k}{2}\alpha}}{e^{\frac{k}{2}} + e^{-\frac{k}{2}}} = e^{\frac{k}{2}(\alpha - 1)} \frac{1 + e^{-k\alpha}}{1 + e^{-k}}.$$

It is seen from this equation that for all the values of α, included between zero and one (or, which is the same, for all the values of z between zero and $\frac{l}{2}$), the function φ vanishes as $k \to \infty$. For $\alpha = 1$ the function φ equals unity regardless of the value of k.

Passing on to the calculation of the diagrams of the normal stresses σ over the cross section, we examine the case where the compressive force is applied at the point 2 (Figure 71). The flexural-torsional bimoment $\bar{B}$ is here calculated by the equation

$$\bar{B} = -P\omega_2.$$

Assuming in this equation $P = 50$ ton and noting that for the point $\omega_2 = -3.72.7$ cm², we obtain:

$$B = 50\,000 \cdot 372.7 = 18.635 \cdot 10^6 \text{kg-cm}^2, \qquad (6.14)$$

$$\sigma_\omega = \varphi(k, \alpha) \frac{B}{J_\omega} \omega(s) = \varphi(k, \alpha) \frac{18.635 \cdot 10^6}{15.868 \cdot 10^6} \omega(s) = 1.174\omega(s)\varphi(k, \alpha).$$

By adding the stresses determined by equation (6.14) to the stresses obtained by the usual equations of the theory of strength of materials for an axial compressive force $N = -P = -50$ ton and for bending moments $M_x = Py_2$ and $M_y = Px_2$, we obtain

$$\sigma_0 = \frac{N}{F} = -\frac{50\,000}{285.6} = -175.1 \frac{\text{kg}}{\text{cm}^2},$$

$$\sigma_x = -\frac{M_y}{J_y} x_k = 10.334 x_k,$$

$$\sigma_y = \frac{M_x}{J_x} y_k = -15.637 y_k.$$

The normal stresses calculated from the three-term equation known from strength of materials: $\sigma = \sigma_0 + \sigma_x + \sigma_y$, for the points 1, 2, 3, 4, and 5 of the cross section, are given in Table 15 below. These stresses depend only on the position of the point on the profile line and do not vary over the length of the beam.

The complementary (secotrial) normal stresses σ_ω determined by equation (6.14) for a beam with $l = 900$ cm, for six values of $\alpha = \frac{2z}{l}$ are given in Table 16. These stresses are functions of the two variables α and s. The variable α determines the position of the point along the beam, and its influence is defined by the function $\varphi(k, \alpha)$ which has the

following values for $k = 1.464$ and for the given values of α:

$$\varphi(k;\ 0) = 0.43917,$$
$$\varphi(k;\ 0.2) = 0.4581,$$
$$\varphi(k;\ 0.4) = 0.5167,$$
$$\varphi(k;\ 0.6) = 0\ 6196,$$
$$\varphi(k;\ 0.8) = 0.7762,$$
$$\varphi(k;\ 1) = 1.$$

The variable s determines the position of the point on the cross section of the middle surface of the given beam, and its influence is determined by the sectorial coordinates ω_k.

Table 15

Points	1	2	3	4	5
$\sigma\ \frac{kg}{cm^2}$	-321.1	-642.5	$+386.6$	$+65.2$	-255.9

Table 16

α	Stress σ_ω at the points				
	1	2	3	4	5
0	$+378.1$	-192.2	-121.6	$+19.7$	$+160.9$
0.2	$+394.5$	-200.5	-126.9	$+20.5$	$+167.8$
0.4	$+445.0$	-226.1	-143.1	$+23.1$	$+189.3$
0.6	$+533.6$	-271.1	-171.5	$+27.7$	$+226.9$
0.8	$+668.4$	-339.7	-214.9	$+34.7$	$+284.4$
1.0	$+861.2$	-437.6	-276.9	$+44.7$	$+366.4$

By adding for the sections the stresses given in Table 15 to the complementary stresses of Table 16, we obtain the total stresses σ due to the compressive force $P = 50$ ton applied at point 2..

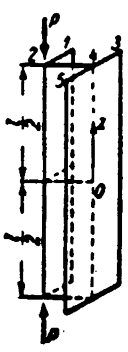
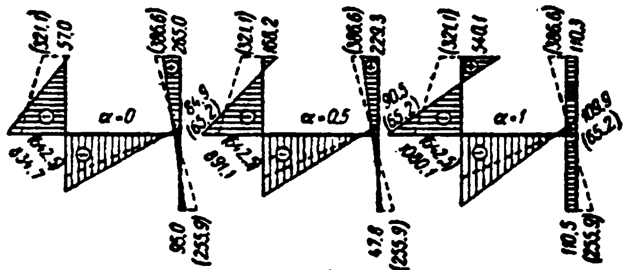


Figure 71

The values of the total stresses for the characteristic points of the cross sections are given in Table 17. The diagrams of the stresses for the sections $\alpha = 0,\ 0.5,\ 1.0$, corresponding to the data of Table 17, are given in Figure 72a, b, c. For comparison, the stresses taken from Table 15 and determined by the usual theory of the strength of materials are indicated by dotted lines in these diagrams. The complementary stresses σ_ω reach, as we see, very large values in this case as well.

When the beam is eccentrically compressed, tangential stresses appear, in addition to the normal stresses, as a result of sectorial warping. These stresses are composed of axial stresses, caused by shear forces which act in the direction of the tangent to the profile line and of stresses leading

at each point of the contour to a twisting couple. The axial shear stresses are determined by equation (6.8), where the flexural-torsional moment $H_\omega(z)$ characterizes the variation of the stresses τ_ω over the length of the beam,



a b c

Figure 72

and the sectorial static moment $S_\omega(s)$ and the thickness of the beam wall $\delta(s)$ characterize the variation of the stresses over the cross section. For a constant value of $\delta(s)$ (as in our example) the variation of the stresses τ_ω over the cross section is determined only by the function $S_\omega(s)$. This moment is calculated for an arbitrary point of the section by equation (I.6.11):

$$S_\omega(s) = \int_{F(s)} \omega \, dF. \tag{6.15}$$

The integration in equation (6.15) extends over the cut-off part of the section, lying to one side of the given point on the contour. The diagram of the sectorial static moments is given in Figure 69c.

Table 17

α	Stress σ at the point				
	1	2	3	4	5
0	+57.0	−834.7	+265.0	+84.9	−95.0
0.2	+73.4	−843.0	+259.7	+85.7	−81.9
0.4	+123.9	−868.6	+243.5	+88.3	−66.6
0.6	+212.5	−913.6	+215.1	+92.9	−29.0
0.8	+347.3	−982.2	+171.7	+99.9	+28.5
1.0	+540.1	−1080.1	+110.3	+109.9	+110.5

The tangential stresses, as evident from equation (6.10), vary over the length of the beam as the odd function $\operatorname{sh}\frac{k}{l}z$, vanishing at the middle of the beam (for $z = 0$), and becoming maximal at the ends $\left(z = \pm\frac{l}{2}\right)$.

Setting $z = \frac{l}{2}$ in equation (6.10), we obtain

$$\tau_\omega = \frac{P\omega_k}{J_\omega \delta}\frac{k}{l} S_\omega(s) \operatorname{th}\frac{k}{2}.$$

As shown above, the axial tangential stresses give rise in any cross section to the moment $H_\omega(z)$ which we call the flexural-torsional moment, as opposed to the torsional moment $H_k(z)$ for pure torsion. In addition to $H_\omega(z)$ a torsional moment $H_k(z)$ also appears in the section of the beam, due

to the nonuniformity of the tangential stresses over the thickness of the wall as calculated by equation (6.11). Figure 73a shows the graphs of the variation of the moments $H_k(z)$ along the beam.

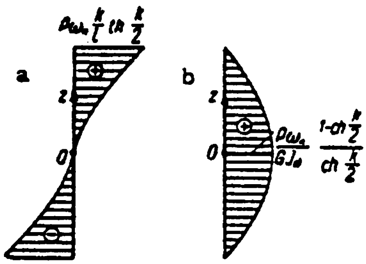


Figure 73

As indicated earlier, a thin-walled beam subjected to an eccentric longitudinal force will undergo a torsional deformation in addition to the axial elongations and the bending deformation. The torsion angle in the case examined (for $z = 0$ in the middle of the beam) is determined from the first of equations (6.5). The graph of the variation of the angle θ over the length of the beam is given in Figure 73b.

§ 7. Torsion of a beam subjected to longitudinal shear force applied at an arbitrary point

1. We shall examine a thin-walled beam subjected to the action of a longitudinal shear force, applied at an arbitrary point of the middle surface of the beam. This force causes, in addition to axial tensile (compressive) and bending stresses, also complementary torsional stresses. Let us suppose that the longitudinal force P, which points along the positive direction of the z-axis, is applied at an arbitrary point $z = t$, $s = k$, of the middle surface of the beam (Figure 74). This force will correspond to an external bimoment $\overline{B} = P\omega_k$ where ω_k is the sectorial coordinate at the point of application of the force P.

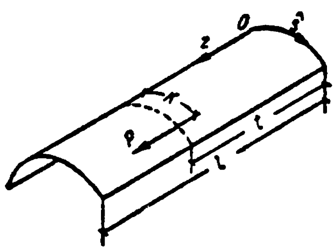


Figure 74

Since an external longitudinal force P is applied at the point $z=t$, $s=k$, producing an external concentrated bimoment $\bar{B}=P\omega_k$, the value of the initial factor B_t for a positive direction of the force P and when $\omega_k>0$ will have an opposite sign to $\bar{B}$ for $z>t$ (see Figure 64d). It is, therefore, necessary to take $B_t=-\bar{B}=-P\omega_k$. In this case the formulas in Table 7 will be expanded as follows:

Table 18

	θ_0	θ'_0	$\frac{1}{GJ_d}B_0$	$\frac{1}{GJ_d}H_0$	$\frac{1}{GJ_d}B_t=-\frac{1}{GJ_d}P\omega_k$
$\theta(z)$	1	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$z-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}(z-t)$
$\theta'(z)$	0	$\operatorname{ch}\frac{k}{l}z$	$-\frac{k}{l}\operatorname{sh}\frac{k}{l}z$	$1-\operatorname{ch}\frac{k}{l}z$	$-\frac{k}{l}\operatorname{sh}\frac{k}{l}(z-t)$
$\frac{1}{GJ_d}B(z)$	0	$-\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$\operatorname{ch}\frac{k}{l}z$	$\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$\operatorname{ch}\frac{k}{l}(z-t)$
$\frac{1}{GJ_d}H(z)$	0	0	0	1	0

These formulas are valid for the part of the beam situated beyond the section $z=t$, i.e., for $z\geqslant t$. For the part of the beam situated on the near side of the section $z=t$, i.e., for $z<t$, it is necessary to retain in the formulas only terms which contain the initial parameters θ_0, θ'_0, B_0 and H_0, discarding terms which contain the external load $\bar{B}=P\omega_k$.

The formulas of Table 18 are valid not only for the case where the external bimoment appears as a result of a concentrated force P only, but also of the action of several forces, concentrated or distributed arbitrarily over the section $z=t$. In those latter cases $\bar{B}$ is calculated by the corresponding equations (6.1) and (6.2).

The initial parameters θ_0, θ'_0, B_0 and H_0 are determined from the boundary conditions. As for the case of transverse forces, we examine particular cases of boundary conditions.

a) Beam hinged at both ends. In this case the boundary conditions will be

$$\text{for } z=0 \quad \theta=0, \quad B=0;$$
$$\text{for } z=l \quad \theta=0, \quad B=0.$$

The initial parameters, determined from these boundary conditions,

with the help of Table 18, will have the following values:

$$\left.\begin{aligned}
&\theta_0 = 0,\\
&\theta'_0 = \frac{\overline{B}}{GJ_d}\frac{1}{l}\left[1 - k\frac{\operatorname{ch}\frac{k}{l}(l-t)}{\operatorname{sh} k}\right],\\
&B_0 = 0,\\
&H_0 = \overline{B}\frac{1}{l}.
\end{aligned}\right\} \tag{7.1}$$

The basic design factors will be calculated by the following equations:
For the part $0 \leqslant z < t$

$$\left.\begin{aligned}
&\theta(z) = \theta'_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z + \frac{H_0}{GJ_d}\left(z - \frac{l}{k}\operatorname{sh}\frac{k}{l}z\right),\\
&\theta'(z) = \theta'_0\operatorname{ch}\frac{k}{l}z + \frac{H_0}{GJ_d}\left(1 - \operatorname{ch}\frac{k}{l}z\right),\\
&B(z) = -GJ_d\theta'_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z + H_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z,\\
&H(z) = H_0.
\end{aligned}\right\} \tag{7.2}$$

For the part $t \leqslant z \leqslant l$

$$\left.\begin{aligned}
&\theta(z) = \theta'_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z + \frac{H_0}{GJ_d}\left(z - \frac{l}{k}\operatorname{sh}\frac{k}{l}z\right) - \frac{P\omega_k}{GJ_d}\left[1 - \operatorname{ch}\frac{k}{l}(z-t)\right],\\
&\theta'(z) = \theta'_0\operatorname{ch}\frac{k}{l}z + \frac{H_0}{GJ_d}\left(1 - \operatorname{ch}\frac{k}{l}z\right) + \frac{P\omega_k}{GJ_d}\frac{k}{l}\operatorname{sh}\frac{k}{l}(z-t),\\
&B(z) = -GJ_d\theta'_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z + H_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z - P\omega_k\operatorname{ch}\frac{k}{l}(z-t),\\
&H(z) = H_0.
\end{aligned}\right\} \tag{7.3}$$

The initial parameters θ'_0 and H_0 are determined from (7.1).

b) Beam clamped at both ends. The boundary conditions for this case will be:

$$\text{for } z = 0 \quad \theta = 0,\quad \theta' = 0;$$

$$\text{for } z = l \quad \theta = 0,\quad \theta' = 0.$$

The initial parameters which satisfy these conditions are determined by the equations:

$$\left.\begin{aligned}
&\theta_0 = 0,\\
&\theta'_0 = 0,\\
&B = \overline{B}\,\frac{\operatorname{ch}\frac{k}{l}t + k\operatorname{sh}\frac{k}{l}(l-t) - \operatorname{ch}\frac{k}{l}(l-t) - \operatorname{ch} k + 1}{2 + k\operatorname{sh} k - 2\operatorname{ch} k},\\
&H_0 = -\overline{B}\,\frac{\frac{k}{l}\left[\operatorname{sh}\frac{k}{l}(l-t) + \operatorname{sh}\frac{k}{l}t - \operatorname{sh} k\right]}{2 + k\operatorname{sh} k - 2\operatorname{ch} k}.
\end{aligned}\right\} \tag{7.4}$$

Using Table 18 and (7.4) we derive the equations necessary for calculating the basic design factors:

For the part $0 \leq z < t$

$$\left.\begin{aligned}
\theta(z) &= \frac{B_0}{GJ_d}\left(1-\mathrm{ch}\frac{k}{l}z\right)+\frac{H_0}{GJ_d}\left(z-\frac{l}{k}\,\mathrm{sh}\frac{k}{l}z\right),\\
\theta'(z) &= -\frac{B_0}{GJ_d}\frac{k}{l}\,\mathrm{sh}\frac{k}{l}z+\frac{H_0}{GJ_d}\left(1-\mathrm{ch}\frac{k}{l}z\right),\\
B(z) &= B_0\,\mathrm{ch}\frac{k}{l}z+H_0\frac{l}{k}\,\mathrm{sh}\frac{k}{l}z,\\
H(z) &= H_0.
\end{aligned}\right\} \qquad (7.5)$$

For the part $t \leq z \leq l$

$$\left.\begin{aligned}
\theta(z) &= \frac{B_0}{GJ_d}\left(1-\mathrm{ch}\frac{k}{l}z\right)+\frac{H_0}{GJ_d}\left(z-\frac{l}{k}\,\mathrm{sh}\frac{k}{l}z\right)-\\
&\qquad -\frac{P\omega_k}{GJ_d}\left[1-\mathrm{ch}\frac{k}{l}(z-t)\right],\\
\theta'(z) &= -\frac{B_0}{GJ_d}\frac{k}{l}\,\mathrm{sh}\frac{k}{l}z+\frac{H_0}{GJ_d}\left(1-\mathrm{ch}\frac{k}{l}z\right)+\\
&\qquad +\frac{P\omega_k}{GJ_d}\frac{k}{l}\,\mathrm{sh}\frac{k}{l}(z-t),\\
B(z) &= B_0\,\mathrm{ch}\frac{k}{l}z+H_0\frac{l}{k}\,\mathrm{sh}\frac{k}{l}z-P\omega_k\,\mathrm{ch}\frac{k}{l}(z-t),\\
H(z) &= H_0.
\end{aligned}\right\} \qquad (7.6)$$

c) Beam with one end clamped and the other hinged. We assume that the end $z=0$ is fixed and, consequently,

$$\text{for } z=0 \quad \theta_0=0, \quad \theta_0'=0.$$

The boundary conditions for the end $z=l$ will be

$$\text{for } z=l \quad \theta=0, \quad B=0. \qquad (7.7)$$

The remaining two initial parameters B_0 and H_0, are determined from the conditions (7.7) and will have the values:

$$\left.\begin{aligned}
B_0 &= \bar{B} = \frac{\frac{1}{k}\,\mathrm{sh}\,k-\mathrm{ch}\frac{k}{l}(l-t)}{\frac{1}{k}\,\mathrm{sh}\,k-\mathrm{ch}\,k},\\
H_0 &= \bar{B}\,\frac{\mathrm{ch}\frac{k}{l}(l-t)-\mathrm{ch}\,k}{\frac{l}{k}\,\mathrm{sh}\,k-l\,\mathrm{ch}\,k}.
\end{aligned}\right\} \qquad (7.8)$$

The equations for determining the basic design factors are the same as for the case of the previous boundary conditions. Their initial parameters B_0 and H_0 should be determined by equations (7.8).

d) Beam clamped at one end and having the other end free (cantilever). We assume that the beam is fixed at the end $z=0$; consequently, as in the previous case:

$$\text{for } z=0 \quad \theta(0)=0, \quad \theta'(0)=0.$$

The boundary conditions at the other end are:

$$\text{for } z=l \quad B(l)=0, \quad H(l)=\bar{0}.$$

The initial parameters have the following values:

$$\left.\begin{aligned} &\theta_0=0,\\ &\theta_0'=0,\\ &B_0=\bar{B}\frac{\operatorname{ch}\frac{k}{l}(l-t)}{\operatorname{ch}k},\\ &H_0=0. \end{aligned}\right\} \tag{7.9}$$

By introducing the values found for the initial parameters (7.9) into the relations (7.5) and (7.6), we obtain the equations for determining the basic design factors for this form of boundary conditions as well.

2. The formulas derived above for the flexural-torsional factors due to a concentrated longitudinal force applied at an arbitrary point of the middle surface of the beam, allow the determination, based on the law of the independence of action [superposition] of forces, of the torsion angle, the warping, and the sectorial static factors due to a surface longitudinal load, which can be either concentrated or continuously distributed over the middle surface of the beam.

The equations derived in this section also permit the determination of the sectorial displacements and the forces due to a concentrated bending moment, which acts in the plane tangent to the middle surface of the beam.

Suppose a concentrated moment M, consisting in a longitudinal couple P with the forces a distance Δs apart, is applied at the point $z=t$, $s=k$, on the tangent plane to the middle surface of the beam (Figure 75). The quantities M, P, and Δs are related by

$$P=\frac{M}{\Delta s}. \tag{7.10}$$

From the preceding subsection we can see that in their final form the flexural-torsional factors due to the concentrated external bimoment $\bar{B}=P\omega_k$ are, after determining the values of the initial parameters from the boundary conditions, proportional to this bimoment. Therefore, we may also express them by

$$\left.\begin{aligned} \theta(z)&=\theta_1(z)P\omega_k,\\ \theta'(z)&=\theta_1'(z)P\omega_k,\\ B(z)&=B_1(z)P\omega_k,\\ H(z)&=H_1(z)P\omega_k, \end{aligned}\right\} \tag{7.11}$$

where $\theta_1(z)$, $\theta_1'(z)$, $B_1(z)$ and $H_1(z)$ denote the same flexural-torsional factors, which are due, however, to the action of a unit bimoment applied at the point $z=t$, $s=k$, and ω_k denotes the sectorial coordinate at the point K of the cross section.

Let $\Phi(z)$ denote any of the flexural-torsional factors $\theta(z)$, $\theta'(z)$, $B(z)$

and $H(z)$; then equations (7.11) may be written

$$\Phi(z) = \Phi_1(z) P\omega_k.$$

At a point situated at a distance Δs from K, the sectorial coordinate will have increased. We denote this increment by $\Delta\omega_k$.

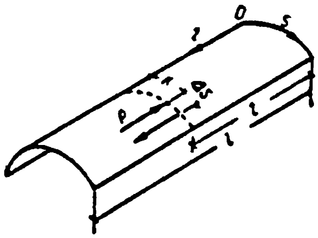


Figure 75

Consequently, taking the formulas of Table 18 as the influence functions, we may write for the factor $\Phi(z)$, due to the action of a couple (Figure 75),

$$\Phi_M(z) = \Phi_1(z)[-P\omega_k + P(\omega_k + \Delta\omega_k)] = \Phi_1(z) P\Delta\omega_k.$$

Substituting P from equation (7.10), and letting $\Delta s \to 0$, we obtain

$$\Phi_M(z) = \Phi_1(z) M \left(\frac{\Delta\omega_k}{\Delta s}\right)_{\Delta s \to 0} = \Phi_1(z) M \left(\frac{d\omega}{ds}\right)_{s=k}.$$

Thus, the expressions for the flexural-torsional factors, due to the action of a concentrated bending moment M in the tangent plane at the point $z=t$, $s=k$, become:

$$\theta(z) = \theta_1(z) M \left(\frac{d\omega}{ds}\right)_{s=k},$$

$$\theta'(z) = \theta'_1(z) M \left(\frac{d\omega}{ds}\right)_{s=k},$$

$$B(z) = B_1(z) M \left(\frac{d\omega}{ds}\right)_{s=k},$$

$$H(z) = H_1(z) M \left(\frac{d\omega}{ds}\right)_{s=k}.$$

It is seen from these expressions that the flexural-torsional factors, due to the action of a concentrated bending moment M lying in the plane tangent to the middle surface at the point $z=t$, $s=k$, are expressed by the same equations as in the case of the action of a concentrated longitudinal force P applied at the same point $z=t$, $s=k$, if $M\left(\frac{d\omega}{ds}\right)_k$ replaces the factor $P\omega_k$, i.e., they are proportional to the value of the derivative of the sectorial coordinate ω with respect to the variable s at the point K.

If the moment M is applied at a zero point of the derivative $\frac{d\omega}{ds}$, all the flexural-torsional factors vanish. From geometrical considerations and from the method of obtaining the principal secotrial area ω it follows that the tangent plane to the middle surface of the beam at the point where $\frac{d\omega}{ds}=0$ passes through the shear center. Thus, at the point where $\frac{d\omega}{ds}=0$ the beam will be in a condition of central transverse bending. We already reached this conclusion when studying the problem of the action of a bending moment, which was replaced by a transverse couple (see Figure 62b*).

* A. and L. Föppl, in their book "Force and deformation" [Russian version of "Zwang und Drang"] in the chapter dealing with torsion of thin-walled beams (v II, p 130), state: "In the case of pure bending the plane of action of the external forces can be shifted parallel to itself without changing the distribution of the stresses in the beam". This statement, as we see, is not correct. The stresses in a thin-walled beam under pure bending depend not only on the orientation of the plane of action of the moment but also on its position and on how this moment arises.

Figures 32, 36a, and 39a show the diagrams of the sectorial areas for I-, Channel, and Z-sections. It is easy to obtain thence the $\frac{d\omega}{ds}$ diagram for their contours. These are shown in Figure 76a, b, and c.

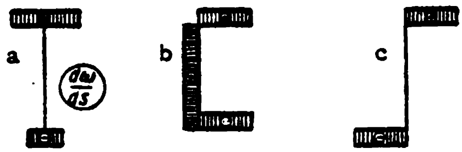


Figure 76

From these diagrams we can conclude that the moments, applied in the plane of the walls of an I- and Z-section, produce only a transverse bending and there is no torsional effect, since the planes of the walls pass through the shear center. The moments which are applied in the planes of the flanges of all three sections, and also in the plane of the web of the channel beam, cause, in addition to transverse bending, also flexural-torsional stresses. Since the diagrams $\frac{d\omega}{ds}$ are constant for each element, the point of application of the moment in each element is arbitrary.

We note that all our considerations refer to the case of a bending moment acting at a point of the cross section of the beam. In this case we may consider the bending moment as a longitudinal couple, and, similarly, its static equivalent as a transverse couple, since they are applied to an infinitely small element $ds\,dz$ of the middle surface of the beam within the boundaries of a single plate.

3. Finally, we examine the case of the longitudinal force applied outside the section of the beam and transmitted to it through a rigid arm fixed at some point D on the contour (Figure 77a).

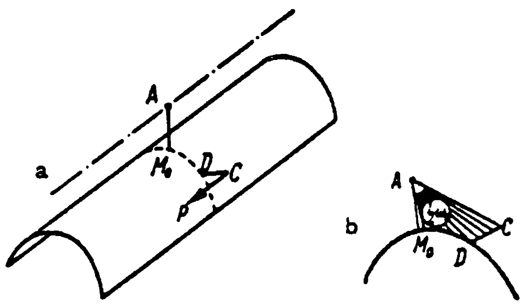


Figure 77

We shall prove the following theorem: *If a rigid arm CD is fixed at some point D, at a right angle to the generator of a thin-walled beam whose cross section has a nondeformable contour and does not undergo shear*

deformations, the longitudinal displacements at the free end of the arm (the point C), in the most general deformation of the beam, consist of deformations determined by the law of plane sections and of a displacement which is proportional to the sectorial area ω_C. The line M_0DC (M_0 is the origin of the principal sectorial areas) serves as the basis for this sectorial area, and its pole is situated in the shear center, A (Figure 77b).

The longitudinal displacement of the end of the arm (the pointt C, Figure 78a) is composed of the longitudinal displacement u_D of the base of the the arm (the point D) and the relative longitudinal displacement u_{CD} of the point C with respect to the point D, which appears as a result of rotation of the arm CD, with respect to the line nn, in the plane P of the cross section t=const and which passes through the point D at right angles to the arm CD:

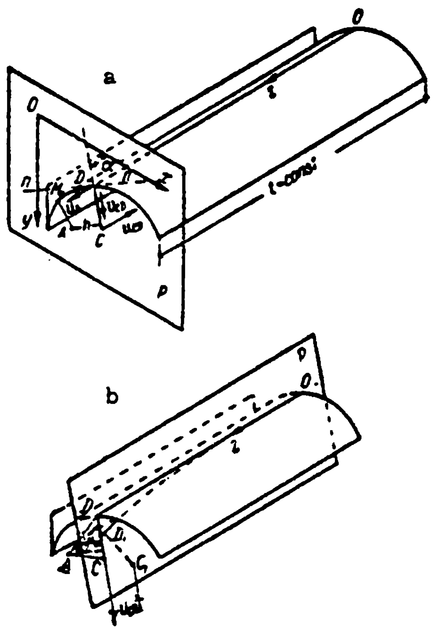


Figure 78

$$u_C = u_D + u_{CD}. \qquad (7.12)$$

Since the point D lies on the middle surface of the beam, u_D is determined by equation (I.3.16)

$$u_D = \zeta - \xi' x_D - \eta' y_D - \theta' \omega_D. \quad (7.13)$$

where x_D, y_D and ω_D are the principal linear and sectorial coordinates of the point D.

By denoting the angle formed by the direction of the arm CD with the axis x by α, and the length of the perpendicular from the shear center to the direction of the arm by h, we can determine from (I.3.9) the projection of the displacement of point D on the direction of the arm:

$$v_D = \xi \cos\alpha + \eta \sin\alpha + \theta h. \quad (7.14)$$

We draw a plane P through the generator of the beam, passing through the point D, and the arm CD (Figure 78b). The projection on the plane P of the generator of the beam, passing through the point D, after the deformation of the beam, is indicated by a dotted line in the figure. Since according to the given condition the arm CD is rigidly fixed to the contour at a right angle to the generator, it will rotate in the longitudinal plane after the deformation through a certain angle, whose tangent is given, up to second order terms, by the derivative of the displacement v_D with respect to z

$$\frac{v_D(z+dz) - v_D(z)}{dz} = \frac{dv_D}{dz} = v_D'.$$

Multiplying ϑ'_D by the length of the arm CD (denoted by S_{CD}) we obtain the value of the longitudinal displacement u_{CD} of the point C as as result of the indicated rotation of the arm CD after the deformation

$$u_{CD} = -\vartheta'_D S_{CD}. \tag{7.15}$$

The minus sign in equation (7.15) is easily explained from Figure 78b.

Introducing expression (7.14) in (7.15), differentiating once with respect to z, and noting that

$$S_{CD}\cos\alpha = x_C - x_D, \quad S_{CD}\sin\alpha = y_C - y_D, \quad S_{CD}h = \omega_{CD},$$

we obtain

$$u_{CD} = -\xi'(x_C - x_D) - \eta'(y_C - y_D) - \theta'\omega_{CD}, \tag{7.16}$$

where x_C and y_C are the principal linear coordinates of the point C, and ω_{CD} is twice the area of the triangle whose base is the length of the arm CD and whose height passes through the shear center.

Introducing expressions (7.13) and (7.16) in equation (7.12) and noting that

$$\omega_D + \omega_{CD} = \omega_C,$$

we obtain

$$u_C = \zeta - \xi' x_C - \eta' y_C - \theta'\omega_C, \tag{7.17}$$

which proves the theorem.

It is now possible to extend the concept of the generalized external forces $\overline{N}$, $\overline{M}_x$, $\overline{M}_y$ and $\overline{B}$, which are the equivalent of the given force P in the sense of the virtual work done in the displacements from the plane of the section $z =$ const, to embrace the case where the force is applied outside the contour of the cross section and transmitted to the section through a rigid arm. For these generalized forces, the equations will be completely analogous to equations (6.3):

$$\left.\begin{aligned} \overline{N} &= P\cdot 1, \\ \overline{M}_x &= Py_C, \\ \overline{M}_y &= -Px_C, \\ \overline{B} &= P\omega_C, \end{aligned}\right\} \tag{7.18}$$

We can directly obtain the last of equations (7.18) from the following theorem: *If a longitudinal force applied outside the beam section is transmitted to the section through a rigid arm fixed at the contour point D (Figure 79), and if this force lies in a cross-sectional plane, the force will cause a bimoment equal to the product of the force P and twice the sectorial area AM_0DC, bounded by the two radius-vectors drawn from the shear center A to the origin of the contour M_0 and to the point C where the force P is applied, and by the section contour and the axis of the arm.*

Let e stand for the length of the arm. We transfer the force P to the point D. In addition to the force P, we should then apply a moment Pe at the point D. It is possible, in this case, to replace the longitudinal

force P by statically equivalent quantities, since we are at the limits of the rigid arm (the longitudinal displacements of the arm obey the law of plane sections). The force P, applied at a point D of the profile

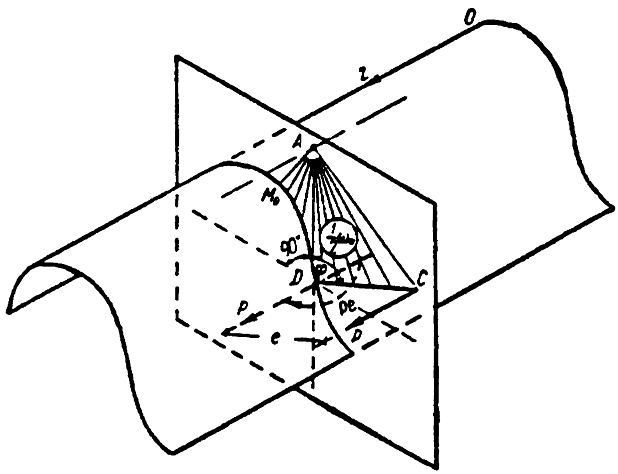


Figure 79

line, causes a bimoment $P\omega_D$, where ω_D is twice the area of AM_0D. We resolve the moment at the fixed point (D), equal to Pe and acting in the plane of the arm, into two components: one acting in the plane of the radius-vector AD, and the other in the plane perpendicular to AD. Since its plane of action passes through the shear center, the first moment does not cause torsion. The second moment $Pe \sin\varphi$ causes a bimoment equal to $\overline{AD} \cdot Pe \sin\varphi$ where AD is the distance between the points A and D; $\overline{AD} \cdot e \sin\varphi$ is twice the area of $\triangle ADC$. Adding the two bimoments we obtain

$$\overline{B} = P\omega_D + P\omega_{\triangle ADC} = P\omega_C,$$

where ω_C is twice the area of AM_0DC. The theorem is proved and in a different way we obtained the same equation as before.

4. Corresponding to the four external statical factors (7.18), the stresses in a thin-walled beam subjected to the action of a longitudinal force P will be determined also in the general case by the four-term equation (I.8.5). As already stated many times, in this work we shall be mainly interested in the fourth term of this equation which refers to the complementary torsional stresses due to the action of an external longitudinal bimoment $\overline{B}(z)$.

The bimoment is a new force factor which describes essentially a system of longitudinal forces statically equivalent to zero. Since the bimoment is proportional to the sectorial area ω_C, the stresses and deformations of a beam subjected to a longitudinal force P depend not only on the position of the force in the plane of the section $z = \text{const}$ but also on the mode of transmission of this force to the beam (at D, where the arm joins the beam). When the position of D changes, the sectorial area

ω_C also changes and consequently the magnitude of the bimoment as well as the corresponding stresses and strains of the beam which are determined by the law of sectorial areas.

The bimoment $B(z)$ in an arbitrary section of the beam is determined by integration of the differential equation

$$B''(z)-\frac{k^2}{l^2}B=0; \tag{7.19}$$

for given boundary conditions at the ends of the beam and satisfying the matching conditions of the spans of the beam at section $t=z$. Equation (7.19) is obtained from equation (3.1) by using the first equations (2.5).

5. As an example, we study a beam subjected to the action of an eccentrically applied tensile force P. Let C be the point of application of the force; D_1 and D_2 the points of connection of the rigid arms to the beam in the sections $z=0$ and $z=l$ respectively; A the shear center; M_0 the origin of the sectorial areas (Figure 80).

We shall deal only with one of the four generalized force factors into which the given external force P will be resolved, viz., the external bimoments. These external bimoments are given by:

$$\text{in the section } z=0 \quad \overline{B}_1=P\omega_{1C},$$
$$\text{in the section } z=l \quad \overline{B}_2=P\omega_{2C},$$

where ω_{1C} and ω_{2C} are the principal sectorial coordinates of point C in the sections $z=0$ and $z=l$ (in Figure 80 these are twice the hatched areas AM_0D_1C and AM_0D_2C). Knowing the bimoments at the ends of the beam, we can easily determine the bimoment for an arbitrary cross section $z=$ const. Since the external load is represented by concentrated longitudinal forces, we shall start from the homogeneous differential equation (7.19) in order to determine the bimoment.

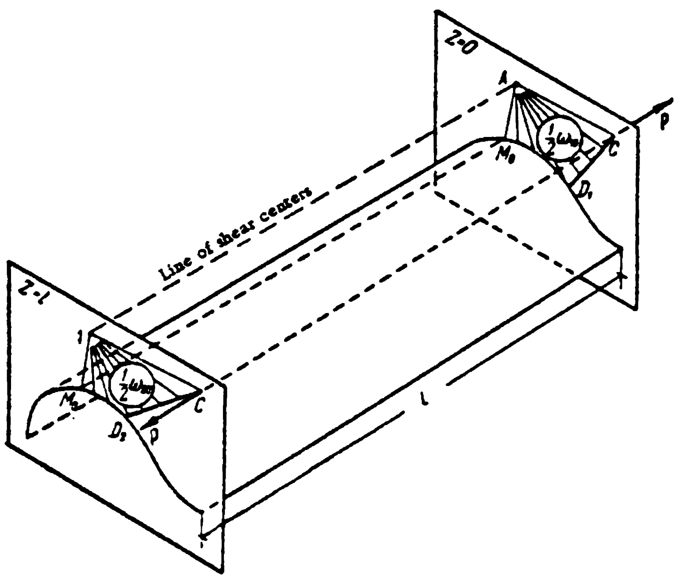


Figure 80

The general integral of this second-order linear differential equation with constant coefficients may be given in the form:

$$B(z) = C_1 \operatorname{sh}\frac{k}{l}z + C_2 \operatorname{ch}\frac{k}{l}z, \tag{7.20}$$

where C_1 and C_2 are arbitrary constants, determined from the following boundary conditions:

$$\text{for } z=0 \quad B(0) = \bar{B}_1 = P\omega_{1C},$$
$$\text{for } z=l \quad B(l) = \bar{B}_2 = P\omega_{2C}.$$

The arbitrary constants will have the values:

$$C_1 = \frac{\bar{B}_2 - \bar{B}_1 \operatorname{ch} k}{\operatorname{sh} k},$$
$$C_2 = \bar{B}_1,$$

and we have an equation for the bimoment in an arbitrary section

$$B(z) = \frac{P}{\operatorname{sh} k}\left[\omega_{2C} \operatorname{sh}\frac{k}{l}z + \omega_{1C} \operatorname{sh}\frac{k}{l}(l-z)\right]. \tag{7.21}$$

From the found bimoment, $B(z)$, it is easy to determine the complementary normal stresses which appear in the beam as the result of torsion due to the action of a longitudinal force:

$$\sigma_\omega(z, s) = \frac{B(z)}{J_\omega}\omega(s).$$

Using the expressions (7.18) and (7.21), we rewrite the general equation (I.8.5) for the calculation of the normal stresses as follows

$$\sigma(z, s) = P\left\{\frac{1}{F} + \frac{x_C}{J_y}x(s) + \frac{y_C}{J_x}y(s) + \right.$$
$$\left. + \frac{\omega(s)}{J_\omega \operatorname{sh} k}\left[\omega_{1C} \operatorname{sh}\frac{k}{l}(l-z) + \omega_{2C} \operatorname{sh}\frac{k}{l}z\right]\right\}. \tag{7.22}$$

If the sectorial coordinates have equal values for C, the point of application of the force P, i.e., $\omega_{1C} = \omega_{2C} = \omega_C$ equation (7.22) will have a simpler form:

$$\sigma(z, s) = P\left[\frac{1}{F} + \frac{x_C}{J_y}x(s) + \frac{y_C}{J_x}y(s) + \frac{\omega_C}{J_\omega}\omega(s)\frac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2} - z\right)}{\operatorname{ch}\frac{k}{2}}\right]. \tag{7.23}$$

If the force is applied, in this case, at the section centroid, $x_c = y_c = 0$ and equation (7.23) is further simplified:

$$\sigma(z, s) = P\left[\frac{1}{F} + \frac{\omega_c}{J_\omega}\omega(s)\,\frac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2} - z\right)}{\operatorname{ch}\frac{k}{2}}\right]. \tag{7.24}$$

6. We shall examine the general case of determination of the bimoment due to a given longitudinal load. In the case of the given longitudinal load forming a self-balancing system of forces, the arising bimoment is invariant with respect to the pole and the origin of the sectorial areas. In other words, the bimoment of any longitudinal self-balancing load, continuously distributed over the section or consisting of concentrated forces, does not depend on the shear center and on the origin of the sectorial areas.

We shall prove this. Let a longitudinal load consist of concentrated forces $P_1, P_2, P_3, \ldots, P_n$ applied at the points ($i = 1, 2, 3, \ldots, n$) of the cross section.

According to the principle of independent action of forces, the bimoment due to the sum of the forces is equal to the sum of the bimoments due to each force separately:

$$\bar{B} = \sum_{i=1}^{n} P_i \omega_i, \tag{7.25}$$

where ω_i is the sectorial area extending from the origin to the point of application of the force P_i, the pole being situated at the shear center.

We denote by $\bar{\omega}_i$ a sectorial area having an arbitrary origin and a pole at an arbitrary point. The sectorial coordinates $\bar{\omega}_i$ and ω_i can then be related, on the basis of (I.7.8), as follows:

$$\omega_i = \bar{\omega}_i + ax_i + by_i + c, \tag{7.26}$$

where x_i and y_i are the coordinates of the i-th point; a, b, c are coefficients depending on the position of the pole and of the origin of the sectorial areas $\bar{\omega}_i$.

Substituting the expressions (7.26) in equation (7.25), we have:

$$\sum_{i=1}^{n} P_i \omega_i = \sum_{i=1}^{n} P_i \bar{\omega}_i + \sum_{i=1}^{n} P_i (ax_i + by_i + c). \tag{7.27}$$

The second term on the right-hand side of equation (7.27) represents the moment due to the longitudinal forces P_1, P_2, P_3, ..., P_n with respect to the line determined by the coefficients a, b, and c*. In the case of a load, statically equivalent to zero, this moment vanishes, whatever the line chosen in the plane of the cross section. Hence

$$\sum_{i=1}^{n} P_i \omega_i = \sum_{i=1}^{n} P_i \bar{\omega}_i,$$

which confirms the above assumption.

* As a matter of fact, it is known from analytical geometry that the distance from an arbitrary point (x_i, y_i) in the plane the straight line $ax + by + c = 0$ is determined up to a coefficient of proportionality by the expression

$$ax_i + by_i + c.$$

§ 8. Saint-Venant's principle in the theory of thin-walled beams

1. The considerations presented in the previous section show that departure from the law of plane sections and the appearance of torsion in a beam can take place not only under the action of a transverse load which passes off the shear center, but also in the absence of any torsional load. The law of linear distribution of normal stresses over a section, in extension or compression, of a beam which is not stiffened by transverse plates at the ends, proves to be valid only in the particular case of a transverse load, i. e. when the bimoments due to this load vanish at the supports (Figure 81).

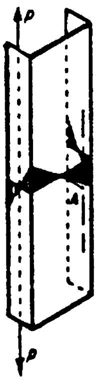


Figure 82

It follows from equation (7.24) that torsion can also occur in the beam upon central extension, when the force is applied along the line of centroids. Beam torsion under axial extension is plainly in evidence in the example of a Z-section. It is seen from the diagram of the sectorial areas, given in Figure 39a, that under the conditions of restrained torsion the web of a Z-beam subjected to torsional moments at the ends undergoes a uniform extension (contraction). On the basis of the reciprocity theorem it is also possible to state the opposite, viz. a Z-section subjected to tension (compression) in one wall only will undergo a torsional deformation in addition to the elongations (contractions). Indeed, a longitudinal force applied to only one wall of the beam gives rise to a non-vanishing bimoment. The value of this bimoment in an arbitrary cross section $z=$const, is calculated from equation (7.21):

$$B(z)=\frac{P\omega_c}{\operatorname{sh} k}\left[\operatorname{sh}\frac{k}{l}z+\operatorname{sh}\frac{k}{l}(l-z)\right]=P\omega_c\frac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}, \quad (8.1)$$

where ω_c is the sectorial area at the centroid of the section (the middle of the web), i. e. at the point of application of the axial force P, applied to the web only. A torsional deformation, $\theta(z)$, corresponding to this-bimoment, is determined by the differential equation (2.5):

$$EJ_\omega\theta''(z)=-B(z).$$

Under the influence of extension of the web, shearing forces appear in the flanges along the lines where they join the web (Figure 82a), as a result of which the flanges in addition to being extended also undergo flexure in their respective planes so that they cave in towards the web. As a result of the bending of the flanges on different sides, the beam is twisted without the agency of external torsional moments (Figure 82b).

Such a centrally applied longitudinal tensile load causes additional stresses $\sigma_\omega=\frac{B}{J_\omega}\omega$ besides the stresses $\sigma_0=\frac{P}{F}$. The stresses σ_ω, which appear as the result of the bimoment B, do not, as opposed to the axial stresses $\sigma_0=\frac{P}{F}$, remain constant over the length of the beam but vary according to the same law as the bimoment B (8.1). According to Saint-Venant's principle the stresses called forth in the section by a balanced external load are local in character and fall-off quickly away from the point of application of this load. This statement is not valid for thin-walled

beams. Although the bimoments represent a balanced system of forces, theoretical and experimental studies show that the stresses σ_{ω}, caused by the bimoments applied at the ends of a thin-walled beam with an open contour, as a rule do not have the character of local stresses owing to the three-dimensional deformation of such a beam which is in torsion. As applied to thin-walled beams of open cross section which does not vary along the beam (and in general to shells) Saint-Venant's principle is not actually verified and has a limited range of application.

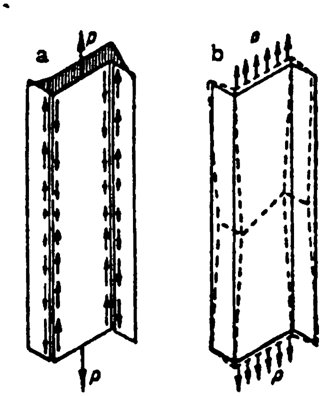


Figure 82

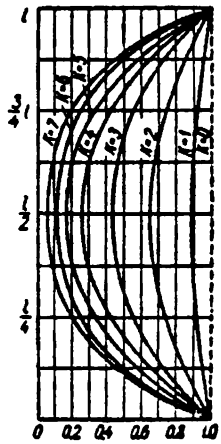


Figure 83

In the case of loading we have just considered the sectorial stresses σ_{ω} fall-off away from the beam ends towards the center. This is seen from their variation over the length of the beam (8.1). The rate of fall-off depends on the relative elastic characteristic $k^2 = l^2\frac{GJ_d}{EJ_{\omega}}$.

The smaller the value of k, the slower the fall-off and, conversely, the larger k the sharper the fall-off. In the limiting case, for $k \to 0$, the stress diagram along the beam will tend towards a straight line parallel to the beam axis. The stresses σ_{ω} along the whole beam remain constant. The problem of pure flexure of beams corresponds to this case in the theory of bending. As shown in § 6 for the example of an asymmetrical Y-section, the stresses σ_{ω} have a local character when $k \to \infty$.

It is possible to get an idea of the nature of the variation of the bimoment (or of the longitudinal normal stresses σ_{ω}) along the span of the beam for various values of the elastic characteristic from Figure 83. This figure shows graphs of the variation of the function $\dfrac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}$ which enters in the equation of the bimoment (and, consequently, of the stresses σ_{ω}) and which determines the law of variation of $B(z)$ and of $\sigma_{\omega}(z)$ as a function of z for values of $k = 1$, 2, 3, 4, 5, 6 and 7. For comparison the graph of the limiting case, $k = 0$, is also given.

Numerical values of the function $\dfrac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}$ in the sections $z=\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}l$ are given in Table 19.

Table 19

k	$\dfrac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2}-z\right)}{\operatorname{ch}\frac{k}{2}}$	
	at $z=\frac{1}{4}l$ and $\frac{3}{4}l$	at $z=\frac{1}{2}l$
1	0.915	0.887
2	0.731	0.648
3	0.550	0.425
4	0.410	0.266
5	0.308	0.163
6	0.234	0.099
7	0.179	0.060

Concerning the value of the relative elastic characteristic k, it is obvious, from the following relation

$$k=l\sqrt{\frac{GJ_d}{EJ_\omega}},$$

that k depends mainly on the thickness δ of the beam, on its length l and on the relative position of the plates (web, flanges, etc) which form the beam.

Leaving the contour of the cross section and the length of the beam constant and varying only its thickness, it is possible to show that the smaller the thickness δ, the smaller will k be, and vice versa; this is so because the rigidity GJ_d is of the third degree in the thickness while the sectorial rigidity EJ_ω is linear in it. Thus, the smaller the thickness of the component plates the slower the fall-off of the sectorial stresses (and also of the bimoments) over the length of the beam.

Other things being equal, k varies linearly with the length of the beam. The stresses fall-off more sharply the longer the beam. In § 6 this dependence is studied in the example of a Y-section and the graphs are shown in Figure 70.

We shall clarify the dependence of k on the relative position of the component plates (other things being equal), in the example of a Z-beam. We consider this beam as a thin-walled system composed of three narrow plates, which are joined to one another (web to flanges) at right angles. When bent in their plane, under the action of a centrally applied longitudinal load (only to the web) the flanges of such a beam (Figure 82b) will encounter little resistance from the web since the latter will undergo torsion, the flanges being on opposite sides. Due to the torsion, to which an open section with thin web and flanges offers very little resistance, the fall-off of the bimoment stresses in this beam will be slower.

The situation will be different if all the three plates of a Z-section are in the same plane, i. e. if we rotate each flange by 90°. We shall have a plate composed of three narrow strips (Figure 84a). In this case the balanced longitudinal load, applied at the ends of the middle plate, causes stresses only in the direct neighborhood of the point of application of the load, for the extreme strips, which are connected to the middle plate so as to form one piece, cannot be deflected in their plane (Figure 84b). These stresses are due mainly to the shear deformation. An intermediate relative position of the plates would naturally give a rate of fall-off of the stresses σ_ω intermediate between the fall-off rates for plates at the 0° and 90° positions.

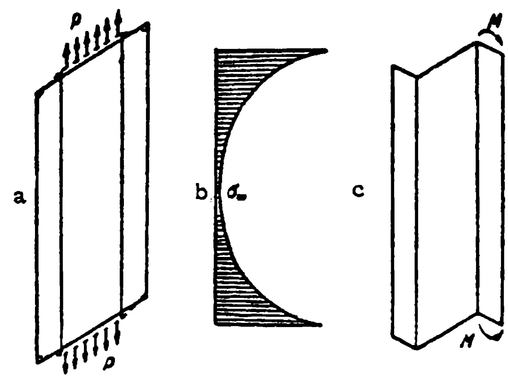


Figure 84

The phenomenon of torsion and the associated sectorial warping can also take place in the case of bending moments applied at the ends of the beam. Thus, for example, if the moments in the case of a Z-section are applied at the ends of a single flange (Figure 84c) the bimoments (and hence also the torsion angles $\theta(z)$ and the stresses σ_ω) will be different from zero. It follows that the concept of pure bending, as well as the concept of pure extension examined above, is rendered more exact in the light of the theory developed here.

We make here the general observation that the phenomenon of beam torsion under the action of a longitudinal force as described here and in § 6 takes place only when the ends of the beam are not fixed so as to eliminate longitudinal deformations, i. e. when the end sections of the beam are allowed to warp freely (be twisted away from their planes). Such boundary conditions are largely of theoretical interest. In practice, the ends of beams in metallic structures usually have stiffening ribs or stringers, whatever the manner of support, which to a larger or smaller extent restrain the sections from warping. With increasing rigidity of the transversal stiffening end-plate, the influence of the sectorial factors for eccentric longitudinal force decreases. If the stiffeners possess infinite rigidity, the end sections and, consequently, also the sections at an arbitrary point $z = \text{const}$, remain plane under the action of a longitudinal force. The additional sectorial factors vanish in the limiting case.

The torsion of thin-walled beams under the action of a tensile force or bimoment, as described here and in the previous two sections, was experimentally confirmed on the models shown in Figures 85 and 86.

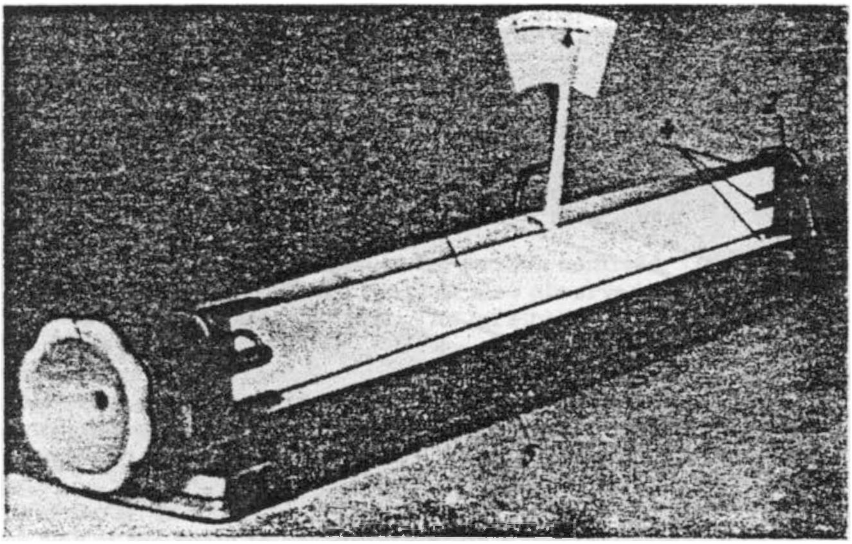


Figure 85a

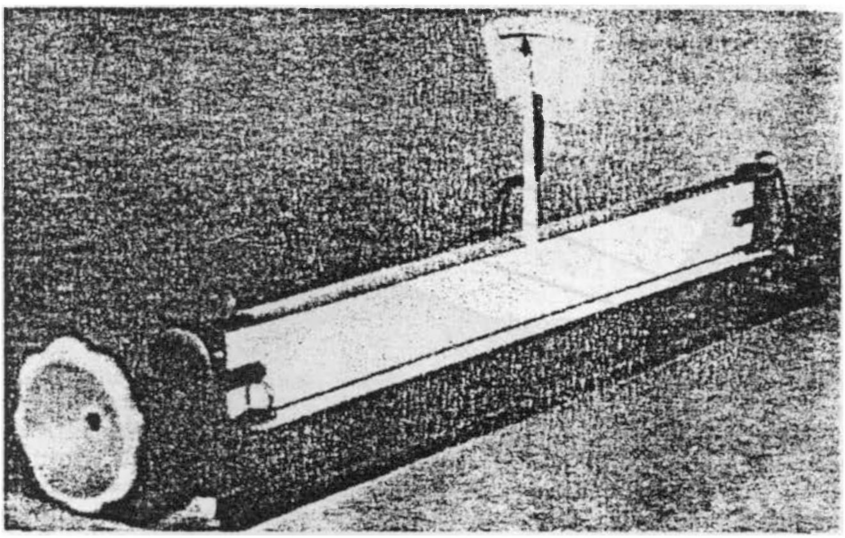


Figure 85b

Figure 85 shows the action of a tensile force, the model in this case (Figure 85a) is a thin-walled Z-beam, 1, which has support angles at the ends: angle 2 is securely fixed to the base 5; another angle, 3, can move along a guide fixed to the base.

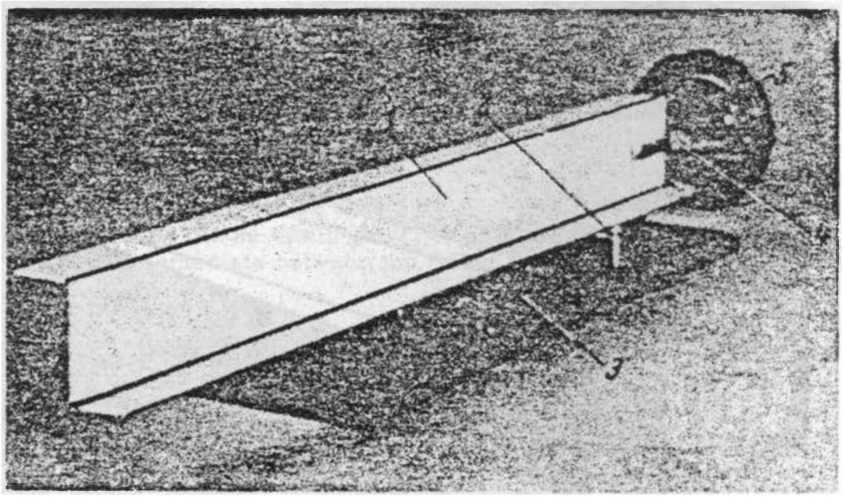


Figure 86a

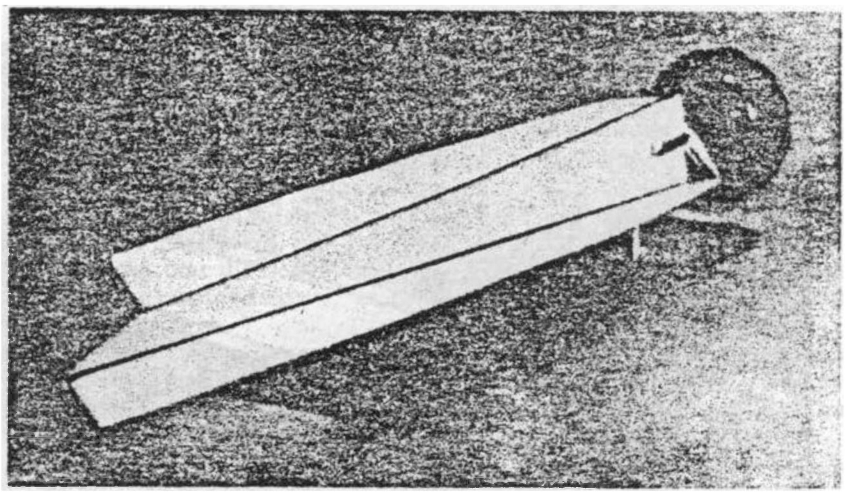


Figure 86b

The support angles have three bolts on which the bushings 4 fixed to the ends of the beam are freely mounted. The axes of the bushings are situated at the section centroid and at the points of its flanges where the principal sectorial areas are equal to zero. The bushings and the bolts are cross-bored to receive the pins by which the beam is securely fixed in

assembly to the support angles. The movable angle is displaced along a special guide of a pallet section, which has a vertical wall, 6, at one end. Through this wall a turnbuckle is freely introduced, and fastened co-axially at one end to the movable angle (the support of the beam). A splined slot is cut in the screw for the guide which is fixed to the vertical wall. On the free end of the screw a crank, 7, is mounted, which when turned gives the screw only a translational displacement transferred to the movable support and extending the thin-walled beam. There are two ways to extend the beam:

a) By a single force applied at the section centroid. In this case, besides extension, torsion of the beam is also observed, appearing as a result of the warping of the sections. The torsion angle is shown by the pointer of the instrument placed at mid-span (Figure 85a).

b) By two forces, applied to the flanges at two points, for which the sectorial areas are equal to zero. In this case only extension without torsion occurs and the torsion indicator remains at rest (Figure 85b).

Figure 86 shows a model which allows the loading of a beam by a bimoment only.

The model is a thin-walled Z-beam, 1, free at one end. At the other end the web of the beam is securely fixed to a vertical angle, 2, connected to the base 3 (Figure 86a). To the same end face of the beam a screw is coaxially fastened with a block, 4, freely mounted on it. This block can occupy any position with respect to the web and the flanges of the Z-section. A crank, 5, is mounted on the end of the screw. By turning the crank we stretch the beam along its axis and at the same time press the block against those points of the beam cross section touched by it (actually c o m p r e s s i n g the corresponding fiber of the beam). Thus, the bimoment of the load is transferred to the end face of the Z-section. The value of the bimoment is maximal when the block is most inclined.

Thw warping causes a torsion which is plainly evident on the model (Figure 86b) and can be determined by an indicator placed at the free end of the beam.

§ 9. Analogies with the elementary theory of beam flexure

1. In the foregoing sections we developed the theory of the restrained torsion which takes place for any distribution law of the external torsional moment along the beam and for various boundary conditions, and characterized by the appearance not only of tangential but also of normal stresses in the cross section of the beam. This theory is identical in structure with the existing elementary theory of beam flexure. As repeatedly indicated before, the theory of beam flexure, based on the law of plane sections, follows as a particular case from the proposed new theory, which is based on the more general law of sectorial areas. By examining separately the theory of free torsion of a thin-walled beam, depending, in the previous formulas and equations, on the sole function of the torsional angles $\theta(z)$ we see that this theory is analogous in its mathematical formulation to the elementary theory of beam flexure.

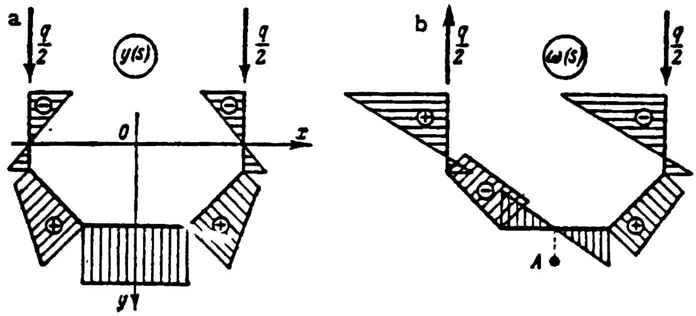


Figure 87

This analogy is presented below in a summary of the basic equations of the theory of flexure and of the theory of restrained torsion (Table 20).

Table 20

Bending in the plane Oyz (law of plane sections, Figure 87a)	Restrained torsion (law of sectorial areas, Figure 87b)
$J_x = \int_F y^2\, dF$	$J_\omega = \int_F \omega^2\, dF$
$S_x = \int_F y\, dF$	$S_\omega = \int_F \omega\, dF$
$\eta = \eta(z)$	$\theta = \theta(z)$
$\eta' = \frac{d\eta}{dz}$	$\theta' = \frac{d\theta}{dz}$
$M_x = -EJ_x\eta''$	$B = -EJ_\omega\theta''$
$Q_y = M'_x = -EJ_x\eta'''$	$H_\omega = B' = -EJ_\omega\theta'''$
$\sigma_x = \frac{M_x(z)}{J_x} y(s)$	$\sigma_\omega = \frac{B(z)}{J_\omega}\omega(s)$
$\tau_x = -\frac{Q_y(z)S_x(s)}{J_x\delta(s)}$	$\tau_\omega = -\frac{H_\omega(z)S_\omega(s)}{J_\omega\delta(s)}$
$q_y = q_y(z)$	$m = m(z)$

As seen from Table 20, the analogy between the theory of flexure and the theory of restrained torsion consists in the fact that the basic factors η, η', M_x, Q_y representing, in the theory of flexure, the deviation (angle of rotation of the section), the moment and the transverse force respectively, correspond in the theory of restrained torsion to the basic factors θ, θ', B and H_ω, which represent the torsion angle, the measure of warping, the bimoment and the flexural-torsional moment, respectively.

The external specific transverse load q_y corresponds to an external specific torsional moment m. The normal and tangential stresses σ_ω and τ_ω are expressed by equations which are completely analogous to the equations of σ_x and τ_x in the case of bending. The same can also be said of the geometrical characteristics J_x, S_x, J_ω and S_ω.

The difference between the theory of torsion developed here and the theory of bending consists in the fact that instead of the differential equation

$$EJ_x\eta^{IV} = q_y, \tag{9.1}$$

which we use in the theory of bending, we have to use in the theory of restrained torsion a somewhat more complicated differential equation

$$EJ_\omega\theta^{IV} - GJ_d\theta'' = m. \tag{9.2}$$

The presence in equation (9.2) of a term containing the second derivative of the torsion angle θ is due, as mentioned earlier, to the nonuniformity of distribution across the wall of the beam of the tangential stresses parallel to the profile line.

2. However, research shows that for cylindrical and prismatical open shells and for beams consisting of thin plates, with a ratio of the thickness to the characteristic dimension of the cross section of the order of 0.02 and less, the torsional rigidity GJ_d, which is proportional to the cube of the thickness, can be set equal to zero without an appreciable error. This assumption is similar to the assumption that the torsional moments, which occur as the result of nonuniform distribution of tangential stresses across the wall of the beam, are equal to zero.

Dropping in equation (9.2) the term containing θ'' for structural ribbed arch-shells and for hipped systems, i. e., assuming that the tangential stresses (parallel to the middle surface) are uniformly distributed across the wall and thus give rise to a torsional moment

$$H = H_\omega = -EJ_\omega\theta''',$$

we obtain

$$EJ_\omega\theta^{IV} = m. \tag{9.3}$$

The last equation has the same form as equation (9.1) for the bending of an ordinary beam.

The following important proposition follows from the analogy noted above:

All the analytical methods based on the law of plane sections which are used in the theory of strength of materials and structural engineering for calculating beams and systems in flexure, can be completely generalized and extended to include the theory of restrained torsion of thin-walled beams, of systems consisting of such beams and of ribbed arches.

Thus, all the four basic quantities of the theory of restrained torsion (the torsion angle θ, corresponding to the deflection η; the derivative θ' which determines, according to (I.3.16), the extent of the absolute warping of the section and the corresponding deviation η' in the theory of bending; the bimoment B corresponding to the bending moment M_x and the torsional moment H_ω corresponding to the transverse force Q_y) can be determined by well-known analytical or graphical methods of the theory of bending. Thus, for example, if a beam of length l is subjected to the action of a uniformly distributed load $q = q_y$, directed parallel to the axis Oy and applied with an eccentricity e with respect to the line of shear centers, and if the boundary conditions at the ends $z = 0$ and $z = l$ correspond to a hinge support (i. e. in the case of bending, the deflections η and the moments M_x vanish, and in the case of restrained torsion the torsional angle θ and the bimoment B vanish), we shall have completely identical equations for the

bending moment M_x and the deflection η and for the bimoment B and the torsional angle θ respectively:

$$\left.\begin{aligned} M_x(z) &= \frac{qz}{2}(l-z), \quad \eta(z) = \frac{qz}{24EJ_x}(l^3 - 2lz^2 + z^3), \\ B(z) &= \frac{mz}{2}(l-z), \quad \theta(z) = \frac{mz}{24EJ_\omega}(l^3 - 2lz^2 + z^3). \end{aligned}\right\} \qquad (9.4)$$

The last two equations of (9.4) are obtained from the first two on the basis of the proposition stated above, by replacing q by $m = qe$, J_x by J_ω, η by θ and M_x by B.

The bending moment M_x and the deflection η are maximal in the middle of the span:

$$\max M_x = \frac{ql^2}{8}, \quad \max \eta = \frac{5}{384}\frac{ql^4}{EJ_x}.$$

Similarly, the bimoment B and the torsion angle θ will also be maximal in the middle of the span:

$$\max B = \frac{ml^2}{8}, \quad \max \theta = \frac{5}{384}\frac{ml^4}{EJ_\omega}.$$

The longitudinal normal stresses $\sigma(z, s)$, at any point (z, s) will be calculated in the examined case of loading by the two-term equation

$$\sigma(z, s) = \frac{M_x(z)}{J_x} y(s) + \frac{B(z)}{J_\omega} \omega(s).$$

3. In the same way, starting from the analogy given here and not performing all the calculations needed to determine the constants of integration in equation (9.3), we can immediately write down the equation for the three bimoments for a continuous thin-walled beam (or cylindrical shell), whose sections at the supports (extreme and intermediate) are fixed so as to eliminate bending:

$$l_n B_{n-1} + 2(l_n + l_{n+1}) B_n + l_{n+1} B_{n+1} = -6\,(R_n^{\text{f.r.}} + R_n^{\text{f.l.}}), \qquad (9.5)$$

where l_n, l_{n+1} are the spans; B_{n-1}, B_n, B_{n+1} are the bimoments at the supports; $R_n^{\text{f.r.}}$, $R_n^{\text{f.l.}}$ are the right and left reactions, of the fictitious support, due to "loading" by the diagrams of the bimoments of the single span beams of the fundamental system. These beams are statically determinate.

Equation (9.5) is obtained from the equations for the three moments by replacing the quantities of the theory of bending by their corresponding quantities of the theory of torsion.

4. For thin-walled metal beams with a ratio of wall thickness to the characteristic dimension of the cross section of the order of 0.1, the torsional moments, which appear as a result of a nonuniform distribution of tangential stresses across the wall, are most important. It is impossible, for such beams, to assume that the rigidity GJ_d is equal to zero.

The problem of flexural torsion of a beam in this more general case is stated by equation (9.2), which is mathematically analogous to the equation of the theory of transverse bending of an initially extended beam, i.e. to

$$EJ\eta^{\mathrm{IV}} - P\eta'' = q,$$

where P is the tensile force.

It follows from this analogy that, when solving the problem of flexural torsion of a beam using Saint-Venant's pure torsional rigidity, the analytical methods of calculating initially extended beams, as known from strength of materials and the applied theory of elasticity, are not restricted in application. Thus, for example, the equations for three bimoments for a single-span thin-walled beam, which has a constant nonzero rigidity GJ_d, can be obtained from the equations for the three moments of a single-span beam, extended by a force P and having the same conditions with respect to the external load and to the fixed supports as those of a beam in the problem of flexural torsion.

These equations have the form

$$a_{n,\,n-1}B_{n-1}+a_{n,\,n}B_n+a_{n,\,n+1}B_{n+1}=\theta'_{B(n)}-\theta'_{A(n+1)},\qquad (9.6)$$

where B_{n-1}, B_n, B_{n+1} are the bimoments at the supports of three successive sections; $a_{n,\,n-1}$, $a_{n,\,n}$ and $a_{n,\,n+1}$ are coefficients calculated in the most general case by the equations:

$$a_{n,\,n-1}=\frac{\operatorname{sh} k_n-k_n}{k_n^2\operatorname{sh} k_n}\,\frac{l_n}{EJ_{\omega,\,n}},$$

$$a_{n,\,n}=\frac{k_n\operatorname{ch} k_n-\operatorname{sh} k_n}{k_n^2\operatorname{sh} k_n}\,\frac{l_n}{EJ_{\omega,\,n}}+\frac{k_{n+1}\operatorname{ch} k_{n+1}-\operatorname{sh} k_{n+1}}{k_{n+1}^2\operatorname{sh} k_{n+1}}\,\frac{l_{n+1}}{EJ_{\omega,\,n+1}},$$

$$a_{n,\,n+1}=\frac{\operatorname{sh} k_{n+1}-k_{n+1}}{k_{n+1}^2\operatorname{sh} k_{n+1}}\,\frac{l_{n+1}}{EJ_{\omega,\,n+1}},$$

k_n, k_{n+1} are the elastic characteristics of the beam in the spans n, $n+1$, on both sides of the n-th support; $J_{\omega,\,n}$, $J_{\omega,\,n+1}$ are the principal sectorial moments of inertia of the section of the beam in these spans.

The quantities $\theta'_{B(n)}$, $\theta'_{A(n+1)}$ on the right-hand side of equations (9.6), are a measure of the warping of the n-th and $(n+1)$th spans at the n-th supported section of the basic system (cut at the supports). The indexes B, A denote the right- and left-ends of the spans n, $n+1$ of the basic system.

The quantities $\theta'_{B(n)}$, $\theta'_{A(n+1)}$ can be determined for each particular case of loading by the method of initial parameters described above for the boundary conditions of a beam hinged at both ends. Foɒ fixed n, each of equations (9.6) expresses the condition of continuity of warping at the n-th support, analogous to the conditions of continuity of the deviation (inclination angle of the tangent to the flexure line) in the theory of continuous beams. Assigning to the index n successively the values 1, 2, 3, ..., we obtain a complete system of three-term equations having a structure similar to that of the three-moment equations in the theory of continuous beams. These equations determine the bimoments at the supports.

§ 10. Practical method for designing hipped systems and shells which are reinforced by transverse ribs

Theoretical and experimental studies show that shell structures and hipped systems, reinforced in the transverse direction by ribs can be considered in practice with a sufficient degree of accuracy as three-dimensional thin-walled systems with a rigid (nondeformable) cross section /40, 42/.

Also, in ribbed shells and hipped structures whose thickness is very small in comparison with the other dimensions of their cross section, it is possible to assume that the torsional rigidity GJ_d is zero and, consequently calculate the torsion from the differential equation (9.3)

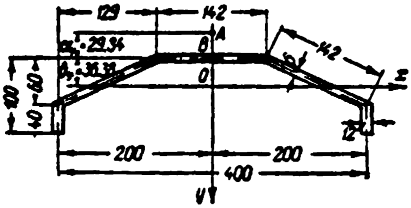


Figure 88

As an example, we examine the thin-walled hipped section with five facets shown in Figure 88. We determine the stresses in this section due to a vertical uniformly distributed load, which acts in the plane of the extreme right facet.

Since the cross section has a single axis of symmetry, the shear center lies on this axis. The distance, on the symmetry of axis (the y axis), from the shear center to the point of intersection of the symmetry axis with the contour line of the section is determined by the second of equations (I.7.5):

$$a_y = a_y - b_y = -\frac{J_{\omega_B x}}{J_y} = -\frac{\int_F \omega_B x\, dF}{\int_F x^2\, dF}.$$

Figure 89 shows the diagrams of the sectorial areas ω_B and of the distances x and y for the points of the section contour. From these diagrams we obtain the following values for the geometrical characteristics:

The sectorial product of inertia:

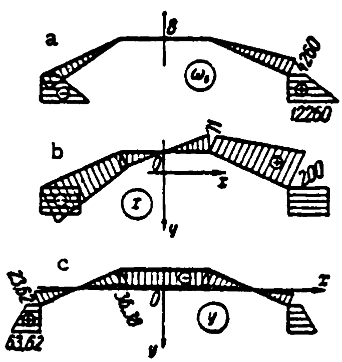


Figure 89

$$J_{\omega_B x} = \int_F \omega_B x\, dF = 2156 \cdot 10^4 \text{cm}^5,$$

The axial moments of inertia are

$$J_x = \int_F y^2 dF = 36.58 \cdot 10^4 \text{cm}^4,$$

$$J_y = \int_F x^2 dF = 73.48 \cdot 10^4 \text{cm}^4,$$

The coordinates of the shear center are

$$a_y = a_y - b_y = -29.34 \text{cm}.$$

Figure 90a shows the diagram of the principal sectorial areas ω, which describes the variation of the torsional stresses σ_ω.

Integrating the squared ordinate of this diagram over the area F, we obtain the sectorial moment of inertia

$$J_\omega = \int_F \omega^2 dF = 18040 \cdot 10^6 \text{cm}^6.$$

Figure 90b shows the diagram of the sectorial statical moments $S_\omega(s)$ which characterize the variation of the complementary shear stresses τ_ω over the section.

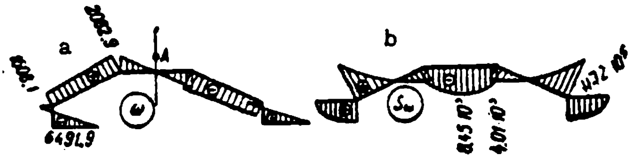


Figure 90

We now have all the data for determining the complementary stresses (normal and tangential) due to torsion.

We denote by M, Q, q and e the bending moment, the transverse force, the external uniformly distributed external load and the eccentricity of this lead, respectively.

Drawing on the mathematical analogy between flexure and torsion and using equations (9.4), we write

$$B(z)=\frac{m}{q}M(z)=eM(z),$$

$$H_\omega(z)=\frac{m}{q}Q(z)=eQ(z).$$

Whence we obtain the equations for the stresses σ_ω and τ_ω,

$$\sigma_\omega=\frac{Me}{J_\omega}\omega(s), \qquad \tau_\omega=-\frac{Qe}{J_\omega\delta}S_\omega(s).$$

The total stresses σ and τ, due to a load that causes bending in the vertical plane, and to torsion (respectively) are determined from

$$\left.\begin{aligned}\sigma&=\left(\frac{y}{J_x}+\frac{e\omega}{J_\omega}\right)M,\\ \tau&=-\left[\frac{S_x(s)}{J_x}+\frac{eS_\omega(s)}{J_\omega}\right]\frac{Q}{\delta}.\end{aligned}\right\} \qquad (10.1)$$

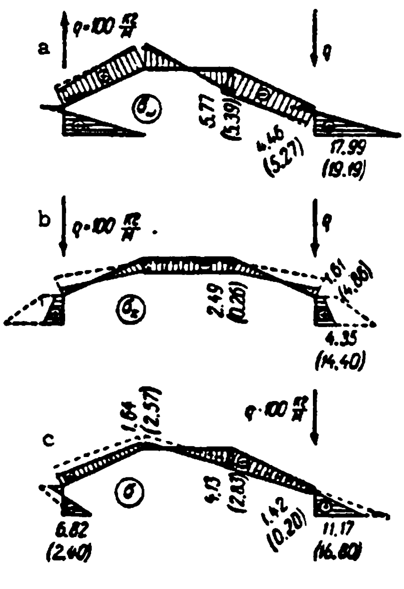


Figure 91

Since we have used the analogy between flexure and torsion, equations (10.1) can be applied to all cases, where the conditions of constraint at the supported sections are of the same type with respect to flexural and flexural-torsional factors. For example, these equations are valid when the component strips are separately hinged, so that the whole section, fixed to a rigid diaphragm, can be freely displaced from its plane (become warped). It is also possible to use these equations for the case of fixing that restrains transverse and longitudinal displacement.

Figure 91a shows the diagrams of the maximum stresses σ_ω due to torsion when the two extreme strips of the examined hipped section are antisymmetrically loaded by a uniform load

$q = 100$ kg/m. The span of the beam is $l = 10$ m. The terminal sections are hinged.

Figure 91b shows the diagram of σ_x due to bending by the same transverse load, applied symmetrically.

Figure 91c shows the total stresses $\sigma = \sigma_x + \sigma_\omega$ due to a unilateral load, $q = 100$ kg/m, obtained by superposition of the diagrams of Figures 91a and b. The loading scheme is shown on each of the three figures. In the considered case the largest normal stresses in bending and torsion appear in the middle cross section. The stresses obtained by a more exact theory of hipped sections which allows for deformation of the section contour /40/ are indicated by dotted lines on all the diagrams. The numerical values of the characteristics of these stresses are given in brackets.

A comparison of the solid and dotted lines shows that the deformation of the section contour has a much greater effect on bending in the plane of symmetry than on torsion.

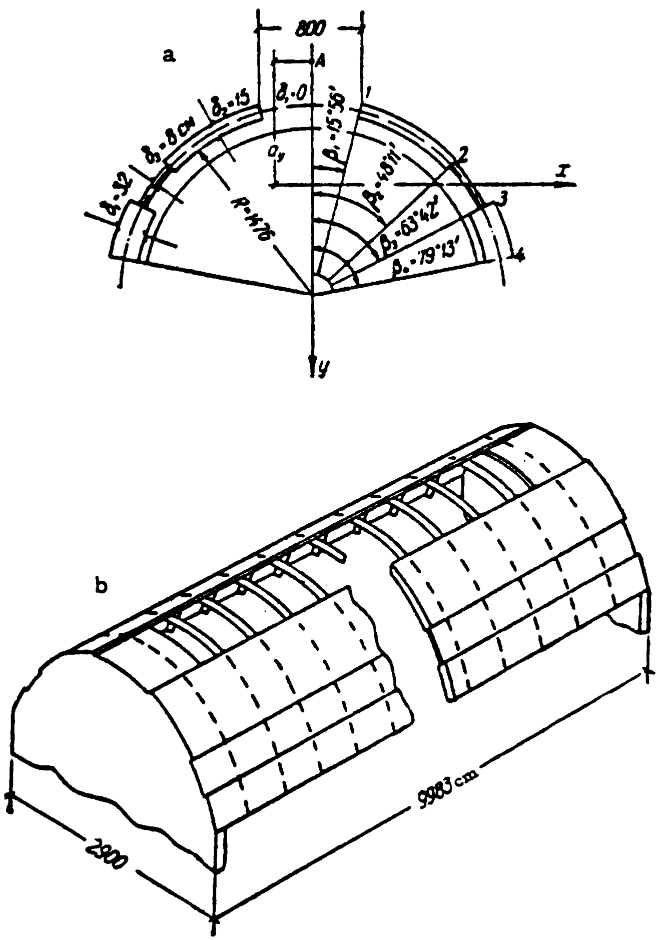


Figure 92

Thin-walled hipped plate systems which are not reinforced by transverse rigid ribs should be designed with particular attention to the deformation of the section contour under the action of a transverse bending load. For torsional loads these systems can be designed according to the theory of sectorial areas developed here with a degree of accuracy sufficient in practice. If the hipped system is reinforced by transverse ribs, the influence of the section contour deformation is found to be greatly reduced by the use of stiffer ribs. The dotted lines in the diagrams then approach the solid lines.

Hipped-plate ribbed systems, as used in thin-walled reinforced concrete roofs, behave under the action of transverse loads (symmetrical or asymmetrical) like three-dimensional thin-walled solid systems possessing an inflexible contour. Such systems are designed by the theory developed here, based on the law of sectorial areas.

We bring as an example some design results for a ribbed, cylindrical wooden shell with a span $l = 100$ m, constructed in the USSR in 1935 /40/*. The cross section of this shell has the form of a circular arc and consists of parts of varying thickness; the upper (central) part is weakened by the opening which serves as skylight; the section has a single axis of symmetry (Figure 92a). A general view of the shell is given in Figure 92b. The curved ends of the shell rest on walls which are rigid in their plane. The coordinate of the shear center a_y and the sectorial moment of inertia J_ω are computed by equations (1.21) and (1.22). We give the basic geometrical characteristics of the section with reference to the principal axes:

$$F = 39\,080 \cdot 10^{1}\,\text{cm}^2, \quad J_x = 6120 \cdot 10^{6}\,\text{cm}^4,$$
$$a_x = 0, \qquad J_y = 49\,340 \cdot 10^{6}\,\text{cm}^4,$$
$$a_y = -985\,\text{cm}, \quad J_\omega = 12\,600 \cdot 10^{9}\,\text{cm}^6.$$

Figure 93a shows the diagram of the normal stresses σ_z in the middle cross section of the shell due to bending by a uniformly distributed transverse load $q = 100$ kg/m, acting symmetrically with respect to the axis of symmetry (the loading diagram is also shown).

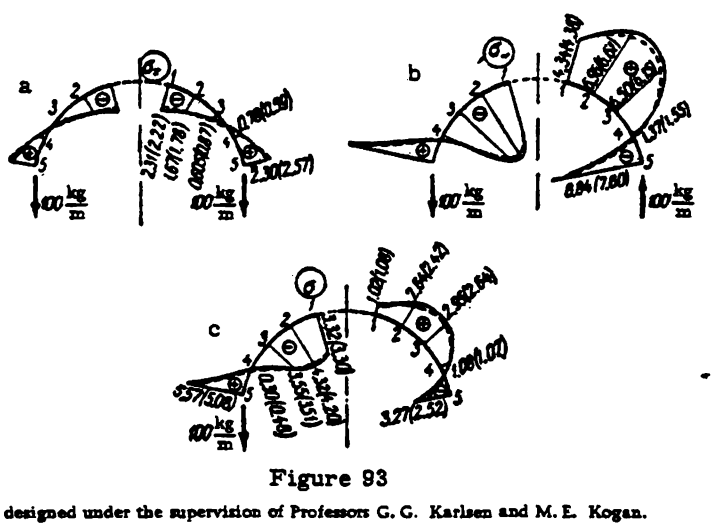


Figure 93

* This shell was designed under the supervision of Professors G. G. Karlsen and M. E. Kogan.

Figure 93b gives the diagrams of the stresses σ_ω due to a vertical antisymmetrical load (the loading diagram is shown on the same figure) which causes torsion only.

For a unilateral load $q = 100$ kg/m, the diagrams of the total stresses $\sigma = \sigma_z + \sigma_\omega$ are shown in Figure 93c. These are obtained by superposition of the diagrams of bending and torsion. The stresses found by the author's more exact method which enables the deformation of the section contour to be taken into account /51/ are indicated on these diagrams by dotted lines and by numbers in brackets.

In this example we see once more that the complementary torsional stresses which arise owing to the eccentricity of a transverse load (with respect to the shear center) reach very large values.

The fact that the cross sections of the shell do not remain plane after deformation is a major factor in the spatial behavior of thin-walled beams and shells.

§ 11. Beams and shells with cross sections having only one degree of freedom

1. We shall apply our theory to the design of three-dimensional structures of the thin-walled beam type with cross sections having but one degree of freedom in transverse displacements. We shall assume that the structures examined have stiffeners which restrain them from bending in the usual manner of beams. The cross sections, thus fixed, can only rotate about a certain straight line which generally represents the instantaneous axis of rotation. Examples of such structures are ribbed cylindrical and prismatical roofs which have a hinge running along one alteral edge, fixed to an immovable longitudinal supporting wall; the other lateral edge is free.

As an example, we examine a thin-walled three-dimensional structure of a cylindrical shell type of open section, fixed by transverse ribs and subjected to the action of a given transverse load. We assume that along one lateral edge the shell is fixed to an immovable supporting wall by means of a cylindrical hinge, whereas along the other edge the shell is not fixed at all (Figure 94a). At the terminal ends the shell may be fixed in any way. We directly see that the cross section $z =$const of this ribbed shell has only one degree of freedom with respect to displacements in its plane. The displacements of an arbitrary point of the section can be expressed by the angle of rotation of this section $\theta = \theta(z)$ about a stationary point C. As before, we shall take the positive sense of this angle to be clockwise, observing the cross section $z =$const in the direction of negative

According to the theory of thin-walled beams the torsional deformations are associated with warping of the section. In the present case this warping obeys the law of sectorial areas with a pole at the fixed point C. The origin of the sectorial areas $\omega(s)$ depends on the way in which the points of the supported lateral edge of the shell are fixed with respect to longitudinal displacements. If the hinged edge of the shell is also fixed to restrain longitudinal displacements so that this edge cannot be extended, the fixed point C of the cross section of the shell is also the sectorial origin

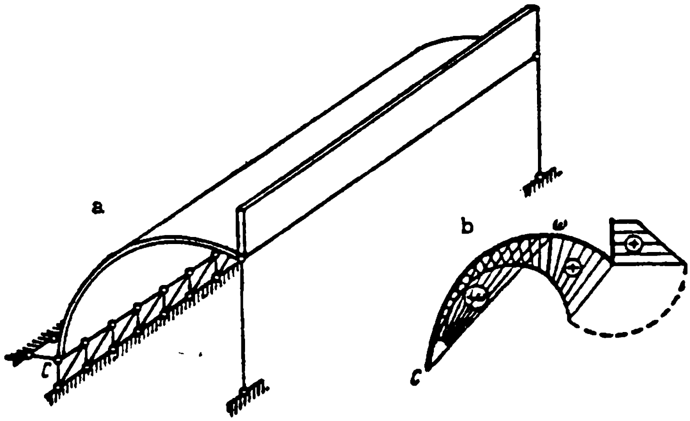


Figure 94

(Figure 94b). For the considered modes of fixing the shell to the supporting wall not only normal and transverse forces but also shear forces acting in the longitudinal direction may appear. When only an external transverse load is acting on the shell, these forces will be proportional to the sectorial static moment (I.6.11),

$$S_\omega = \int_F \omega \, dF,$$

which are computed for the whole cross section. In case the lateral edge of the shell is fixed at each of its points so that it cannot be displaced in the plane of the cross section but is free to move in the longitudinal direction (Figure 95a), the shear stresses on the supported edge should vanish. It follows that, when only a transverse load is acting on the shell, the resultant of the longitudinal normal stresses vanishes in any cross section.

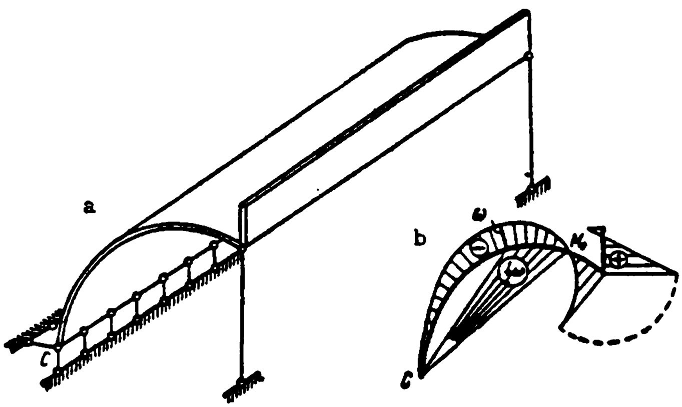


Figure 95

Since the resultant is proportional th the sectorial static moment, the sectorial origin C', for the case of a longitudinal movable supported edge,

is determined from the condition

$$S_\omega = \int_F \omega\, dF = 0.$$

The diagram of the sectorial areas for a shell with movable supported edges, satisfying this condition, is shown on Figure 95b.

The system of differential equations of equilibrium of an isolated elementary transverse strip of the beam-shell leads in our case to a single differential equation for the function of the torsion angle $\theta(z)$:

$$EJ_\omega \theta^{IV} - GJ_d \theta'' = m. \tag{11.1}$$

In this equation the rotation angle $\theta(z)$ and the external line-distribution of torsional moments $m(z)$ are considered with respect to the fixed point C, in the plane of the cross section, which is the center of rotation of the section $z =$ const. The pole of the sectorial area is chosen at the same point C. The point C is taken as the origin of the sectorial areas, if the supported edge is fixed so as to restrain longitudinal displacements. If this edge is free to move longitudinally, the sectorial origin is found from the condition that the sectorial static moment is equal to zero.

We have mentioned earlier that for hipped prismatic and cylindrical systems, the torsional rigidity GJ_d can be taken to vanish. In this case equation (11.1) assumes the simpler form

$$EJ_\omega \theta^{IV} = m.$$

Once the torsion function $\theta(z)$ is determined from this equation for the corresponding boundary conditions on the transverse edges, we can find the longitudinal normal and tangential stresses from the following equation:

$$\left.\begin{aligned} \sigma(z, s) &= \frac{B(z)}{J_\omega}\,\omega(s), \\ \tau(z, s) &= -\frac{H_\omega(z)}{J_\omega \delta(s)}\,S_\omega(s). \end{aligned}\right\} \tag{11.2}$$

In order to calculate these stresses it is also possible to use the analogy between flexure and torsion. In this case the bimoment $B(z)$ and the flexural-torsional moment $H_\omega(z)$, entering in equation (11.2), are determined in the same way as the corresponding bending moment and transverse force in the theory of beams, replacing the transverse linear load distribution $q(z)$ by the external torsional moment distribution $m(z) = q(z)\,e$. Here e is the eccentricity of the applied load $q(z)$ with respect to the supported edge.

If the shell, subjected to the action of a transverse load, has a continuous multiple-span structure resting on transverse partitions which are rigid in their planes, and is hinged along the longitudinal supported edge, it is easy to obtain the solution of the problem by using the equation for the three bimoments (9.5).

2. We now proceed to examine more complicated systems with a single degree of freedom for transverse displacements of the section $z =$ const. Such a system, having a cross section composed of two rigid frames interconnected by a hinge, is shown in Figure 96. It consists of a cylindrical shell and a prismatical hipped shell connected along a common edge. As before, we shall consider that a transverse load only is acting on the system, and that the system may be fixed at the transverse ends in any manner.

We assume initially, that the hinged lateral edge CC is fixed so as to restrain longitudinal displacements, that the hinged edge C_2C_2 is free to move longitudinally, and that the cylindrical hinge in the middle, C_1C_1, restrains relative longitudinal displacements of the hipped and round shells. The longitudinal edges thus fixed, the shear forces which appear in the supported edge $\bar{C}C$ are counteracted by the stationary bearing wall.

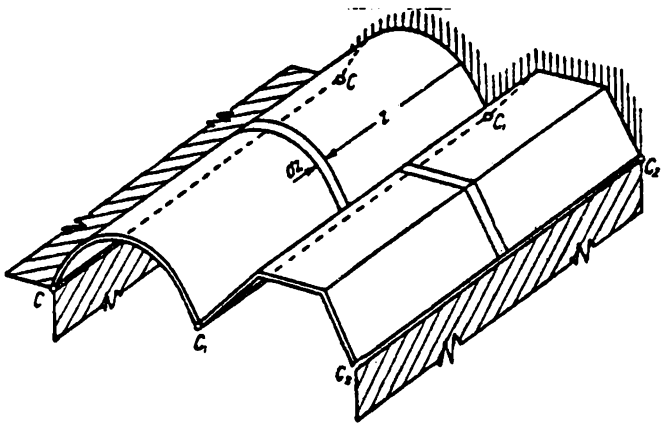


Figure 96

In order to solve the problem we must first of all establish the variation of the longitudinal displacements over the cross section of the system resulting from the warping of the section.

Figure 97

Since the edge CC has no longitudinal mobility, we choose the pole and the origin of the sectorial area of the shell in the fixed point C, which is the center of rotation of the cross section of the shell. We shall choose the pole of the sectorial area for the hipped shell at the point C_2, its center of rotation (Figure 97a).

Since our system has one degree of freedom in the plane of the cross section, the diagrams of the sectorial areas of the hipped and round shells are subjected to additional constraints and, consequently, the angle of rotation of the cross section of the hipped shell θ_2 is determined by the angle of rotation θ_1 of the round shell cross section (Figure 97a). In the following we use the deflection v of the cylindrical middle hinge instead of the generalized transverse displacement (i. e., that of any point of a section, $z=$const of our system). In this case, we shall obviously obtain the following equations for the angles of rotation θ_1 and θ_2,

$$\theta_1 = \frac{v}{a}, \quad \theta_2 = \frac{v}{b},$$

where a and b are the chords of the round and the hipped shells, respectively.

By determining the relation between the angles of rotation θ_1 and θ_2 it is easy to construct the diagram of the longitudinal displacements in the cross section of our system. This diagram for round shell section will vary as the function $\varphi_1(s)=\frac{\omega_1(s)}{a}$, where $\omega_1(s)$ is the sectorial area taken within the boundaries of the shell; the pole and the origin of this area are situated at C. The diagram of the longitudinal displacements of the hipped shell section will vary as the function $\varphi_2(s)=\frac{\omega_2(s)}{b}$, where $\omega_2(s)$ is the diagram of the sectorial area for the hipped shell. The pole of this area is situated at C_y and the origin M (Figure 97b) is found from the condition that the ordinates of the diagrams of φ_1 and φ_2 at the cylindrical middle hinge C_1 are equal, since the round and the hipped shells are fixed to each other along the longitudinal line C_1C_1. The distribution of the longitudinal displacements $\varphi(s)$ of the system as a whole is shown in Figure 97b*.

We now proceed to derive the differential equation of equilibrium of an elementary transverse strip, isolated from our system. For this purpose, we apply Lagrange's principle of virtual displacements. According to this principle the sum of the work of all the forces acting on an elementary transverse strip vanishes for an arbitrary virtual displacement of this strip.

As virtual displacement of the strip we choose a unit deflection of the cylindrical middle hinge, $v=1$. The work done in this deflection by the shear forces exerted by the "removed" parts of the system on the transverse strip of unit width is

$$A=\int\limits_{S_1}\frac{\partial(\tau\delta)}{\partial z}\theta_1 h_1\,ds+\int\limits_{S_2}\frac{\partial(\tau\delta)}{\partial z}\theta_2 h_2\,ds=$$
$$=\int\limits_{S_1}\frac{1}{a}\frac{\partial(\tau\delta)}{\partial z}d\omega_1+\int\limits_{S_2}\frac{1}{b}\frac{\partial(\tau\delta)}{\partial z}d\omega_2\,. \qquad (11.3)$$

In this expression, the first integral is evaluated over the contour line of the round shell and the second over the contour line of the hipped shell, $h_1(s)$ is the length of the perpendicular from the point C to the tangent to the round shell at the given point on the contour line, $h_2(s)$ is the length of the perpendicular from the sectorial center of the hipped shell (the point M) to the tangent to its contour at the given point.

Integrating by parts, we transform (11.3) as follows

$$A=\left|\frac{1}{a}\frac{\partial(\tau\delta)}{\partial z}\omega_1\right|_{C}^{C_1}-\int\frac{1}{a}\frac{\partial^2(\tau\delta)}{\partial s\,\partial z}\omega_1\,ds+\left|\frac{1}{b}\frac{\partial(\tau\delta)}{\partial z}\omega_2\right|_{C_1}^{C_2}-\int\frac{1}{b}\frac{\partial^2(\tau\delta)}{\partial s\,\partial z}\omega_2\,ds. \qquad (11.4)$$

In this expression the integrated terms vanish since at the point C the sectorial area ω_1 is equal to zero. At the point C_1 the distribution of the longitudinal displacements is continuous and at the point C_2 there are no longitudinal shear forces.

With the help of equation (I.5.10) we write the virtual work of the shear forces (11.4) as follows

$$A=\int\frac{1}{a}\frac{\partial^2\sigma}{\partial z^2}\omega_1\,dF+\int\frac{1}{b}\frac{\partial^2\sigma}{\partial z^2}\omega_2\,dF. \qquad (11.5)$$

* We draw attention to the fact that a positive sectorial area corresponds to a common sense of rotation of the variable radius vector and of the torsional angle θ.

Since on that part of the contour which belongs to the cylindrical shell the longitudinal normal stresses vary according to the law

$$\sigma(z, s) = -E\vartheta_1''(z)\,\omega_1(s),$$

and on the part which belongs to the prismatical shell according to

$$\sigma(z, s) = -E\vartheta_2''(z)\,\omega_2(s),$$

it follows that, on substituting these expressions in equation (11.5), we finally obtain

$$A = -EJ_\varphi v^{IV}(z), \tag{11.6}$$

where the constant J_φ is determined by the equation

$$J_\varphi = \int \varphi^2\,ds = \int \frac{1}{a^2}\,\omega_1^2\,dF + \int \frac{1}{b^2}\,\omega_2^2\,dF.$$

We noted earlier that for prismatical and cylindrical systems possessing a rigid contour of the cross section the torsional rigidity GJ_d can be taken as equal to zero. It remains, therefore, to determine the work of the external transverse load exerted on the isolated transverse strip of unit width in the deflection $v=1$ of the cylindrical middle hinge.

We assume that vertical concentrated and distributed loads are acting on this strip. We shall consider that in the longitudinal direction any one of these loads is given by a known function of z (Figure 98).

For the virtual work of these loads, denoted by $m(z)$, we obtain the expression

$$m(z) = \frac{1}{a}\left[\sum_{i=1}^{k} p_i(z)\,s_i + \int p(z,s)\,ds\right] + \\ + \frac{1}{b}\left[\sum_{j=1}^{m} p_j'(z)\,s_j' + \int p'(z,s')\,ds'\right]. \tag{11.7}$$

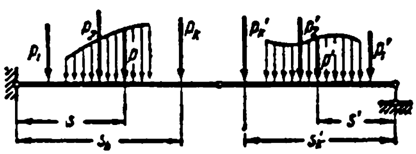


Figure 98

In this equation $p_i(z)$ are the external load distributions, concentrated in this direction and applied at the point s_i of the cross section $z=$const of the shell (Figure 98); $p(z, s)$ is the external surface load distributed over the shell (the coordinate s is taken from the fixed point C) $p_j'(z)$ are the external load distributions concentrated in the transverse direction and applied at the point s_j' of the cross section $z=$const of the hipped shell; $p'(z, s')$ is the external surface load distributed over the hipped shell (the coordinate s' is measured from the point C_2).

Adding expressions (11.6) and (11.7), we obtain the following differential equation for the generalized deflection $v(z)$:

$$EJ_\varphi v^{IV}(z) = m(z). \tag{11.8}$$

Using this equation to find the required function $v(z)$ for the given boundary conditions at the transverse ends, we can apply equations (11.2) to determine the distribution of the longitudinal normal and tangential stresses in the system. For that purpose we determine the bimoment B

by the equation

$$B = -EJ_{\varphi}v''(z).$$

In solving our problem we assume the torsional rigidity $GJ_d = 0$. If the transverse dimensions of our system are such that the torsional moment H_k has to be considered in the section $z = \text{const}$, it is necessary to add to the left-hand side of equation (11.8) a term which expresses the virtual work done in a deflection $v = 1$ by the torsional moments of the system acting on a transverse strip of unit width.

We find the following value for this work

$$A' = G\left(\frac{1}{a^2}J'_d + \frac{1}{b^2}J''_d\right)v''(z) = GJ_dv''(z), \tag{11.9}$$

where J'_d and J''_d represent the moments of inertia for pure torsion of the cylindrical and hipped shells, respectively.

The differential equation (11.8) is then

$$EJ_{\varphi}v^{IV}(z) - GJ_dv''(z) = m(z). \tag{11.10}$$

3. We shall now examine other forms of fixing the longitudinal edges of the investigated system. We note that the whole analysis remains unchanged except for the determination of the sectorial origin for the cylindrical and hipped shells.

Thus, for example, when both the lateral hinged edges are free to move longitudinally, and the cylindrical middle hinge eliminates longitudinal displacements of the hipped shell relative to the cylindrical shell, the sectorial origin for the cylindrical part will be found from the condition

$$\int \varphi\, ds = 0,$$

where the integral is taken over the entire cross section of our system. The diagram $\varphi(s)$ should then be continuous at the point C_1 (Figure 99a).

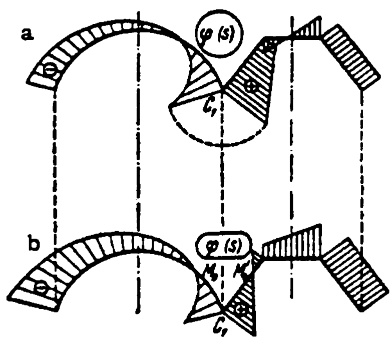


Figure 99

If we assume that not only the lateral edge but also the cylindrical middle hinge do not have shear forces (the hipped and cylindrical shells can move relative to one another along the line C_1C_1), the sectorial origins for the parts CC_1 and C_1C_2 are correspondingly found from the conditions

$$\int \varphi_1 ds = 0, \quad \int \varphi_2 ds = 0.$$

Here the first integral is taken over the boundary of the cylinder and the second over the boundary of the hipped shell. It is obvious that in this case the longitudinal displacements, which vary as $\varphi(s)$ in the cross section, have a discontinuity on the longitudinal line C_1C_1 (Figure 99b).

4. Since equations (11.10), or in the simpler case (11.8), are identical in form with the corresponding equations of the theory of thin-walled beams, it follows that the previously described methods for the design of these single or multiple-span beams can be completely extended to include three-dimensional systems of the type examined here. In particular, the method

of mathematical analogy with the classical theory of bending, as described in § 9, can be successfully used. It is only necessary to keep in mind that, to adapt ourselves to the case of complex thin-walled systems possessing a single degree of freedom in the cross section, the basic geometrical and statical quantities are understood to be the generalized displacements and the generalized internal forces which obey the law of distribution of the displacements of points in the cross section and the law section of warping adopted for the given system. We illustrate the above by an example. We assume that the shell shown in Figure 96 is simply supported at the transverse ends where the deflections and longitudinal normal strains vanish. Let such a shell be subjected to the action of a uniform load distribution $q=\text{const}$ over its length. To solve the problem we can use, by mathematical analogy, the available equations (5.9) and (5.10), replacing in these equations θ and θ' by v and v' respectively, J_ω by J_φ and determining the quantities J_d and m from equations (11.9) and (11.7).

5. If the system shown in Figure 96 has at the junction of the two shells not a hinge but a rigid connection, its deformations occur as a result of the transverse deflection of the shells. Such a deflection will correspond to elastic transverse bending moments which play an essential role in the author's theory of cylindrical and prismatical shells of intermediate length /51/. Maintaining for such a system the law of warping as adopted above, and employing the additional condition of continuity of the angular deformation in the intermediate point C_φ we obtain for the generalized deflection $v=v(z)$ the equation

$$EJ_\varphi v^{IV} - GJ_d v'' + kv = m. \tag{11.11}$$

Here the subsidiary elastic characteristic k is calculated by the equation:

$$k=\left(\frac{1}{a}+\frac{1}{b}\right)^2\frac{1}{\delta_{11}},$$

in which δ_{11} is the relative angular displacement at the hinge C_1 due to a unit moment. This displacement is determined by the usual methods of structural mechanics for frames of unit width.

Equation (11.11) leads to the equation proposed by the author for the general theory of elastic beams, plates and shells with two generalized elastic characteristics:

$$v^{IV} - 2r^2v'' + s^4v = \frac{1}{EJ_\varphi}m.$$

The solution of this equation in elementary functions by the method of initial parameters is given in /51/.

§ 12. Flexural torsion of a cylindrical shell with a long rectangular cutout (approximate solution)

1. We shall examine a structure of the cylindrical or prismatical shell type, weakened in its middle part by one or several considerably long rectangular cutouts. We shall assume the shell to be fixed in the longitudinal direction by stringers and plates and in the transverse direction by

frames. Examples of such structures are the fuselage of a plane, the body of a ship, roofs of the arch-shell type with skylights at the top. Without reducing the general character of the method described below, we shall consider the examined three-dimensional shell to have two planes of symmetry: a longitudinal-vertical plane and a transverse-vertical plane. With a single longitudinal cutout, such a structure consists of three parts: one middle shell of length $2l$ of open section and two terminal shells of length a each of closed section (Figure 100).

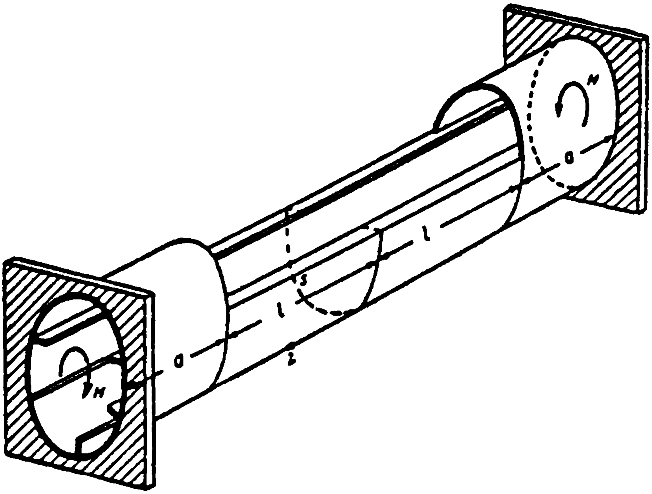


Figure 100

We measure the coordinate z from the middle cross section which coincides with the plane of symmetry.

We assume that the shell has plate stiffeners at the transverse ends $z = \pm(l + a)$ which restrain there the deformation of the contour and the warping of the section. Such plates, together with the intermediate frames, bring the examined shell closer to a three-dimensional system of the thin-walled beam type with a longitudinally varying cross section.

2. Suppose the shell to be subjected to the action of the torsional moments H, applied in the planes of the supported sections. Such a load causes a flexural-torsional deformation in the shell, associated with warping of the cross sections.

In the middle part (the contour of the shell is shown in Figure 101a) the flexural-torsional deformation obeys the law of sectorial areas with origin at the contour point of the axis of symmetry. The point A is the pole of these areas. It also lies on the axis of symmetry. The position of the point A is determined from the orthogonality condition of the functions $x = x(s)$ and $\omega = \omega(s)$.

The diagram of the sectorial areas is constructed for the whole section in the middle part of the beam including the longitudinal stringers and plates (Figure 101b). The geometrical characteristics of the section of this part of the beam are also calculated with respect to the principal coordinates for all the parts of the section. This is done as explained in sub-section 7 of § 1.

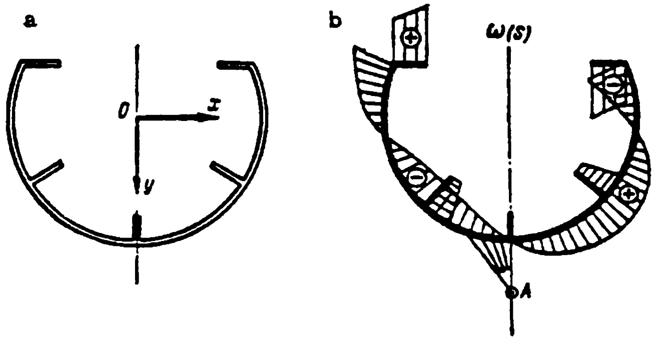


Figure 101

All the basic design quantities for flexural torsion for the cutout part $-l \leqslant 0 \leqslant l$ can be determined by the general equations of the method of initial parameters given in Table 3 of § 3. Assuming that in the initial section $z=0$, $\theta_0=0$, $B_0=0$ and noting the fact that the torsional moment H in the examined case remains constant along the beam, we obtain

$$\left.\begin{aligned} \theta &= \theta_0' \frac{l}{k}\,\mathrm{sh}\,\frac{k}{l} z + \frac{H}{GJ_d}\left(z - \frac{l}{k}\,\mathrm{sh}\,\frac{k}{l} z\right), \\ \theta' &= \theta_0'\,\mathrm{ch}\,\frac{k}{l} z + \frac{H}{GJ_d}\left(1 - \mathrm{ch}\,\frac{k}{l} z\right), \\ \frac{k}{l} B &= -\theta_0' GJ_d\,\mathrm{sh}\,\frac{k}{l} z + H\,\mathrm{sh}\,\frac{k}{l} z. \end{aligned}\right\} \tag{12.1}$$

Here the quantity θ_0' which expresses the absolute warping in the initial section $z=0$, remains undetermined.

Thus, the functions $\theta(z)$, $\theta'(z)$, $B(z)$ are determined for the middle part, up to a constant parameter θ_0', by equations (12.1) for a given generalized beam characteristic as determined by equation (2.2), and for a given torsional moment H. The longitudinal normal stresses $\sigma=\sigma(z, s)$ and the longitudinal displacements $u=u(z, s)$ for flexural torsion caused by the torsional moments at the supports are determined from equations (I.3.16) and (I.5.6)

$$u = -\theta'\omega, \quad \sigma = -E\theta''\omega. \tag{12.2}$$

The structure under consideration consists of a closed cylindrical short shell at each of the end sections. Since there is a rigid plate at the end of the section, which restrains the deformation of the contour and the warping of the section, such a shell can be considered, in the case of torsion, as a moment-less system [like a membrane], i. e., we can assume that the normal and tangential stresses are uniformly distributed across the wall. The normal stresses which refer to the areas of the longitudinal section vanish in the absence of a surface load*.

The shear forces T can be determined approximately by the equation for pure torsion. Taking these forces to be constant (not dependent on the coordinates z and s) and equating the torsional moment due to these forces

* It is easy to reduce these stresses to zero, by evaluating the sum of the projections, on the normal to the surface, of all the forces applied to an infinitely small surface element $dz\,ds$.

with the given torsional moment H of the shell in any cross section, we obtain

$$\oint Th\,ds = H, \tag{12.4}$$

where h is the perpendicular on the tangent to the contour line.

From equation (12.3) we have

$$T = \frac{H}{\Omega},$$

where Ω is twice the area inside the contour of the shell.

From the equilibrium equation

$$\frac{\partial(\sigma\delta)}{\partial z} + \frac{\partial T}{\partial s} = 0$$

it is possible to obtain a simple equation for the longitudinal normal stress $\sigma = \sigma(s)$ showing that this stress does not depend on the coordinate z.

From Hooke's law $\sigma = E\frac{\partial u}{\partial z}$ we obtain an equation for the longitudinal displacement $u = u(z, s)$ for the rigidly supported section of the shell at $z = l + a$:

$$u = -\frac{\sigma(s)(l + a - z)}{E}. \tag{12.3}$$

The minus sign in the equation indicates that in the part $l \leqslant z \leqslant l + a$ the displacement μ will be negative, for a positive tensile stress and for a stiff edge at $z = l + a$ which shows that the point is displaced in the direction of the negative Oz.

From equation (12.4) we obtain for $z = l$

$$\sigma = -\frac{E}{a}u. \tag{12.5}$$

This equation approximately determines the proportionality between the stresses σ and the displacements u at points of the end adjoining the middle shell. The equation (12.5) recalls Winkler's hypothesis in the elementary bending theory of a beam on an elastic foundation.

Given equations (12.1), (12.2), (12.4), we can develop the condition for an elastic articulation of the shells in the plane of their contact $z = l$. For longitudinal displacements, the conditions of continuity at the points of the line $z = l$, along the contact boundary leads to the equation

$$u_k = u_c,$$

where u_k and u_c are the longitudinal displacements referring to the extreme and middle shell respectively.

By determining u_k in equation (12.5) from (12.2), we obtain

$$\sigma_k = \frac{E\vartheta'\omega}{a}. \tag{12.6}$$

We now separate an arbitrarily small transverse strip in the neighborhood of the contact (boundary) line $z = l$ and apply to this strip longitudinal normal forces $\sigma_k\delta_k$ and $\sigma_c\delta_c$, which act from the side of the extreme and middle shell, respectively.

In accordance with the basic propositions of the theory of thin-walled

beams, which result from the method of variation of constants and the law of sectorial areas, we have to present the equilibrium condition of the elementary transverse strip at the contact zone in the form of equations for the work done by all longitudinal forces in virtual displacements of the strip, determined for flexural torsion according to the particular law of sectorial warping. This condition is written

$$\int_F (\sigma_c \delta_c - \sigma_k \delta_k) \omega \, ds = 0, \tag{12.7}$$

where δ_k and δ_c are the thickness of the shell elements at the extreme and middle parts, respectively.

By substituting the stresses, determined for $z = l$ by equations (12.2) and (12.6), in equation (12.7) and performing suitable transformations, we obtain

$$\theta'' + \frac{\nu}{a} \theta' = 0. \tag{12.8}$$

The dimensionless quantity ν in this equation is calculated from

$$\nu = \frac{\int_{F_k} \omega^2 \, dF_k}{\int_{F_c} \omega^2 \, dF_c} = \frac{J_{\omega k}}{J_{\omega c}}.$$

Here the numerator is the sectorial moment of inertia $J_{\omega k}$ which refers to the extreme shell. This moment should be calculated for the section at the junction of the extreme shell with the middle one, taking into consideration the areas of the plates and of the stringers of the extreme shell. The denominator is the sectorial moment of inertia of the contour of the middle shell. If the cross sections of the extreme and intermediate shells are such that the moments of inertia $J_{\omega k}$ and $J_{\omega c}$ are equal at the junction $\nu = 1$.

Applying the generalized boundary condition (12.8) with the help of equations (12.1), where it is necessary to assume $z = l$, and by taking into consideration equation (2.2), we obtain after simple transformations

$$\theta_0' = \frac{Hl^2}{EJ_\omega k^2} (1 - k_1 \nu),$$

where the dimensionless coefficient k_1 is calculated from

$$k_1 = \frac{1}{a \left(\frac{k}{l} \operatorname{sh} k + \frac{\nu}{a} \operatorname{ch} k \right)}. \tag{12.9}$$

By now substituting the quantity θ_0' found in the first of equations (12.1), we obtain

$$\theta(z) = \frac{Hl^2}{EJ_\omega k^2} \left(z - k_1 \nu \frac{l}{k} \operatorname{sh} \frac{k}{l} z \right). \tag{12.10}$$

This general equation determines the torsion angle of the middle shell in any cross section $z = \text{const}$.

The equation of the bimoments (2.5) for this part will be

$$B = -EJ_\omega \theta'' = Hk_1 \nu \frac{l}{k} \operatorname{sh} \frac{k}{l} z.$$

It is seen from this equation that the bimoment is maximal for $z=l$ i. e., on the line where middle shell joins the terminal shell. The diagram of the bimoments, like that of the torsion angles, for the middle part $l \geq z \geq -l$ will be skew-symmetrical with respect to the origin of the coordinate z.

After determining the bimoments, the normal stresses in the examined case of flexu al torsion of a shell are calculated from the one-term equation

$$\sigma=\frac{B(z)}{J_{\omega}}\omega(s).$$

The absolute values of these stresses are maximal at the supported sections $z=\pm l$ of the middle shell in those points, for which the ordinates of the principal sectorial area $\omega(s)$ are maximal (Figure 101b).

3. The equations for the torsion angle and for the bimoment given above are obtained by considering the rigidity GJ_d for a pure torsion of the shell in the cutout part, i. e. taking into consideration, as repeatedly mentioned before, the nonuniform distribution of the tangential stresses across the shell wall. Investigation shows that for a thin open shell the rigidity in pure torsion is of secondary importance. It is possible to equate this rigidity for such a shell to zero, which is equivalent to the assumption that the tangential stresses are uniformly distributed across the wall.

The equations of pure torsion for an open shell of zero rigidity can be obtained from the equations given above by passing to the limit.

We shall derive, as an example, the equation for the torsion angle $\theta(z)$. Assuming that for $GJ_d=0$ the coefficient k also equals zero, and by expanding the hyperbolic functions in a series and neglecting terms of higher than second order, we obtain from expressions (12.9) and (12.10)

$$k_1=\frac{1}{\nu\left[1+\frac{ak^2}{\nu}\left(\frac{1}{l}+\frac{\nu}{2a}\right)\right]}=\frac{1}{\nu}\left[1-\frac{ak^2}{\nu}\left(\frac{1}{l}+\frac{\nu}{2a}\right)\right],$$

$$\lim_{k\to 0}\theta(z)=\lim_{k\to 0}\frac{Hl^2}{EJ_{\omega}k^2}\left\{z-\left[1-\frac{ak^2}{\nu}\left(\frac{1}{l}+\frac{\nu}{2a}\right)\right]\left(1+\frac{k^2z^2}{6l^2}\right)z\right\}=$$

$$=\frac{Hl^2}{EJ_{\omega}}\left[\frac{a}{\nu}\left(\frac{1}{l}+\frac{\nu}{2a}\right)-\frac{z^2}{6l^2}\right]z.$$

In this particular case the bimoment $B(z)$ will be

$$B(z)=-EJ_{\omega}\theta''(z)=Hz.$$

§ 13. Experimental confirmation of the theory of thin-walled beams

In 1938-39 the TsNIPS laboratories of structural engineering conducted experiments on thin-walled metal beams with open sections in order to veryif the validity of the four-term equation (I.8.5), proposed by the author for normal stresses under simultaneous flexure and torsion,

$$\sigma=\frac{N}{F}-\frac{M_y}{J_y}x+\frac{M_x}{J_x}y+\frac{B}{J_{\omega}}\omega.$$

We shall examine the results of experiments on a welded channel

beam under simultaneous flexure and torsion. The beam was made of a sheet of St. 3 steel, $\delta = 5$ mm, with a span $l = 2{,}700$ mm. Due to the special arrangement of the supports and the absence of stiffening transverse plates, the section was free to rotate about its vertical and horizontal axes and the separate points of the sections could be freely displaced longitudinally (warping of the sections) but there could be no twisting at the supports. A transverse load was applied centrally at the middle of the span with different eccentricities with respect to the shear center.

Measuring instruments (lever strain gauges) were placed on two sections, located 101 cm from the supports. Sixteen gauges were placed around the contour of each of the two sections. Four strain gauges were placed over the width of each of the three principal elements of the beam. By means of these strain gauges distribution of stress over the width of each element and over the section as a whole was found.

The physical characteristics of the material of the specimen: The modulus of elasticity E, the shear modulus G and Poisson's ratio μ were determined earlier in the laboratory for testing materials of Moscow State University. Their values were:

$$E = 2.15 \cdot 10^6 \frac{\text{kg}}{\text{cm}^2}; \quad G = 0.815 \cdot 10^6 \frac{\text{kg}}{\text{cm}^2}; \quad \mu = 0.334.$$

The measured deformations were processed by the method of least squares and were separated into four components, corresponding to the four forms of action on the beam: vertical bending and torsion (principal stresses), horizontal bending and longitudinal extension (complementary stresses). Figure 102 shows the stresses over the section of the specimen for one case of central and three cases of eccentric load application, with eccentricities $e = 2.29$ cm; $e = 4.66$ cm and $e = 6.82$ cm. The graphs give:

a) results obtained from experiment;

b) results obtained after processing the experimental data by the method of least squares;

c) stresses obtained by theoretical calculations.

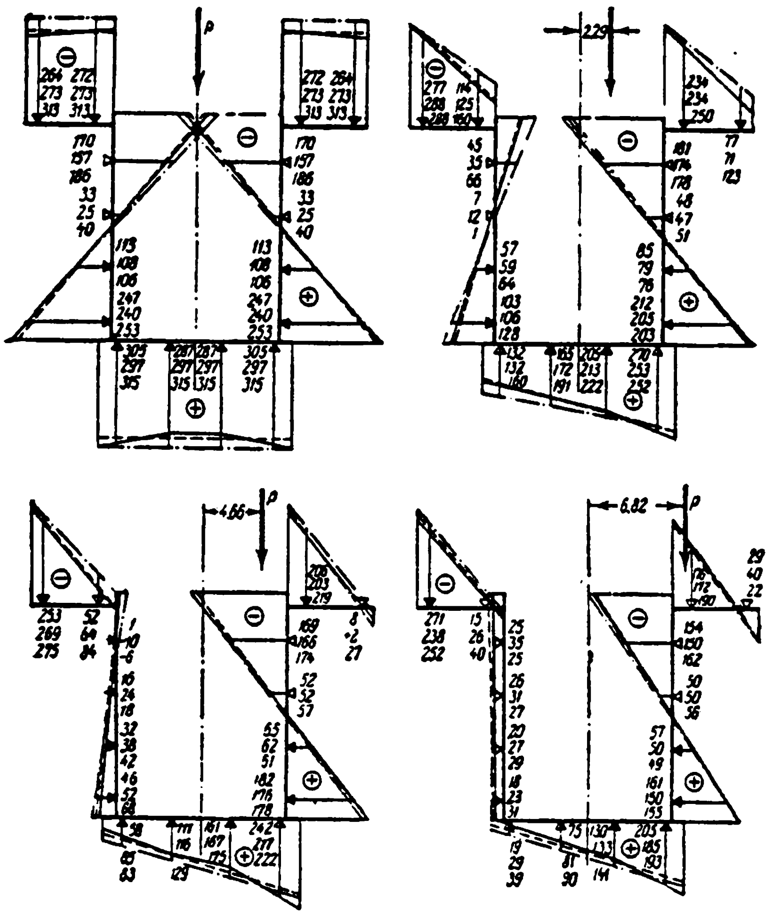
The rather close agreement between the experimental and theoretical curves shows that, for simultaneous bending and torsion, the normal stresses in the cross section of thin-walled girder-beams are distributed according to the law of sectorial areas.

Those interested in more details of the experimental set-up and the results should consult the references /27, 28/.

In addition to the processing of the results of experiments carried out directly in the TsNIPS laboratories, the results of experiments conducted by Bach in the years 1909-10 /211/, were processed with the help of the four-term equation developed by the author.

Carrying out experiments on a metal beam of channel section (Figure 103a), Bach established that a transverse load, acting perpendicularly to the plane of symmetry of the channel and passing through its centroid, produces not only flexure but also torsional deformations. For an arbitrary position of the load, the elongations of four extreme fibers of the channel do not obey the law of plane sections.

Under the action of a transverse load in the plane of the web of the channel (Figure 103b) the torsional deformations in Bach's experiments turned out to be considerably less than for the case of the load being applied in the section centroid (Figure 103c) (elongations in the wall of the channel were not measured).



experimental data (upper figures of vertical triads)
same, processed by the method of least squares (middle figures)
theoretical values (lower figures)

Figure 102

However, observing experimentally the deviation from the law of plane sections under the action of a transverse load which does not pass through the shear center, Bach did not give an exhaustive explanation of this phenomenon, ascribing it to the asymmetry of the section.

The results of Bach's experiments, computed for stresses in kg/cm^2 are given in Table 21. Tests I refer to the case of the load passing through the middle surface of the channel wall (Figure 103b). Tests II refer to the case when the load passes through the centroid of the section. In the same table are also given, for comparison, stress values obtained in Weber's investigations and by the author's theory.

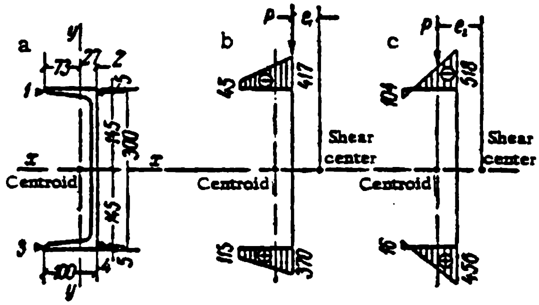


Figure 103

Table 21

Test	I				II			
Point No	1	2	3	4	1	2	3	4
Bach's data	+205.7	-108.7	-205.7	+108.7	+357.9	-189.0	-357.9	+189.0
Weber's data	+200.0	-123.0	-200.0	+123.0	+337.0	-207.0	-337.0	+207.0
By the author's theory	+201.0	-106.2	-201.0	+106.2	+337.0	-178.0	-337.0	+178.0

The good agreement between the theoretical results obtained by the author and the experimental data shows that evaluation by the four-term equation of this type of experiment gives correct results.

§ 14. Calculation of beams, allowing for longitudinal bending moments

The theory expounded above refers to thin-walled beams of the long shell type which are very thin. This theory is based on the assumption that the normal stresses are uniformly distributed across the wall. This assumption is equivalent to neglecting the longitudinal bending moments in view of their insignificant role under the general stress conditions of a thin-walled beam. Such an assumption forms also the basis of the author's general technical theory of shells and three-dimensional hipped systems and allows for considerable simplification in their design methods. Many theoretical and experimental investigations on thin cylindrical and prismatic shells of intermediate length, conducted by the author himself and by other workers, served as the basis for this assumption.

However, the field of application of this theory (as every theory which is based on such assumptions) cannot be clearly defined, since it depends on many conditions. The concept of a "thin-walled beam" is sometimes ambiguous and depends on the nature of the problem, the purpose of the calculation, the nature of the loads, etc.

The basic factor in our theory is the calculation of the warping of the section and its influence on the behavior of the beam. In the theory of

thin-walled beams of open section this appears in the form of the law of sectorial areas which is a generalization of the law of plane sections. In the theory of thin-walled beams of closed section, as well as in the theory of the warping of solid beams, as will be shown in § 1 of Chapter IV and in § 1 of Chapter V, the law of sectorial areas is not applicable, and the influence of the deformations is taken into account not by the generalized sectorial coordinate $\omega(s)$, but by another generalized coordinate of warping $\varphi(x, y) = xy$ There are cases in which the generalized warping coordinate $\varphi(x, y)$ will also have another analytical expression.

It will be shown below how the expressions of the generalized coordinates in the cross section of an elastic beam of open section should be changed in order to take into account the influence of longitudinal bending moments or, which is the same, to consider the influence of longitudinal normal stresses nonuniformly distributed across the beam wall. The longitudinal bending moment per unit length is given, as known from the theory of plates, by the following equation

$$M_1 = -\frac{E\delta^3}{12}\left(\frac{\partial^2 w}{\partial z^2} + \mu\frac{\partial^2 w}{\partial s^2}\right).$$

Here the expression $\frac{\partial^2 w}{\partial s^2}$ stands for the change of the curvature of the beam in the transverse direction. For thin-walled beams this change of the curvature is equal to zero, since the contour of the beam is considered to be absolutely rigid; therefore

$$M_1 = -\frac{E\delta^3}{12}\frac{\partial^2 w}{\partial z^2}.$$

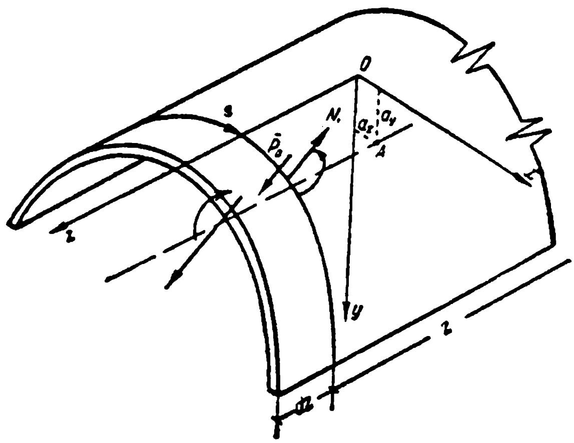


Figure 104

For the normal component of the total displacement vector we have equation (I.3.10),

$$w(z, s) = -\xi(z)\sin\alpha(s) + \eta(z)\cos\alpha(s) + \theta(z)\,t(s),$$

where $\alpha(s)$ is the angle between the tangent to the contour line at a given point (s). and the axis Ox. The perpendicular drawn from an arbitrary sectorial center A to the normal to the profile line at the given point (s)

is denoted by $t(s)$; ξ and η are the translational displacements of the point A; θ is the angle of rotation of the section in its plane with respect to the point A.

Thus

$$M_1(z, s) = -\frac{E\delta^3(s)}{12}\left[-\xi''(z)\sin\alpha(s) + \eta''(z)\cos\alpha(s) + \theta''(z)\,t(s)\right].$$

The transverse force per unit length N_1 connected with this moment, will be

$$N_1 = \frac{\partial M_1}{dz} = -\frac{E\delta^3(s)}{12}\left[-\xi'''(z)\sin\alpha(s) + \eta'''(z)\cos\alpha(s) + \theta'''(z)\,t(s)\right]^{*}).$$

In order to take into account the influence of the longitudinal bending moments on the behavior of a thin-walled beam, we have to introduce correction terms in the differential equation (I.6.14), resulting from the projection of the transverse force N_1 on the corresponding axis. This is equivalent to introducing a certain additional normal surface load $\bar{p}_n$, which acts on a strip of unit width $dz = 1$ and which is equal to the difference of the transverse forces applied at the sections $z = \text{const}$ and $z + dz = \text{const}$ (Figure 104)

$$\bar{p}_n = \frac{\partial N_1}{\partial z} = -\frac{E\delta^3(s)}{12}\left[-\xi^{\mathrm{IV}}(z)\sin\alpha(s) + \eta^{\mathrm{IV}}(z)\cos\alpha(s) + \theta^{\mathrm{IV}}(z)\,t(s)\right].$$

It is clear that $\bar{p}_n$ has no component along the Oz axis. Therefore, the first equation of (I.6.14) remains unchanged. The additional terms are obtained from the three last equations of (I.6.14).

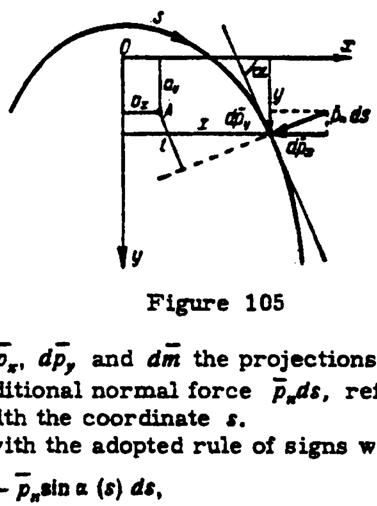


Figure 105

We denote by $d\bar{p}_x$, $d\bar{p}_y$ and $d\bar{m}$ the projections on the axes x, y and the moment of the additional normal force $\bar{p}_n ds$, referred to an arbitrary point of the section with the coordinate s.

In accordance with the adopted rule of signs we have (Figure 105)

$$\begin{aligned} d\bar{p}_x &= -\bar{p}_n \sin\alpha(s)\,ds, \\ d\bar{p}_y &= \bar{p}_n \cos\alpha(s)\,ds, \\ d\bar{m} &= d\bar{p}_y(x - a_x) - d\bar{p}_x(y - a_y) = \\ &= \bar{p}_n\left[(x - a_x)\cos\alpha + (y - a_y)\sin\alpha\right]ds = \bar{p}_n t\,ds. \end{aligned}$$

* The equation for the transverse force N_1, known from the theory of plates, has the following form $N_1 = \frac{\partial M_1}{\partial z} + \frac{\partial H}{\partial s}$, where H is the torsional moment, calculated by the equation $H = \frac{E\delta^3}{12}(1-\mu)\frac{\partial^2 w}{\partial z\,\partial s}$. In the case of thin-walled beams $\frac{\partial H}{\partial s} = 0$ since the derivative of the torsional moment with respect to s includes the distortion of the curvature of the beam contour $\frac{\partial^2 w}{\partial s^2}$.

By passing to the generalized forces, and by integrating these expressions over the whole contour of the cross section, we obtain for the additional terms of the second, third and fourth equation of the system (I.6.14) the corresponding expressions:

$$\begin{aligned}\bar{p}_x &= -E\xi^{IV}\int \sin^2\alpha(s)\,dJ + E\eta^{IV}\int \sin\alpha(s)\cos\alpha(s)\,dJ + \\ &\qquad + E\theta^{IV}\int t(s)\sin\alpha(s)\,dJ,\\ \bar{p}_y &= E\xi^{IV}\int \sin\alpha(s)\cos\alpha(s)\,dJ - E\eta^{IV}\int \cos^2\alpha(s)\,dJ - \\ &\qquad - E\theta^{IV}\int t(s)\cos\alpha(s)\,dJ,\\ \bar{m} &= E\xi^{IV}\int t(s)\sin\alpha(s)\,dJ - E\eta^{IV}\int t(s)\cos\alpha(s)\,dJ - E\theta^{IV}\int t^2(s)\,dJ.\end{aligned}$$

Including these additional terms in the corresponding equations of (I.6.14) and combining the coefficients of ξ^{IV}, η^{IV} and θ^{IV}, we obtain the same equations. Here, however, the geometrical characteristics J_x, J_y, J_{xy} and J_ω, $J_{\omega x}$, $J_{\omega y}$ will be determined by equations differing from the corresponding ones (I.6.9)-(I.6.13) by the presence of additional terms which account for the influence of the longitudinal bending moment, i. e.,

$$\left.\begin{aligned}J_x &= \int y^2(s)\,dF + \int \cos^2\alpha(s)\,dJ,\\ J_y &= \int x^2(s)\,dF + \int \sin^2\alpha(s)\,dJ,\\ J_{xy} &= \int x(s)y(s)\,dF - \int \sin\alpha(s)\cos\alpha(s)\,dJ,\\ J_{\omega x} &= \int \omega(s)x(s)\,dF - \int t(s)\sin\alpha(s)\,dJ,\\ J_{\omega y} &= \int \omega(s)y(s)\,dF + \int t(s)\cos\alpha(s)\,dJ,\\ J_\omega &= \int \omega^2(s)\,dF + \int t^2(s)\,dJ.\end{aligned}\right\} \qquad (14.1)$$

In these equations the following notations are adopted:

$$dF = \delta\,ds, \quad dJ = \frac{\delta^3}{12}\,ds = \frac{\delta^2}{12}\,dF,$$

where the thickness is taken as a function of the variable s, $\delta = \delta(s)$. The integrals are evaluated over the whole cross section.

We can obtain the same results by starting from other considerations. We assume that the longitudinal normal stresses vary linearly across the wall (Figure 106a, b). The position of an arbitrary point M of the cross section will now be a function of two variables: the coordinate s, measured along the contour line from some origin to the point M_1 and of the distance n measured from the contour line along the normal (Figure 106a).

We have here essentially the phenomenon examined in § 7, namely, the case of a longitudinal force not directly applied at some point of the contour line but applied outside the contour, its action being conveyed to the beam through a rigid arm normal to the contour line. In this case we should consider the normal to any point of the contour line as an arm, extended on both sides of the contour line. The length of the arm varies from zero to half the thickness of the beam at the given point, and each point of the arm is considered as a point of application of a longitudinal force.

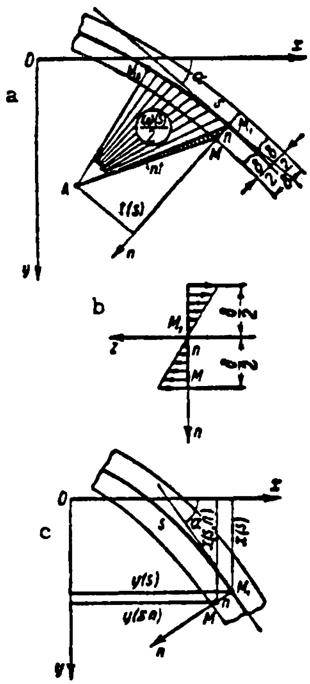


Figure 106

We denote by $x(s, n)$, $y(s, n)$, $\omega(s, n)$ respectively the generalized linear and sectorial coordinates of an arbitrary points (s, n) of the cross section of the beam. By $x(s)$, $y(s)$ and $\omega(s)$ we denote the same coordinates which refer, however, to the point s of the profile line. From Figure 106a, bewe find the following relations:

$$\left.\begin{aligned} x(s, n) &= x(s) - n\sin\alpha(s),\\ y(s, n) &= y(s) + n\cos\alpha(s),\\ \omega(s, n) &= \omega(s) + nt(s). \end{aligned}\right\} \qquad (14.2)$$

The functions $\sin\alpha(s)$ and $\cos\alpha(s)$ can be presented differently, i.e., as derivatives with respect to the variable s of the corresponding coordinates:

$$\sin\alpha(s) = \frac{dy(s)}{ds} = y',$$

$$\cos\alpha(s) = \frac{dx(s)}{ds} = x'.$$

Since the generalized coordinates are functions of the two variables s and n, the geometrical characteristics are expressed by double integrals, in which $ds\,dn$ represents the element of area of the cross section, the integration being performed over the whole area of the cross section and the limits of integration being, with respect to the variable n, from $-\frac{\delta}{2}$ to $+\frac{\delta}{2}$ ($\delta = \delta(s)$ is the beam thickness). The integration with respect to the variable s is performed over the whole contour of the cross section. In view of the fact that the variable n appears in the expression of the generalized coordinates in a very simple linear form, as seen from (14.2), we can transform the double integrals to a contour integral by performing the integration with respect to n. After such transformations, the geometrical characteristics will be given by the following expressions:

$$\left.\begin{aligned} S_x &= \int_{(s)}\int_{-\delta/2}^{+\delta/2} y(s, n)\,ds\,dn = \int y(s)\,dF,\\ S_y &= \int_{(s)}\int_{-\delta/2}^{+\delta/2} x(s, n)\,ds\,dn = \int x(s)\,dF,\\ J_x &= \int_{(s)}\int_{-\delta/2}^{+\delta/2} y^2(s, n)\,ds\,dn = \int y^2(s)\,dF + \int \cos^2\alpha\,dJ, \end{aligned}\right| \qquad (14.3)$$

$$
\left.\begin{aligned}
J_y &= \int\limits_{(s)}\int\limits_{-\delta/2}^{+\delta/2} x^2(s, n)\,ds\,dn = \int x^2(s)\,dF + \int \sin^2\alpha\,dJ,\\
J_{xy} &= \int\limits_{(s)}\int\limits_{-\delta/2}^{+\delta/2} y(s, n)\,x(s, n)\,ds\,dn = \int x(s)\,y(s)\,dF -\\
&\qquad - \int \sin\alpha\cos\alpha\,dJ,
\end{aligned}\right\} \qquad (14.3)
$$

$$
\left.\begin{aligned}
S_\omega &= \int\limits_{(s)}\int\limits_{-\delta/2}^{+\delta/2} \omega(s, n)\,ds\,dn = \int \omega(s)\,dF,\\
J_{\omega x} &= \int\limits_{(s)}\int\limits_{-\delta/2}^{+\delta/2} x(s, n)\,\omega(s, n)\,ds\,dn = \int \omega(s)\,x(s)\,dF -\\
&\qquad - \int t(s)\sin\alpha(s)\,dJ,\\
J_{\omega y} &= \int\limits_{(s)}\int\limits_{-\delta/2}^{+\delta/2} y(s, n)\,\omega(s, n)\,ds\,dn = \int \omega(s)\,y(s)\,dF +\\
&\qquad + \int t(s)\cos\alpha(s)\,dJ,\\
J_\omega &= \int\limits_{(s)}\int\limits_{-\delta/2}^{+\delta/2} \omega^2(s, n)\,ds\,dn = \int \omega^2(s)\,dF + \int t^2(s)\,dJ.
\end{aligned}\right\} \qquad (14.4)
$$

In the equations obtained above (14.1), S_x, S_y and S_ω have, generally speaking, retained the same form which they had without taking the longitudinal bending moments into consideration. It should be noted that, in a sense, we have obtained more exact expressions only for the characteristics $J_{\omega x}$, $J_{\omega y}$ and J_ω. For the moments of inertia J_x, J_y and J_{xy}(14.1) we have the usual expressions from the theory of strength of materials.

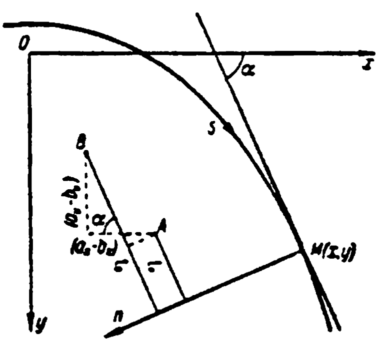


Figure 107

As a result of the above considerations the principal central axis, after satisfying the conditions $S_x = S_y = J_{xy} = 0$ is determined by the usual methos of the theory of strength of materials. The principal sectorial coordinates of the shear center, taking into account longitudinal bending moments, should be determined by the more exact equations which are obtained in the same way as in § 7 of Chapter I.

Let B and A be poles with the coordinates (b_x, b_y) and (a_x, a_y). In this case, equation (I.4.3) for the sectorial area assumes, when the pole and

the origin are shifted, the form:

$$\omega_A = \omega_B + (a_y - b_y)x - (a_x - b_x)y + C.$$

From Figure 107 we find the expression for the function $t(s)$ when the pole is shifted,

$$t_A = t_B - (a_x - b_x)\cos\alpha(s) - (a_y - b_y)\sin\alpha(s).$$

Taking into account that the pole B and the origins are arbitrarily chosen and that the point A is the required shear center, and establishing for ω_A and t_A the conditions $S_\omega = J_{\omega x} = J_{\omega y} = 0$ by means of the inverted equations (14.4), we obtain three equations from which we find the coordinates a_x and a_y of the shear center and the constant C which determines the direction of the initial radius-vector.

The equations for determining the indicated quantities will be

$$\left.\begin{aligned} a_x &= b_x + \frac{\int \omega_B(s)\,y(s)\,dF + \int t_B(s)\cos\alpha(s)\,dJ}{J_x}, \\ a_y &= b_y - \frac{\int \omega_B(s)\,x(s)\,dF - \int t_B(s)\sin\alpha(s)\,dJ}{J_y}, \\ C &= \quad -\frac{\int \omega_B(s)\,dF}{F}. \end{aligned}\right\} \qquad (14.5)$$

As in the general theory of thin-walled beams, the integrals can be calculated by the graphical and analytical methods of structural mechanics. It is necessary to add to the diagrams $x(s)$, $y(s)$ and $\omega_B(s)$ the diagrams:

$$\cos\alpha(s) = x'(s), \quad \sin\alpha(s) = y'(s) \text{ and } t_B(s),$$

which, in the majority of cases in practice, are also quite simple.

The general differential equations of equilibrium of a thin-walled beam remain as before (I.7.3))

$$\begin{aligned} &EF\zeta'' = 0, \\ &EJ_y\xi^{IV} = q_x, \\ &EJ_x\eta^{IV} = q_y, \\ &EJ_\omega\theta^{IV} - GJ_d\theta'' = m. \end{aligned}$$

§ 15. Transverse bending moments in thin-walled beams

1. When considering thin-walled beams and ribbed shells as three dimensional web-frames whose cross sections have a rigid contour, we implicitly take into consideration, in addition to the normal and shear forces in the section, the transverse bending moments which tend to deform the contour of the cross section. It is easy to calculate these moments if the axial shearing forces, acting in the longitudinal line of the cross section, are known. Separating, for this purpose, an infinitely small element in the form of a strip of the beam or shell, bounded by two cross sections $z = \text{const}$ and $z + dz = \text{const}$, the shear forces T increase at each point of the contour. These increments, neglecting infinitesimal terms of higher

order, can be expressed as partial differentials of $T(z, s)$ with respect to one of the unknown variables:

$$dT = \frac{\partial T}{\partial z} dz = \frac{\partial(\tau\delta)}{\partial z} dz. \qquad (15.1)$$

Replacing τ by its expression (8.9), we obtain an equation from which it is easy to determine the magnitude of the tangential load in the case of combined bending and torsion. Since the tangential load, dT, is known from the condition of equilibrium for the individual strip, we can determine the transverse bending moment G for any longitudinal section of the beam and similarly the transverse and normal forces Q and N (Figure 108). The graphical methods of statics, which are based on the theory of the force polygon, may be used in order to calculate the force factors G, N, Q of the longitudinal section of the beam, caused by an external surface load and by a difference in shear forces. We note here that in the case of a concentrated transverse load applied at some point of the span, the transverse forces Q_x, Q_y and the flexural-torsional moments H_ω (equation (8.9)) remain constant in the unloaded part of the beam. The tangential load due to shear forces as determined by equation (15.1) for $Q_x = \text{const}$, $Q_y = \text{const}$ and $H_\omega = \text{const}$, will be equal to zero. Consequently, the transverse bending moments in the examined case of a concentrated load will be also equal to zero. The deformations of the contour section are local and do not influence the distribution of the forces in the walls of the beam.

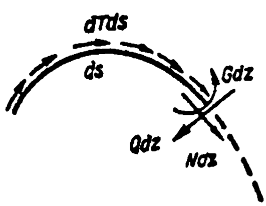


Figure 108

2. The transverse bending moments G can be also calculated by other, purely analytical methods. This is done by starting from the differential equations of equilibrium of an infinitely small element $dz\,ds$ of the middle surface of the shell which we examine in the light of definite statical and geometrical hypotheses. The statical hypotheses are based of the following assumptions. Cylindrical shells of arbitrary shape, fixed by sufficiently closely placed longitudinal and transverse ribs (stringers and frames), we consider as thin-walled three-dimensional systems having only normal and shear forces in their cross sections. Longitudinal flexural and torsional moments are taken to vanish since they hardly influence the state of stress of the shell. Along the longitudinal sections of the shell, transverse forces may appear, in addition to normal and shear forces. Based on such statical hypotheses, a thin-walled three-dimensional system, consisting longitudinally (along the generator) of an infinite number of transverse elementary flexible strips is accepted as the design model for a shell. Each of these strips resembles the two-dimensional curved beam, which undergoes, in each of its sections, not only extension (or compression) but also transverse bending or shear. The interaction between adjacent cross sections in the shell consists in the transmission of only shearing and normal forces from one strip to another. We shall call these kinds of three-dimensional systems, i.e. systems having different physical properties in the longitudinal and transverse directions, *orthotropic three-dimensional systems*.

The statical structure of the described design model is shown in Figure 109. The connections, through which the longitudinal normal and

shear forces are transmitted from one transverse strip to another, are shown schematically as small elementary rods placed in the middle surface of the shell. The differential equations of equilibrium of a cylindrical shell, in view of the adopted statical conditions, will be expressed as follows (Figure 110):

$$\left.\begin{aligned} \frac{\partial(\sigma\delta)}{\partial z}+\frac{\partial T}{\partial s}+p=0,\\ \frac{\partial T}{\partial z}+\frac{\partial N}{\partial s}-\frac{Q}{R}+q=0,\\ \frac{\partial Q}{\partial s}+\frac{N}{R}+r=0,\\ \frac{\partial G}{\partial s}-Q=0, \end{aligned}\right\} \qquad (15.2)$$

where p, q, r are respectively the projections of the surface load vector on the axis z, on the direction s and on the normal to the surface at the given point. $R=R(s)$ is the radius of curvature of the contour line.

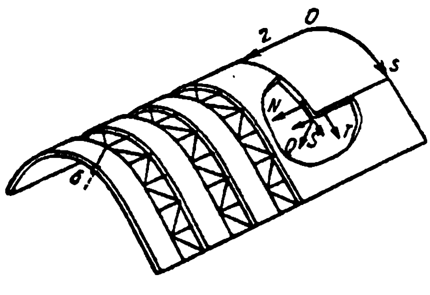


Figure 109

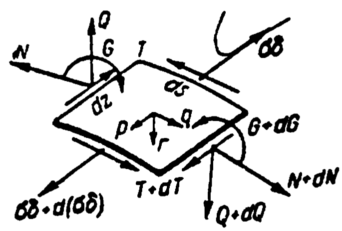


Figure 110

Eliminating the forces T, N, Q, the system of equation (15.2) can be written as a single equation

$$\frac{\partial^2(\sigma\delta)}{\partial z^2}+\Omega G=P, \qquad (15.3)$$

where P is a function depending on the component of the external surface load, thus:

$$P=-\frac{\partial p}{\partial z}+\frac{\partial q}{\partial s}-\frac{\partial^2}{\partial s^2}(Rr); \qquad (15.4)$$

Ω being a differential operator with respect to the variable s

$$\Omega=\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2}{\partial s^2}\right)+\frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial}{\partial s}\right). \qquad (15.5)$$

As will be shown below, this operator is connected with the law of sectorial areas. If there are no surface forces, the internal forces of interest to us in the shell can be expressed by a single function $F=F(z, s)$ in the equations

$$\left.\begin{aligned} \sigma\delta=\Omega F,\quad T=-\Omega_1\frac{\partial}{\partial z}F,\quad N=R\frac{\partial^4 F}{\partial z^2\partial s^2},\\ G=-\frac{\partial^2 F}{\partial z^2},\quad Q=-\frac{\partial^3 F}{\partial s\partial z^2}, \end{aligned}\right\} \qquad (15.6),$$

where $\mathfrak{L}_1$ is a differential operator of the third order with respect to the variable s,

$$\mathfrak{L}_1 = \frac{\partial}{\partial s}\left(R\frac{\partial^2}{\partial s^2}\right) + \frac{1}{R}\frac{\partial}{\partial s}; \tag{15.7}$$

the operators $\mathfrak{L}$ and $\mathfrak{L}_1$ are connected by the relation

$$\mathfrak{L}F = \frac{\partial}{\partial s}(\mathfrak{L}_1 F). \tag{15.8}$$

For $R\frac{\partial^2 F}{\partial s^2} = \varphi$ and $R \to \infty$ the first three of the relations (15.6) transform into Airy's well known equations for the plane problem of the theory of elasticity. The function F can be called the *stress function* for the cylindrical shell. This function, as in the plane problem, plays the role of a fundamental statically indeterminate quantity in our design model, which rests on statical hypotheses only. If the surface load is different from zero, it is necessary to add to the right-hand side of (15.6) the corresponding particular solutions of the inhomogeneous statical equations. These particular solutions can be obtained from equations (15.2) and from the assumption that $G = 0$ or $(\sigma\delta) = 0$. In the second case (for $\sigma\delta = 0$) the particular integrals for G, N and Q can be easily obtained from (15.3) on the basis of the law of sectorial areas for the moments G.

3. From the statical hypotheses we obtain a single equation (15.3) with two unknown functions $(\sigma\delta)$ and G. In order to avoid this indeterminacy in the solution of the problem, it is necessary to obtain one more equation. We find this equation by examining the geometrical aspect of the problem and applying geometrical hypotheses.

According to geometrical hypotheses, the transverse extensions of a shell and the shearing strains, quantities that have a slight influence on the state of the fundamental internal forces of the shell, are taken equal to zero. The deformations of the shell in our design model occur in such a way, that the lines of the middle surface perpendicular to the generator remain unextended at each point and the angles between the lines of principal curvatures (the coordinate lines), which initially were right angles, remain so also after the deformation.

The relations between the strains of the shell ε_1, ε_2, γ and $\varkappa$ and the displacements u, v and w are expressed by equations

$$\left.\begin{aligned} \varepsilon_1 &= \frac{\partial u}{\partial z},\\ \varepsilon_2 &= \frac{\partial v}{\partial s} - \frac{w}{R},\\ \gamma &= \frac{\partial u}{\partial s} + \frac{\partial v}{\partial z},\\ \varkappa &= \frac{\partial}{\partial s}\left(\frac{v}{R} + \frac{\partial w}{\partial s}\right). \end{aligned}\right\} \tag{15.9}$$

The relative extension of the element along the generator is expressed by the first of these equations.

The second equation refers to the extension of the element in the direction perpendicular to the generator. This extension is obtained as a sum of two quantities, of which the first $\left(\frac{\partial v}{\partial s}\right)$ gives the relative extension of

the arc ds, resulting from the increment v in passing from the point M to the point N; the second $\left(-\frac{w}{R}\right)$ refers to the contraction of the arc which takes place as a result of moving the point M along the inward normal to the surface by a distance w (Figure 111).

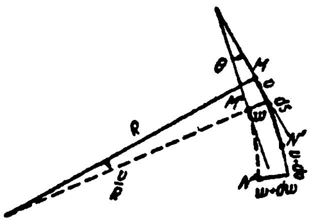


Figure 111

The third equation gives the shearing strain in the middle surface of the beam.

The last (fourth) equation refers to the bending strain of the element along the arc ds. This curvature-producing deformation is obtained as a partial derivative of the angle $\theta = \frac{v}{R} + \frac{\partial w}{\partial s}$ with respect to the arc-length s. The tangent to the arc of the cross section at the point M rotates through this angle as the element is deformed and this is indicated by a dotted line in Figure 111.

Eliminating from these equations the displacements u, v and w, we obtain a single differential equation for the continuity of strain [compatibility]

$$\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \varepsilon_1}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial \varepsilon_1}{\partial s}\right) - \left[\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \gamma}{\partial z \partial s}\right) + \right.$$
$$\left. + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial \gamma}{\partial z}\right)\right] + \frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \varepsilon_2}{\partial z^2}\right) + \frac{\partial^2 \varkappa}{\partial z^2} = 0. \quad (15.10)$$

Starting from the geometrical hypotheses of absence of deformation of the cross section and absence of shearing strain, i. e. by considering that

$$\varepsilon_2 = 0 \text{ and } \gamma = 0, \quad (15.11)$$

we obtain

$$\Omega\varepsilon_1 + \frac{\partial^2 \varkappa}{\partial z^2} = 0. \quad (15.12)$$

This strain continuity equation which is very important in our theory, and was already proposed by the author /35, 36/, shows that the bending deformations $\varkappa = \varkappa(z, s)$ of a transverse elementary strip (deformation of the shell contour) is generally associated, according to the hypotheses (15.12), with an extension of the shell along the generator (warping of the cross section).

For ribbed shells and thin-walled beams we adopt another hypothesis, of the lack of deformation of the contour of the cross section, i. e. we assume $\varkappa = 0$. In this case equation (15.12) is simplified and reads

$$\Omega\varepsilon_1 = 0, \quad (15.13)$$

where the operator Ω is expressed by equation (15.5). We now show that the operator Ω is connected with the law of sectorial areas. Equation (15.13) will be written in full as follows:

$$\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \varepsilon_1}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial \varepsilon_1}{\partial s}\right) = 0. \quad (15.14)$$

This is a linear differential equation of the fourth order with variable coefficients which depend on the radius of curvature $R = R(s)$. We introduce a new function $\varphi(z, s)$, defined by the equation

$$\varphi = \frac{\partial \varepsilon_1}{\partial s}. \quad (15.15)$$

Whence ε_1 is expressed by φ in the following way:

$$\varepsilon_1 = \int_0^s \varphi(z, s)\, ds + Z_1(z), \tag{15.16}$$

where $Z_1(z)$ is an arbitrary function of the coordinate z.

Replacing ε_1 in equation (15.14) by a new function φ, we obtain

$$\frac{\partial^2}{\partial s^2}\left(R\frac{\partial\varphi}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{\varphi}{R}\right) = 0. \tag{15.17}$$

We integrate this equation once with respect to s and denote the new arbitrary function of z by $Z_4(z)$. As a result we obtain

$$\frac{\partial}{\partial s}\left(R\frac{\partial\varphi}{\partial s}\right) + \frac{\varphi}{R} = Z_4(z). \tag{15.18}$$

If we denote by $\alpha = \alpha(s)$ the angle that the tangent to the contour line at the point s makes with the x axis, the following geometrical relations hold:

$$ds = R\, d\alpha; \quad \frac{dx}{ds} = \cos\alpha, \quad \frac{dy}{ds} = \sin\alpha. \tag{15.19}$$

By using (15.19) it is possible to replace in equation (15.18) the independent variable s by α, and to obtain

$$\frac{\partial^2\varphi}{\partial\alpha^2} + \varphi = RZ_4. \tag{15.20}$$

The homogeneous equation

$$\frac{\partial^2\varphi}{\partial\alpha^2} + \varphi = 0, \tag{15.21}$$

as known from the theory of differential equations, is satisfied by the function

$$\varphi = A\sin\alpha + B\cos\alpha. \tag{15.22}$$

We find the solution of the inhomogeneous equation (15.20) from the solution of the homogeneous equation by the method of variation of constants. For that purpose we differentiate the function (15.22) with respect to α, noting that the arbitrary functions A and B depend not only on z but also on α:

$$\frac{\partial\varphi}{\partial\alpha} = \frac{\partial A}{\partial\alpha}\sin\alpha + \frac{\partial B}{\partial\alpha}\cos\alpha + A\cos\alpha - B\sin\alpha,$$

and we put

$$\left.\begin{aligned} &\frac{\partial A}{\partial\alpha}\sin\alpha + \frac{\partial B}{\partial\alpha}\cos\alpha = 0, \\ &\frac{\partial\varphi}{\partial\alpha} = A\cos\alpha - B\sin\alpha. \end{aligned}\right\} \tag{15.23}$$

In the same manner, differentiating the last equation once more with respect to α:

$$\frac{\partial^2\varphi}{\partial\alpha^2} = \frac{\partial A}{\partial\alpha}\cos\alpha - \frac{\partial B}{\partial\alpha}\sin\alpha - A\sin\alpha - B\cos\alpha.$$

Substituting this result, together with expression (15.22), in equation

(15.20), we obtain

$$\frac{\partial A}{\partial a}\cos a-\frac{\partial B}{\partial a}\sin a=RZ_4. \tag{15.24}$$

Solving this equation and the first of equations (15.23) with respect to $\frac{\partial A}{\partial a}$ and $\frac{\partial B}{\partial a}$, we find

$$\left.\begin{aligned}\frac{\partial A}{\partial a}&=Z_4R\cos a,\\ \frac{\partial B}{\partial a}&=-Z_4R\sin a.\end{aligned}\right\} \tag{15.25}$$

Whence, by integrating with respect to a, we obtain

$$\left.\begin{aligned}A&=Z_4\int_0^s R\cos a\,da+Z_2(z),\\ B&=-Z_4\int_0^s R\sin a\,da+Z_3(z),\end{aligned}\right| \tag{15.26}$$

where $Z_2(z)$ and $Z_3(z)$ are new arbitrary functions which depend only on the variable z. The integrals on the right-hand side of equations (15.26) are easily evaluated using the relation (15.19). The solution of equation (15.20) can be written

$$\varphi=Z_2\sin a+Z\ \ \cos a+Z_4(x\sin a-y\cos a), \tag{15.27}$$

and, consequently, we have for ε_1 as determined by equation (15.16)

$$\varepsilon_1=Z_1+Z_2\int_0^s\sin a\,ds+Z_3\int_0^s\cos a\,ds+Z_4\int_0^s(x\sin a-y\cos a)\,ds=$$

$$=Z_1+Z_2\int_0^s dy+Z_3\int_0^s dx+Z_4\int_0^s(x\,dy-y\,dx),$$

$$\varepsilon_1=Z_1+Z_2y(s)+Z_3x(s)+Z_4\omega(s), \tag{15.28}$$

where $\omega(s)=\int_0^s(x\,dy-y\,dx)$.

All the arbitrary functions of the integration are referred to $Z_1(z)$.

It is more convenient to clarify the geometrical meaning of $\omega(s)$ by transforming to polar coordinates ρ and ψ (Figure 112). Since

$$x=\rho\cos\psi,\quad y=\rho\sin\psi,$$

we have

$$\begin{aligned}dx&=-\rho\sin\psi\,d\psi+d\rho\cos\psi,\\ dy&=\rho\cos\psi\,d\psi+d\rho\sin\psi,\end{aligned}$$

and, consequently,

$$\omega(s)=\int_0^s(x\,dy-y\,dx)=\int_0^s\rho^2\,d\psi,$$

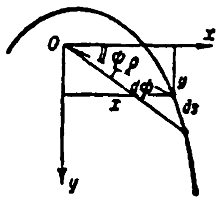


Figure 112

where $\int_0^s \rho^2 d\phi$ (and hence also $\omega(s)$) represents twice the area of the sector included between two radius-vectors and the arc s, i. e. the value of the sectorial area. Equation (15.28) which is the solution of the differential equation (15.14), is identical with equation

$$\varepsilon = \zeta'(z) - \xi''(z)\,x(s) - \eta''(z)\,y(s) - \theta''(z)\,\omega(s).$$

which was derived earlier in § 3 of Chapter I.

In this equation the role of the arbitrary functions of integration $Z_1(z)$, $Z_2(z)$, $Z_3(z)$, $Z_4(z)$ of equation (15.14) is taken over by the quantities $\zeta'(z)$, $-\xi''(z)$, $-\eta''(z)$, $-\theta''(z)$ which have a definite geometrical meaning. The two equations are identical and describe the law of sectorial areas of which the law of plane sections is a particular case.

4. Regarding a shell stiffened by ribs as a reduced orthotropic elastic system, we present Hooke's law under hypotheses (15.11) in the following simplified form (Poisson's ratio μ is taken equal to zero)

$$\varepsilon_1 = \frac{1}{A}\sigma; \quad \varkappa = -\frac{1}{D}G, \tag{15.29}$$

where A is the modulus of extension of the shell along the generator; D is the flexural rigidity (reduced, taking the transverse ribs into account) of the shell along the contour. If there are no ribs, obviously

$$A = E, \quad D = \frac{E\delta^3}{12}, \tag{15.30}$$

where E is the [Young's] modulus of elasticity in extension or compression; δ is the thickness of the shell.

Substituting (15.29) in equation (15.12) and combining it with the equation (15.3) obtained earlier, we obtain a system of two differential equations for the shell with two unknown functions σ and G:

$$\frac{\partial^2(\sigma\delta)}{\partial z^2} + \Omega G = P, \quad \Omega\sigma - \frac{A}{D}\frac{\partial^2 G}{\partial z^2} = 0. \tag{15.31}$$

The system (15.31) has a symmetrical structure in regard to terms containing derivatives with respect to s. This is in full accordance with the fundamental theorems of the theory of elasticity.

For R=const (for a circular shell) equation (15.31) will have constant coefficients.

If we assume $D = \infty$, in the second equation (15.31), i. e. if we assume that we are dealing with a ribbed shell or a thin-walled beam in which the contour is assumed to be rigid ($\varkappa = 0$), we obtain equation (15.13) for $\sigma = A\varepsilon$

$$\Omega\varepsilon = 0. \tag{15.32}$$

For a longitudinal deformation expressed as a function of the contour coordinate s, $\varepsilon = \varepsilon(z, s)$ the geometrical law of sectorial areas, which we examined in detail earlier, is obtained as a result.

If we assume $\sigma\delta = 0$ in the first equation of the system (15.31), i. e. if we assume that only shear forces are acting in the cross section of the shell (as in the case, for example, with a corrugated shell) we obtain,

since $P=0$ (homogeneous problem), the equation

$$\Omega G = 0, \qquad (15.33)$$

which is completely analogous to equation (15.32). This means that the transverse bending moments $G = G(z, s)$, regarded as functions of the contour coordinate s, will obey the law of sectorial areas. In contrast to its analogous geometrical law for extensions $\varepsilon = \varepsilon(z, s)$, this law can be called the pure statical law of sectorial areas. Denoting the general integral of the homogeneous equation (15.33) by $G(z, s)$ we may, on the basis of the foregoing write it in the form:

$$G(z, s) = Z_5(z) + Z_6(z)x(s) + Z_7(z)y(s) + Z_8(z)\omega(s), \qquad (15.34)$$

where $Z_5(z)$, $Z_6(z)$, $Z_7(z)$ and $Z_8(z)$ are arbitrary functions of integration; $x(s)$, $y(s)$ the coordinates of a point on the profile line; $\omega(s)$ the sectorial coordinate of this point.

It is quite easy to give a mechanical interpretation to the separate terms in expression (15.34). The first term represents pure bending of a transverse rib due to the moment G_0, applied at the initial lateral edge $s=0$ (Figure 113); the second and third terms represent the moments, at the point of the contour $x(s)$, $y(s)$, due to the forces q_{x0} and q_{y0}, which are parallel to the coordinate axes and applied at the initial lateral edge $s=0$; the last term refers to the moment due to the shear forces $T = T(z)$ which are constant along the contour line. The arbitrary functions $Z_5(z)$, $Z_6(z)$, $Z_7(z)$ and $Z_8(z)$ which have a perfectly definite mechanical meaning, are determined from the boundary conditions on the lateral edges of the shell.

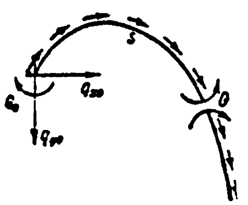


Figure 113

Determining the force $\sigma\delta \neq 0$ from equations (15.32) and substituting it in the first equation of the system (15.31), we obtain

$$\Omega G - \left[P - \frac{\partial^2 (\sigma\delta)}{\partial z^2}\right] = 0 \qquad (15.35)$$

The general integral of this inhomogeneous equation (15.35) can be given in the form of the sum of twc integrals

$$G = g_1(z, s) + g_2(z, s), \qquad (15.36)$$

where $g_1(z, s)$ is the general integral of the hmoogeneous equation given by equation (15.34), and $g_2(z, s)$ is a particular integral of the inhomogeneous equation (15.35). Thus, we see that in thin-walled beams in addition to axial forces there appear also transverse bending moments G and normal forces Q and N, which act on the areas of the longitudinal section. One of the fundamental assumptions of the elementary bending theory of beams, viz. that under bending the longitudinal fibers of the beam do not press on one another is invalidated in the case of thin-walled beams. For thin-walled beams or shells it is more natural not to adopt the hypothesis of the absence of normal forces and moments on the longitudinal sections, but that of the absence of extension and flexure, i. e. the hypothesis of the undeformability of the section contour.

We return to the diagrams shown on Figures 91a, b (see the worked-out example in § 10) and we note that in the case of bending in the plane of symmetry the contour deformations play a more important role than in the

torsion. This is explained by the fact that under bending there appear in the cross sections shear forces which are represented by symmetrical diagrams. The moments G, due to these forces, will be also distributed symmetrically with respect to the y axis and will cause a symmetrical deformation of the section contour. The distance between the extreme points of the contour will change as a result of such a deformation.

In the case of torsion the variation of the shear forces over the section, as seen from the diagram for the sectorial statical moment S_{ω} (Figure 90b), is given by an odd (antisymmetrical) function. The transverse bending moments due to these forces will vary according to a law of a skew-symmetrical diagram. As a result of this, the projection on the x-axis of the distance between two symmetrical points of the contour will not vary. The contour deformation of the section under torsion due to an antisymmetrical load is thus smaller than the deformation due to bending in the vertical plane under a symmetrical load.

Chapter III

THIN-WALLED BEAM-SHELLS REINFORCED BY TRANSVERSE CONNECTIONS

§ 1. The method of spatial design of multiply supported structures

The bimoment theory of flexural torsion together with the methods of structural mechanics allows the relatively simple design of three-dimensional systems of the thin-walled beam type, having intermediate supports at arbitrary points along the span. To such systems belong, in particular, girder bridges with oblique piers (Figure 114) and roof structures with ribbed domes, which have beam supports along the lateral edges (Figure 115).

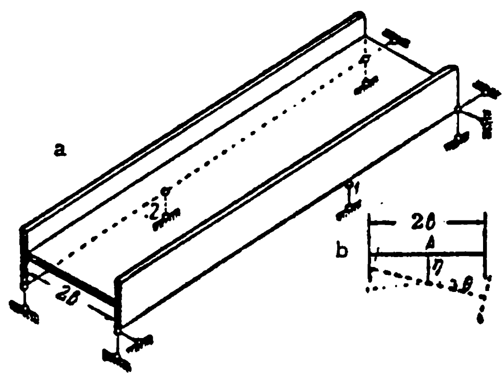


Figure 114

For simplicity we shall assume that the axis of symmetry in the cross section of the examined system is vertical. We assume further that such a system is subjected to the action of a vertical load, whose resultant, for a cross section of unit width, lies in the plane of symmetry of the section. An example of such a load is the dead weight of the structure. Such a load, according to the theory developed in the preceding chapters, causes only bending deformation when there are no asymmetrical support connections. Torsional deformations will be absent since the load, lying also in the longitudinal plane of symmetry, passes through the line of shear centers. When connections do exist, and are situated at a certain distance from the axis of symmetry of the section, the structure, in addition to bending, will also undergo a torsional deformation, caused by the actions of the supporting connections. In order to determine these actions we can use the methods

of structural mechanics, extending and generalizing them so as to apply to the thin-walled three dimensional systems examined here. We shall start with the method of forces. Discarding the intermediate support connections, we obtain a thin-walled single-span structure which we choose as a fundamental system. The strained state of such a system, under any complex load, can be determined by the superposition of elementary states of flexure and flexural torsion. Since, in our example, the external load and the support connections are vertical, we shall be interested in only two types of the elementary states of the fundamental system, those of flexure and of flexural torsion (Figure 114b; 115b).

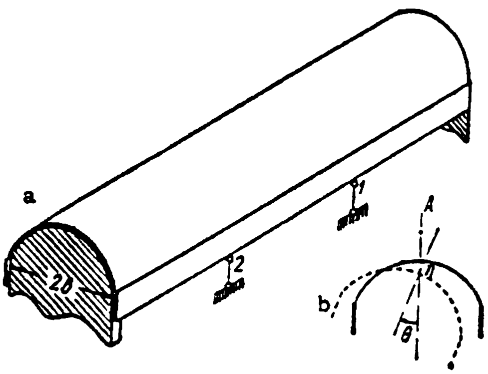


Figure 115

The effect of the reactions of any supported beam, which do not pass through the shear center, can be reduced to the loading of the fundamental system by a concentrated vertical load, lying in the plane of symmetry and causing pure bending deformation, plus a concentrated torsional moment, which causes flexure and torsion. The vertical deflection $\eta = \eta(z, t)$ of the beam in any section $z =$ const, due to a concentrated vertical load, P, acting along the axis of symmetry in any other section $t =$ const, is determined by the following equations, for the case of a beam with hinged ends:

For $l \geqslant z \geqslant t$

$$\eta(z, t) = \frac{P(l-z)t}{6EJ_x l}[l^2 - (l-z)^2 - t^2], \tag{1.1}$$

For $t > z \geqslant 0$

$$\eta(z, t) = \frac{P(l-t)z}{6EJ_x}[l^2 - (l-t)^2 - z^2]. \tag{1.2}$$

Equations (1.1) and (1.2) can be obtained either by applying the method of initial parameters to the differential equation of beam flexure in the equations of the system (I.7.3), or by applying Mohr's formula for displacements

$$\eta(z, t) = \int_0^l \frac{M_z(s) M_t(s)}{EJ} ds, \tag{1.3}$$

where $M_z(s)$ and $M_t(s)$ are the bending moments due to unit forces, applied at two different points on the axis of the beam with abscissas z and t.

Equations (1.1) and (1.2) represent the influence function (Green's function) in the problem of bending of a single-span beam with hinged ends.

The torsion angle $\theta=\theta(z,t)$, in any section $z=$ const, due to a concentrated torsional moment $H=Pe$, acting in the plane of any other section $t=$const, is determined by equations (II.5.4) and (II.5.5):

For values $l\geqslant z\geqslant t$

$$\theta(z,t)=\frac{H}{GJ_d}\left[\frac{t}{l}(l-z)-\frac{l}{k}\frac{\operatorname{sh}\frac{k}{l}t}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}(l-z)\right]; \tag{1.4}$$

For values $t>z\geqslant 0$

$$\theta(z,t)=\frac{H}{GJ_d}\left[\frac{l-t}{l}z-\frac{l}{k}\frac{\operatorname{sh}\frac{k}{l}(l-t)}{\operatorname{sh}k}\operatorname{sh}\frac{k}{l}z\right]. \tag{1.5}$$

These equations describe the influence function (Green's function) in the problem of combined bending and torsion of a single-span thin-walled beam, which is hinged at the ends.

Equation (1.4) passes into (1.5) when z is replaced by t, which is in accordance with the symmetry of Green's function, and with Betti's reciprocity theorem, $\theta(z,t)=\theta(t,z)$.

In the case of ribbed domes as used in construction, and also in the case of cylindrical and prismatical shells with open section, as widely used in aviation and ship building, the rigidity GJ_d of Saint-Venant's theory of torsion is a factor of secondary importance. We can simplify equations (1.4) and (1.5) by taking this rigidity and, therefore, also the quantity k, equal to zero. For this purpose it is necessary to expand in series the hyperbolic functions in equations (1.4) and (1.5) and retain but the first two terms of the expansion for each function before passing to the limit.

We simplify, for example, equation (1.4):

$$\begin{aligned}\lim_{k\to 0}\theta(z,t)&=\lim_{k\to 0}\frac{H}{GJ_d}\times\\ &\times\left\{\frac{t}{l}(l-z)-\frac{l}{k}\,\frac{\frac{k}{l}t\left(1+\frac{k^2}{6l^2}t^2\right)\frac{k}{l}(l-z)\left[1+\frac{k^2}{6l^2}(l-z)^2\right]}{k\left(1+\frac{k^2}{6}\right)}\right\}=\\ &=\lim_{k\to 0}\frac{H}{GJ_d}\frac{t(l-z)}{l}\left\{1-\left(1+\frac{k^2}{6l^2}t^2\right)\left[1+\frac{k^2}{6l^2}(l-z)^2\right]\left(1-\frac{k^2}{6}\right)\right\}=\\ &=\lim_{k\to 0}\frac{Hk^2}{6GJ_d}\frac{t(l-z)}{l^3}[l^2-(l-z)^2-t^2]=\\ &=\frac{H}{6EJ_\omega}\frac{t(l-z)}{l}[l^2-(l-z)^2-t^2].\end{aligned} \tag{1.6}$$

Equation (1.6) is valid for the values $l\geqslant z\geqslant t$. For values $t>z\geqslant 0$ we obtain from equation (1.5)

$$\theta(z,t)=\frac{H}{6EJ_\omega}\frac{z(l-t)}{l}[l^2-(l-t)^2-z^2]. \tag{1.7}$$

Equations (1.6) and (1.7) have the same structure as equations (1.1) and (1.2) and differ from them by quantities which are related to the displacement to be determined, to the load and to the generalized geometrical characteristic. The similarity of these equations is also a result of

the general circumstance, discussed earlier, of the mathematical analogy between flexure and flexural torsion in the theory of beams.

We shall now present an equation for the bending of the fundamental system by an external, transverse, uniformly distributed load q, acting in the plane of symmetry

$$\eta(z) = \frac{qz}{24EJ_x}(l^3 - 2lz^2 + z^3). \tag{1.8}$$

Given the deformation of the fundamental system in its elementary states examined here, we can now, by statical methods, easily obtain the required equation for the reactions of the intermediate vertical supports. These equations, for n intermediate supports, have the following form

$$\left.\begin{array}{l} \delta_{11}X_1 + \delta_{12}X_2 + \ldots + \delta_{1n}X_n + \delta_{1p} = 0, \\ \delta_{21}X_1 + \delta_{22}X_2 + \ldots + \delta_{2n}X_n + \delta_{2p} = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \delta_{n1}X_1 + \delta_{n2}X_2 + \ldots + \delta_{nn}X_n + \delta_{np} = 0. \end{array}\right\} \tag{1.9}$$

Here, the required reactions X_i $(i = 1, 2, 3, \ldots, n)$ are positive if they correspond to an extension of the support connections. The coefficients and the free terms of equations (1.9) are computed from the general expressions

$$\begin{array}{ll} \delta_{ij} = \eta_{ij} + b\theta_{ij} & (i, j = 1, 2, 3, \ldots, n), \\ \delta_{iq} = \eta_{iq} & (i = 1, 2, 3, \ldots, n), \end{array}$$

where i and j are the indexes of the equations and the unknowns; $\eta_{ij} = \eta(z_i, t_j)$ is the deflection of the section of the i-th support due to a single concentrated force, corresponding to the reaction at the j-th support; $\theta_{ij} = \theta_{ij}(z_i, t_j)$ is the torsion angle in the section of the i-th support due to a single concentrated torsional moment $H_j = b$, acting in the plane which passes through the j-th support.

The quantity b is half the width of the structure (the arm of the torsional moment). In accordance with the adopted rule of signs, this quantity will be positive for supports situated, according to Figures 114, 115 on the right of the axis of symmetry and will be negative for supports which are on the left of this axis.

The free term δ_{iq} of the i-th equation is computed by equation (1.8) as the deflection $\eta_i = \eta(z_i)$ due to a given load in the section passing through the i-th support. Determining from equations (1.9) the reactions at the support we can by superposition obtain the deflections and torsional angles of a given three-dimensional multiple support system with the help of the previously given equations for the elementary states of the fundamental system. The normal and tangential stresses which appear in the cross sections of the system are here calculated from the two-term equations

$$\sigma(z, s) = -E[\eta''(z)y(s) + \theta''(z)\omega(s)],$$

$$\tau(z, s) = E\left[\eta'''(z)\frac{S_x(s)}{\delta(s)} + \theta'''(z)\frac{S_\omega(s)}{\delta(s)}\right].$$

§ 2. Beams reinforced by braces

1. The method of calculating the effect of a longitudinal force, applied at an arbitrary point of the cross section of a beam, which was developed in § 7, Chapter II, is of great significance in designing thin-walled members (columns and beams) reinforced by transverse braces. This method, combined with the universally known methods of structural mechanics, allows the evaluation of the effect of the planks.

Basically, the purpose of the braces is to increase the rigidity of the beam when subjected to torsion. From this point of view the beam reinforced by braces occupies an intermediate position between the beams with an open contour of the cross section which we have examined till now, and thin-walled beams with a closed contour of the cross section which we shall examine in more detail in the following. Here we note only that if the theory of thin-walled beams with an open section is based on the hypotheses of undeformability of the section contour and negligible shear deformations then, in investigating thin-walled beams of closed section precisely those factors, the contour deformations and the shear deformations, neglected in beams of open section, are of essential importance. Taking this into consideration when studying the behavior of beams reinforced by braces, we shall proceed from the following assumptions:

a) the beam itself has, as before, a rigid section contour, and shear deformations are neglected;

b) the braces are deformable in their plane and their shear deformations is taken into consideration.

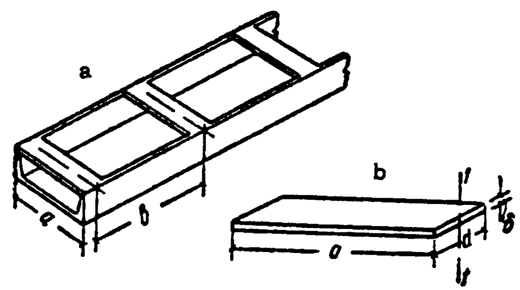


Figure 116

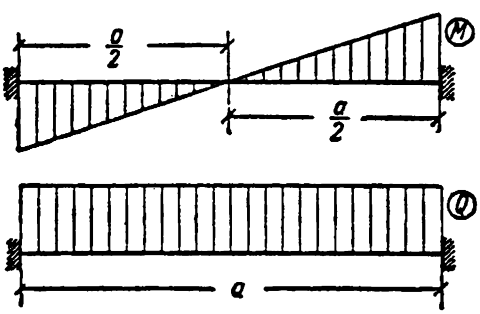


Figure 117

2. We now examine a thin-walled beam of open section, reinforced by n braces (Figure 116a); the distances between the braces, as well as their dimensions can vary, and we shall assume that their connections to the body of the beam are rigid. It is obvious that such a beam can be regarded as a composite three-dimensional structure, consisting of a cylindrical shell and a plane beam system of the Vierendeel [open-frame] girder type. The change in the degree of statical indeterminacy of such a structure, due to the braces only, will be equal to $3n$. If we take into consideration that the contour of the beam is considered to be nondeformable [and, therefore, the axial (longitudinal) deformations of the braces equal zero], and that, since the braces are rigidly connected with the beam, the diagram of the moments, when the braces are subjected to bending, will have the form of a skew-symmetrical trapezium (Figure 117) with a zero point at the middle of the brace, we obtain, upon cutting the brace in the middle, a system of n equations as the fundamental system. The transverse forces $Z_1, Z_2, \ldots, Z_i, \ldots, Z_n$ which appear in the braces, (Figure 118), are taken as the unknowns. The following canonical system of n equations is obtained in order to determine these forces by the methods of structural mechanics:

$$\left.\begin{array}{l}(\delta_{11}+\bar{\delta}_{11})Z_1+\delta_{12}Z_2+\ldots+\delta_{1n}Z_n+\delta_{1q}=0,\\ \delta_{21}Z_1+(\delta_{22}+\bar{\delta}_{22})Z_2+\ldots+\delta_{2n}Z_n+\delta_{2q}=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \delta_{n1}Z_1+\delta_{n2}Z_2+\ldots+(\delta_{nn}+\bar{\delta}_{nn})Z_n+\delta_{nq}=0.\end{array}\right\} \qquad (2.1)$$

The coefficients of these equations have the following physical significance:

δ_{ik} is the relative displacement of the ends of the i-th brace due to a unit force applied to the k-th brace. Such a displacement is in the direction of the acting forces, and is determined without taking the deformations of the braces since it results only from the warping of the beam.

$\bar{\delta}_{ii}$ is the relative displacement of the ends of the i-th brace, in the direction of the acting force, due to a unit force resulting only from the deformation of the brace in its plane. It is determined by taking into account the shear deformation of the brace.

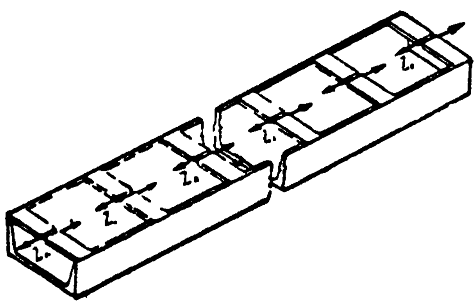


Figure 118

δ_{iq} is the relative displacement, in the same direction, of the ends of the i-th brace, due to the given external load.

The coefficients $\bar{\delta}_{ii}$ are calculated by the general rules of structural mechanics and are expressed by the equation

$$\bar{\delta}_{ii}=2\left(\int_0^{\frac{a}{2}}\frac{M^2}{EJ}\,ds+\nu\int_0^{\frac{a}{2}}\frac{Q^2}{GF}\,ds\right), \qquad (2.2)$$

where a is the length of the brace; $J=\frac{\delta d^3}{12}$ is the moment of inertia of the brace with respect to the axis 1–1 (Figure 116b); $F=\delta d$ is the area of the cross section of the brace; ν is a coefficient, depending on the form of the brace (for a rectangular brace $\nu=1.2$).

By using the diagrams of M and Q, shown for this case in Figure 117, we obtain from (2.2)

$$\bar{\delta}_{ii}=\frac{a^3}{12EJ}+\frac{1.2a}{GF}. \qquad (2.3)$$

As apparent from this equation, the coefficients $\bar{\delta}_{ii}$ do not depend on the position of the braces along the beam.

We now turn to the computation of the coefficients δ_{iq} which describe the relative displacement in the direction of the forces Z_i, of the ends of the i-th brace at the cut as a result of a given external load. If we denote the ends of the braces at the cut by K and L (initial and final point when moving clockwise on the contour), as shown in Figure 119a, we have:

$$\delta_{iq}=u_{qL}-u_{qK}, \qquad (2.4)$$

where u_{qL} and u_{qK} are the displacements $u(z)$, at the points L and K respectively, due to the action of an external load q.

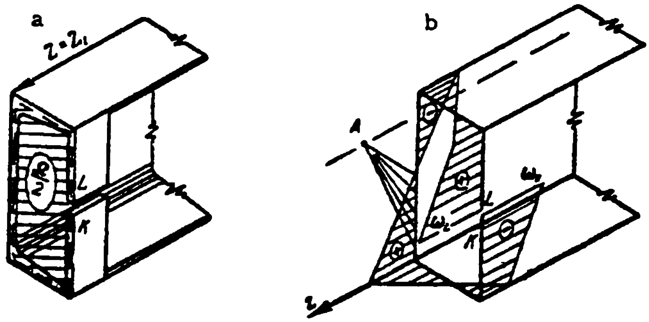


Figure 119

The displacements $u(z,s)$, which result from the torsion of the beam are expressed, according to equation (I.3.16) as follows:

$$u(z,s)=-\theta'(z)\,\omega(s), \qquad (2.5)$$

where $\theta'(z)$ is the relative warping which depends only on the coordinate z and $\omega(s)$ the sectorial area which depends on the position of the point on the contour of the cross section.

From (2.5), expression (2.4) can be written thus:

$$\delta_{iq}=-\theta_q'(z_i)(\omega_L-\omega_K). \qquad (2.6)$$

The diagram of the sectorial areas, with a pole, A, at the shear center, in the section $z_i=$ const which passes through the axis of the i-th brace is given in Figure 119b. The ordinates of the diagram are on the z axis, and not in the plane of the section, as before. It is seen from this figure that $\omega_L-\omega_K$ represents twice the area included between the contour line of the beam and the axis of the brace. Denoting the double area by Ω we have, by (2.6)

$$\delta_{iq}=-\theta_q'(z_i)\,\Omega. \tag{2.7}$$

Thus, the coefficinet δ_{iq} due to the load (the thee term of the system (2.1)) consists of two factors: a constant, Ω, which does not depend on the position of the brace on the z axis and the relative warping $\theta_q'(z)$ which depends on the external load q and on the boundary conditions at the ends of the beam. By using the method of initial parameters developed in Chapter II, it is easy to find the function $\theta'(z)$ for any form of transverse and longitudinal external load.

It now remains to determine the coefficients δ_{ik}, which represent the relative displacement at the cut of the i-th brace in the direction of the acting forces Z_i and due to a unit force applied on the k-th brace. In this case we regard the braces (with the cut), as rigid arms.

In the section $z_k=$const, where the k-th brace is located, the effect of the brace is replaced by the forces Z_k. These longitudinal forces are transmitted to the thin-walled beam through rigid arms such as the brace-halves mentioned above. These forces, as shown in § 7 of Chapter II, produce the external bimoment

$$\bar{B}_k=Z_k(\omega_L-\omega_K).$$

Recalling that

$$\omega_L-\omega_K=\Omega$$

and that Z_k has been replaced by a unit force, we obtain for the bimoment applied at the position of the k-th brace

$$\bar{B}_k=\Omega. \tag{2.8}$$

Passing on to section $z_i=$const, where the i-th brace is located, the relative displacement at the end of the i-th brace, expressed by δ_{ik}, due to the bimoment $\bar{B}_k$ applied at the section $z_k=$const, we can write in analogy with expression (2.6)

$$\delta_{ik}=-\theta_{\bar{B}_k}'(z_i)(\omega_L-\omega_K)=-\theta_{\bar{B}_k}'(z_i)\Omega, \tag{2.9}$$

where Ω is twice the area included by the contour line of the beam and the axis of the i-th brace, and $\theta_{\bar{B}_k}'(z_i)$ the warping in the section $z_i=$const due to the action of the bimoment $\bar{B}_k$ applied at the section $z_k=$const. $\theta_{\bar{B}_k}'(z_i)$ is easily calculated by the equations of § 7, Chapter II, for the various boundary conditions at the ends of the beam. Thus, for example, in the case of the beam hinged at both ends, when introducing the values of θ_0' and H_0 from (II.7.1), in the second of equations (II.7.3), (taking $z_i>z_k$) and after certain transformations, we obtain:

$$\theta_{\bar{B}_k}'(z_i)=-\frac{\bar{B}_k}{GJ_d}\frac{1}{l}\left[k\frac{\operatorname{ch}\frac{k}{l}(l-z_i)\operatorname{ch}\frac{k}{l}z_k}{\operatorname{sh}k}-1\right]. \tag{2.10}$$

Substituting (2.10) and (2.8) in (2.9), we obtain for δ_{ik} the equation:

$$\delta_{ik}=\frac{\Omega^2}{GJ_d}\frac{1}{l}\left[k\frac{\operatorname{ch}\frac{k}{l}(l-z_i)\operatorname{ch}\frac{k}{l}z_k}{\operatorname{sh}k}-1\right]^{*)}. \tag{2.11}$$

As seen from this equation δ_{ik}, as well as the coefficients δ_{iq}, depend on the coordinate z and also on the relative positions of the i-th and k-th braces.

Determining the values of the unknowns Z_i by solving the system (2.1), we may determine any force and displacement we wish to find in a beam reinforced by braces and having arbitrary boundary conditions at the ends. This can be realized by considering that in addition to the given principal external load a series of bimoment loads ΩZ_i (according to the number of braces), which are applied at the connections of the braces, are also acting on the beam.

3. We illustrate the influence of the braces on the example of a truss girder which consists of two No 12 channels, as were examined in § 1, sub-sec. 4, and in § 5, sub-sec. 2 Chapter II, (Figure 120). The cross section of the girder and the principal dimensions are given in Figure 40. All the necessary data with reference to the load and the geometrical characteristics are given in §§ 1 and 5 of Chapter II.

As before (§ 5, Chapter II), we assume that the constraints at the girder ends correspond to hinge supports. Suppose that one brace is welded across the free edges 1 and 3 at each end of the girder (Figure 120a). We shall consider the braces to be identical. The dimensions of the braces are the following: length $a=11.28$ cm, width $d=10$ cm, thickness $\delta=0.6$ cm.

Using equation (2.3), we calculate the coefficients $\bar{\delta}_{11}$ and $\bar{\delta}_{22}$ of equation (2.1), which express the flexibility of these braces. Since equation (2.1) contains most of its coefficients as factors of the quantities GJ_d, it is more convenient to calculate $GJ_d\bar{\delta}_{11}$ and $GJ_d\bar{\delta}_{22}$ instead of $\bar{\delta}_{11}$ and $\bar{\delta}_{22}$.

Hence we find

$$\left.\begin{aligned} F&=\delta d=6.0\ \text{cm}^2,\\ J&=\frac{\delta d^3}{12}=50.0\ \text{cm}^4,\\ GJ_d\bar{\delta}_{11}&=GJ_d\bar{\delta}_{22}=J_d\left(\frac{a^3}{2.5\cdot 12J}+\frac{1.2a}{F}\right)=19.82^{*}). \end{aligned}\right\}^{**} \tag{2.12}$$

* The present calculations refer to the case where the bimoment $\bar{B}_k$ is applied at the section $z_k=\text{const}$, and the relative displacement is examined in the section $z_i=\text{const}$ (since $z_i>z_k$). In case $\bar{B}_i$ is applied at the section $z_i=\text{const}$ and we require the relative displacement in the section $z_k=\text{const}$, while as before $z_i>z_k$, i.e. when we have to find δ_{ki}, we use the second of equations (II.7.2), and equation (2.10) becomes

$$\theta'_{\bar{B}_i}(z_k)=-\frac{\bar{B}_i}{GJ_d}\frac{1}{l}\left[k\frac{\operatorname{ch}\frac{k}{l}(l-z_i)\operatorname{ch}\frac{k}{l}z_k}{\operatorname{sh}k}-1\right].$$

Substituting $\theta'_{\bar{B}_i}(z_k)$ in equation (2.9), we find $\delta_{ik}=\delta_{ki}$, in accordance with the theorem of reciprocity.

** All forces are in kg, lengths in cm.

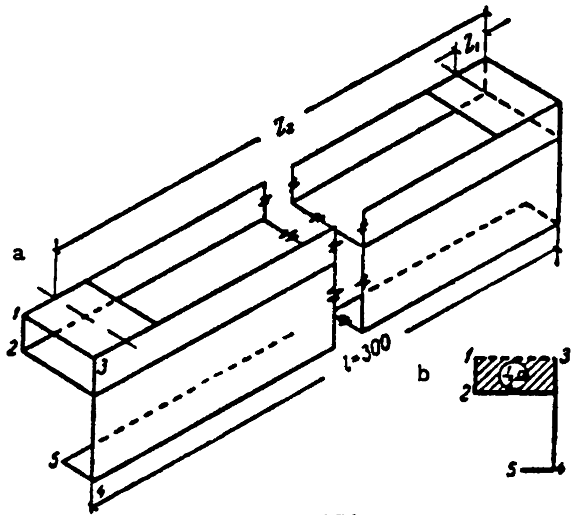


Figure 120

The value of twice the area included between the contour of the section and the covering brace, is equal to (Figure 120b):

$$\Omega = 113.5\text{cm}^2.$$

For a brace width of $d = 10$ cm we have to take one brace as being connected to the beam at the section $z = z_1 = 5$ cm, and the second connected at the section $z = z_2 = 295$ cm.

The coefficients δ_{ik}, computed by equation (2.11), will in this case have the following form

$$\left.\begin{aligned}
\delta_{11} &= \frac{\Omega^2}{GJ_d}\frac{1}{l}\left[k\frac{\operatorname{ch}\frac{k}{l}(l-z_1)\operatorname{ch}\frac{k}{l}z_1}{\operatorname{sh} k} - 1\right],\\
\delta_{22} &= \frac{\Omega^2}{GJ_d}\frac{1}{l}\left[k\frac{\operatorname{ch}\frac{k}{l}(l-z_2)\operatorname{ch}\frac{k}{l}z_2}{\operatorname{sh} k} - 1\right],\\
\delta_{12} = \delta_{21} &= \frac{\Omega^2}{GJ_d}\frac{1}{l}\left[k\frac{\operatorname{ch}\frac{k}{l}(l-z_2)\operatorname{ch}\frac{k}{l}z_1}{\operatorname{sh} k} - 1\right]
\end{aligned}\right\} \tag{2.13}$$

For a girder length of $l = 300$ cm and a value of $k = 6.782$ (§ 5, Chapter II) we have:

$$\operatorname{ch}\frac{k}{l}(l-z_1) = \operatorname{ch}\frac{k}{l}z_2 = 393.8,$$

$$\operatorname{ch}\frac{k}{l}(l-z_2) = \operatorname{ch}\frac{k}{l}z_1 = 1.006,$$

$$\operatorname{sh} k = 440.9,$$

whence

$$\left.\begin{aligned}
GJ_d\delta_{11} &= GJ_d\delta_{22} = 218.8,\\
GJ_d\delta_{12} &= GJ_d\delta_{21} = -42.26.
\end{aligned}\right\} \tag{2.14}$$

Finally, the coefficients δ_{iq}, which play the role of the free terms in equations (2.1), are calculated from equations (2.7) and (5.9) of Chapter II

for the given form of boundary conditions, by the equation

$$\delta_{iq} = -\Omega \frac{1}{GJ_d} \frac{ml}{k} \left[\frac{k}{l} \left(\frac{l}{2} - z_i \right) - \frac{\operatorname{sh} \frac{k}{l} \left(\frac{l}{2} - z_i \right)}{\operatorname{ch} \frac{k}{2}} \right]. \tag{2.15}$$

We shall have in our case

$$\delta_{1q} = -\Omega \frac{1}{GJ_d} \frac{ml}{k} \left[\frac{k}{l} \left(\frac{l}{2} - z_1 \right) - \frac{\operatorname{sh} \frac{k}{l} \left(\frac{l}{2} - z_1 \right)}{\operatorname{ch} \frac{k}{2}} \right]$$

$$\delta_{2q} = -\Omega \frac{1}{GJ_d} \frac{ml}{k} \left[\frac{k}{l} \left(\frac{l}{2} - z_2 \right) - \frac{\operatorname{sh} \frac{k}{l} \left(\frac{l}{2} - z_2 \right)}{\operatorname{ch} \frac{k}{2}} \right].$$

Since in the case of a uniformly distributed external load $q = 9$ kg/cm with an eccentricity $e = 7.11$ cm, the torsional moment has the value $m = -63.99$ kg and since

$$\frac{k}{l} \left(\frac{l}{2} - z_1 \right) = -\frac{k}{l} \left(\frac{l}{2} - z_2 \right) = 3.278,$$

$$\operatorname{sh} \frac{k}{l} \left(\frac{l}{2} - z_1 \right) = -\operatorname{sh} \frac{k}{l} \left(\frac{l}{2} - z_2 \right) = 13.24,$$

$$\operatorname{ch} \frac{k}{2} = 14.86,$$

we obtain

$$GJ_d\delta_{1q} = -GJ_d\delta_{2q} = 766800. \tag{2.16}$$

The system of equations (2.1) takes the following form, by virtue of (2.12), (2.14) and (2.16),

$$\left. \begin{aligned} 238.6Z_1 - 42.26Z_2 &= -766800, \\ -42.26Z_1 + 238.6Z_2 &= 766800. \end{aligned} \right\} \tag{2.17}$$

The solution of this system gives for the forces Z_1 and Z_2, which simulate the acrion of the braces on the beam, the following values

$$-Z_1 = Z_2 = 2730 \text{ kg}.$$

As could be expected considering the symmetry of the problem, these forces are equal in magnitude and opposite in sign. To these forces correspond the external bimoments $\overline{B}_1$ and $\overline{B}_2$ which are applied to the beam at the same sections and which are determined by the equations:

$$\overline{B}_1 = Z_1\Omega, \quad \overline{B}_2 = Z_2\Omega.$$

Denoting the absolute magnitude of these bimoments by $\overline{B}$, we obtain

$$\overline{B} = -\overline{B}_1 = -Z_1\Omega = \overline{B}_2 = Z_2\Omega = 309800 \text{kg}\cdot\text{cm}^2. \tag{2.18}$$

We can now evaluate the influence of the braces on the torsional stresses and the warping which appear in the beam due to the action of the external load. We shall examine the stresses (flexural as well as torsional) in the middle of the span where they will be largest. Since the normal

torsional stresses are determined by the equation

$$\sigma_\omega = \frac{B(z)}{J_\omega}\omega(s)$$

and since the quantities $\omega(s)$ and J_ω, as well known, depend only on the form of the contour line of the beam and the influence of the load is determined only by the magnitude of the bimoment $B(z)$ in the section $z = \text{const}$ of interest to us, we need not compare the stresses σ_ω for the solution of the problem and it is sufficient to limit ourselves to the comparison of the bimoments $B(z)$, which indeed we do.

In the absence of braces at the ends of the beam, the bimoment due to an external load (in our case the external torsion moment of magnitude $m = -63.99$ kg uniformly distributed over the length of the beam) will be determined by equation (II.5.9)

$$B_q(z) = m\frac{l^2}{k^2}\left[1 - \frac{\operatorname{ch}\frac{k}{l}\left(\frac{l}{2} - z\right)}{\operatorname{ch}\frac{k}{2}}\right],$$

for $z = \frac{l}{2}$ we shall have, in particular,

$$B_q\left(\frac{l}{2}\right) = m\frac{l^2}{k^2}\left(1 - \frac{1}{\operatorname{ch}\frac{k}{2}}\right) = -116800\ \text{kg}\cdot\text{cm}^2. \qquad (2.19)$$

In the presence of braces at the ends of the beam, the bimoment due to the action of the same load is obtained by adding to (2.19) the bimoment due to the action of the external concentrated bimoments $\overline{B}_1$ and $\overline{B}_2$ which replace the action of the braces on the beam.

The bimoment in any section $z > z_1$, due to an external bimoment $\overline{B}_1$ applied at the section $z_1 = \text{const}$, is expressed by the third equation (II.7.2),

$$B(z) = -GJ_d\theta_0'\frac{l}{k}\operatorname{sh}\frac{k}{l}z + H_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z - \overline{B}_1\operatorname{ch}\frac{k}{l}(z - z_1), \qquad (2.20)$$

where θ_0' and H_0, according to (II.7.1), take the form

$$\theta_0' = \frac{\overline{B}_1}{GJ_d}\frac{1}{l}\left[1 - k\frac{\operatorname{ch}\frac{k}{l}(l - z_1)}{\operatorname{sh}k}\right]; \qquad H_0 = \frac{\overline{B}_1}{l}.$$

The bimoment in any section $z < z_2$, due to an external bimoment $\overline{B}_2$ applied at the section $z_2 = \text{const}$, is expressed by the third of equations (II.7.3),

$$B(z) = -GJ_d\theta_0'\frac{l}{k}\operatorname{sh}\frac{k}{l}z + H_0\frac{l}{k}\operatorname{sh}\frac{k}{l}z, \qquad (2.21)$$

where θ_0 and H_0, according to (II.7.1), take the form

$$\theta'(0) = \frac{\overline{B}_2}{GJ_d}\frac{1}{l}\left[1 - k\frac{\operatorname{ch}\frac{k}{l}(l - z_2)}{\operatorname{sh}k}\right]; \quad H_0 = \frac{\overline{B}_2}{l}.$$

Combining equations (2.20) and (2.21), keeping in mind equation (1.18) and carrying out certain simple transformations of the hyperbolic functions, we obtain an equation for the bimoment in an arbitrary section $z_1 < z < z_2$

due to the external bimoments $\overline{B}_1$ and $\overline{B}_2$ which simulate the action of both braces on the beam

$$B_{br}(z) = -\frac{\overline{B}}{\operatorname{sh} k}\operatorname{ch}\frac{k}{l}z_1\left[\operatorname{sh}\frac{k}{l}z + \operatorname{sh}\frac{k}{l}(l-z)\right]. \tag{2.22}$$

For $z=\frac{l}{2}$ (in the middle of the span) this bimoment will have the following value

$$B_{br}\left(\frac{l}{2}\right) = 2\overline{B}\frac{\operatorname{sh}\frac{k}{2}}{\operatorname{sh} k}\operatorname{ch}\frac{k}{l}z_1 = \overline{B}\frac{\operatorname{ch}\frac{k}{l}z_1}{\operatorname{ch}\frac{k}{2}}, \tag{2.23}$$

as $\frac{k}{l}(l-z_2) = \operatorname{ch}\frac{k}{l}z_1$.

Introducing in (2.23) the value of $\overline{B}$ from (2.13) and the values

$$\operatorname{ch}\frac{k}{2} = 14.86, \quad \operatorname{ch}\frac{k}{l}z_1 = 1.006,$$

we obtain

$$B_{br}\left(\frac{l}{2}\right) = 20980\ \text{kg}\cdot\text{cm}^2. \tag{2.24}$$

Comparing (2.19) with (2.24) we observe that the braces reduce by 18% the magnitude of the bimoment in the middle of the span and, consequently, the longitudinal stresses σ_ω due to torsion. These braces, each of them having a width $d = 10$ cm while the total span of the beam is $l = 300$ cm, cover only 1/15 of the span.

Figure 121a shows graphs of the bimoments in the absence and in the presence of braces at the ends.

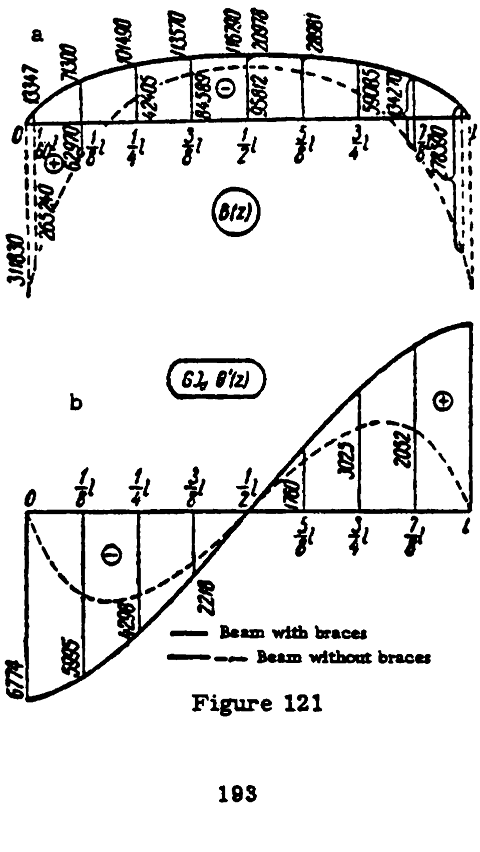


Figure 121

We shall now examine how these braces influence the warping of the cross sections of the beam. Avoiding unnecessary calculations we shall not examine and compare the actual relative warpings $\theta'(z)$, but quantities differing only by a constant factor GJ_d, namely: $GJ_d\theta'(z)$. Consequently, in speaking about warping we shall mean the torsional moment proportional to it.

For the action of an external load m on a beam not reinforced by braces we have in our case, in virtue of (II.5.9),

$$GJ_d\theta'_q(z) = m\frac{l}{k}\left[\frac{k}{l}\left(\frac{l}{2} - z\right) - \frac{\operatorname{sh}\frac{k}{l}\left(\frac{l}{2} - z\right)}{\operatorname{ch}\frac{k}{2}}\right]. \tag{2.25}$$

As seen from equation (2.5), the relative warping vanishes in the middle of the span $\left(z = \frac{l}{2}\right)$, and reaches its largest absolute values at the ends for $z = 0$ and $z = l$. As a result of symmetry, the warpings at the ends of the beam are equal in their absolute value and opposite in sign.

In our case, we obtain the following value for the warping at $z = 0$

$$GJ_d\theta'_q(0) = m\frac{l}{k}\left(\frac{k}{2} - \frac{\operatorname{sh}\frac{k}{2}}{\operatorname{ch}\frac{k}{2}}\right) = -6774\ \text{kg}\cdot\text{cm}. \tag{2.26}$$

The relative warping due to the external bimoments $\overline{B}_1$ and $\overline{B}_2$ which replace the action of two end braces on the beam, is computed, for $z_1 < z < z_2$, by equations (II.7.2),

$$GJ_d\theta'_{\text{br}}(z) = GJ_d\theta'_0 \operatorname{ch}\frac{k}{l}z + H_0\left(1 - \operatorname{ch}\frac{k}{l}z\right),$$

where, by virtue of (II.7.1),

$$GJ_d\theta'_0 = \frac{\overline{B}_1}{l}\left[1 - k\frac{\operatorname{ch}\frac{k}{l}(l - z_1)}{\operatorname{sh} k}\right] + \frac{\overline{B}_2}{l}\left[1 - k\frac{\operatorname{ch}\frac{k}{l}(l - z_2)}{\operatorname{sh} k}\right],$$

$$H_0 = \frac{\overline{B}_1}{l} + \frac{\overline{B}_2}{l}.$$

For the end of the beam, $z = 0$, we shall have by using (2.18)

$$GJ_d\theta'_{\text{br}}(0) = \overline{B}\frac{k}{l}\frac{\operatorname{ch}\frac{k}{l}z_2 - \operatorname{ch}\frac{k}{l}z_1}{\operatorname{sh} k},$$

which in the present case yields

$$GJ_d\theta'_{\text{br}}(0) = 6240\ \text{kg}\cdot\text{cm}. \tag{2.27}$$

Comparing (2.27) with (2.26), we note that braces of width $a = 10$ cm placed at the ends of the beam (referred to the sections z_1 and z_2) reduce the warping there by 92%. Had we not taken into account the flexibility of the braces by neglecting in equations (2.17) the components of (2.12), we would have obtained an absolute value of $GJ_d\theta'_{\text{br}}(0)$ nearer to that of (2.26), viz.

$$GJ_d\beta'_{br}(0) = 6714 \text{ kg}\cdot\text{cm},$$

the absolute value of which is 99% of $GJ_d\beta'_q(0)$.

In the sections $z = z_1$ and $z = z_2$ in the presence of rigid braces the warping due to a load is generally missing, since the warpings due to the load and to the influence of the braces are equal in their absolute value and opposite in sign so that they cancel out.

Figure 121b shows the graphs of the variation of the quantity $GJ_d\beta'(z)$ for a beam which is reinforced at the ends by braces and for an unreinforced beam. It is evident from these graphs how strongly the warping is affected even by two braces at the ends of the beam.

We should do well to clarify still another problem by the example of the truss girder, namely whether the behavior of the beam is considerably changed if we connect by braces not the edges 1 and 3, as done above, but other edges, for example 2 and 5. In this case the area of the closed contour differs considerably from the area of the closed part in the above first case, and has the following value

$$\Omega = 192.8 \text{ cm}^2$$

(the numerical data are given inFigure 40a).

Let the braces be placed along the beam, located as before and having the same transverse dimensions, width and thickness. We shall examine the change in the coefficients of equations (2.17) and in the solution.

Because of the change in length of the brace (now $a = 13.37$ cm) the value of the coefficient $\bar{\delta}_{ii}$, calculated by equation (2.12), changes. For our case we obtain

$$GJ_d\bar{\delta}_{11} = GJ_d\bar{\delta}_{22} = 26{,}33. \tag{2.28}$$

Because of the change in the area Ω, the magnitude of the coefficients δ_{ik}, computed by equations (2.13), is increased $\frac{192.8}{113.5} = 2.9$ times as compared with (1.14) (proportional to the ratio of the squares of the areas Ω, since all the other quantities entering in those coefficients do not vary). The new values of the coefficients will be

$$\left.\begin{aligned} GJ_d\delta_{11} = GJ_d\delta_{22} = 631.3, \\ GJ_d\delta_{12} = GJ_d\delta_{21} = -122.0. \end{aligned}\right\} \tag{2.29}$$

Finally, the coefficients δ_{iq} (2.16) are increased $\frac{192.8}{113.5} = 1.7$ times in comparison with (2.17) (proportional to the ratio of the areas Ω) and will now be

$$GJ_d\delta_{iq} = -GJ_d\delta_{2q} = 1303000.$$

Instead of (2.17) we shall now have the following system of equations

$$\left.\begin{aligned} 657.6Z_1 - 122.0Z_2 = -1303000, \\ -122.0Z_1 + 657.6Z_2 = 1303000. \end{aligned}\right\} \tag{2.30}$$

For Z_1 and Z_2 we obtain the following values:

$$-Z_1 = Z_2 = 1671 \text{ kg},$$

and we obtain for the bimoments which correspond to these forces and which

simulate the action of the braces on the beam

$$\bar{B} = -\bar{B}_1 = -Z_1\Omega = \bar{B}_2 = Z_2\Omega = 322100\ \text{kg}\cdot\text{cm}^2.$$

Comparing the results of (2.31) and (2.18), we note that notwithstanding the large difference in the coefficients of the systems (2.17) and (2.30), the bimoments which replace the action of the brace on the beam in the second case increase the result obtained in the first case by about 2%. This becomes clear if, considering (say) the system (2.17), we assume . at the braces which reinforce the beam are absolutely rigid. Under such a assumption we should obviously hold that the coefficients $\bar{\delta}_{11}$ and $\bar{\delta}_{22}$, which account for the flexibility, are equal to zero. The system of equations (2.17) will then contain only the coefficients δ_{ik} (2.13) which have a factor Ω^2 and the coefficients δ_{iq} which contain the factor Ω, as apparent from (2.15).

If we now consider in this system as unknowns not Z_1 and Z_2, but $B_1 = Z_1\Omega$ and $B_2 = Z_2\Omega$, the whole system of equations (2.17) can be reduced by Ω and the coefficients of equation (2.17) and, consequently, its solution, will not depend on the area Ω formed in the beam due to the presence of the braces. We hence conclude that, if we consider the brace to be absolutely rigid, it is completely indifferent whether a large or a small closed domain is formed in the beam at the connections of the braces.

The fact that we obtained in the example discussed above a large disparity between the results of (2.31) and (2.18) is explained by having taken into consideration the elasticity of the brace whereas its length, though not considerably, differs in both cases, other conditions being equal. Furthermore, even if the braces were of the same length, their relative value in the coefficients (where $\bar{\delta}_{ii}$ is added to δ_{ii}) would be different for various Ω, as easily seen from a comparison of (2.12) and (2.14) with their corresponding expressions (2.28) and (2.29). In the concrete example investigated, even a large difference in the areas Ω had practically a small effect on the results, but carrying out the analysis can also help the reader to assess the usefulness of placing the braces in other cases.

The theory developed in this section which takes into consideration the braces, the numerical example and the graphs in Figure 121 show that the presence of braces has a considerable effect on the stress and strain state of a thin-walled beam.

§ 3. Beams reinforced by closely-spaced strips and diagonal braces

We shall examine the problem of the three-dimensional design of structures of the thin walled beam-shell type of open section, reinforced in any longitudinal plane by transverse connections (strips, diagonals, etc). We assume that these transverse connections, in contrast with the problem examined in § 2, are closely spaced along the beam. The diagrams of such structures, with a channel as the main thin-walled beam, are shown schematically in Figure 122.

We shall consider such a structure as a composite spatial system consisting of a thin-walled beam and of an elastic reduced orthotropic plate

equivalent in its mechanical properties to transverse connections* (Figure 123). Making an imaginary cut along the line connecting the reduced plate to the beam, we must simulate by shear forces the action of this plate on the beam (Figure 124a). We denote the shear forces per unit length by $T(z)$.

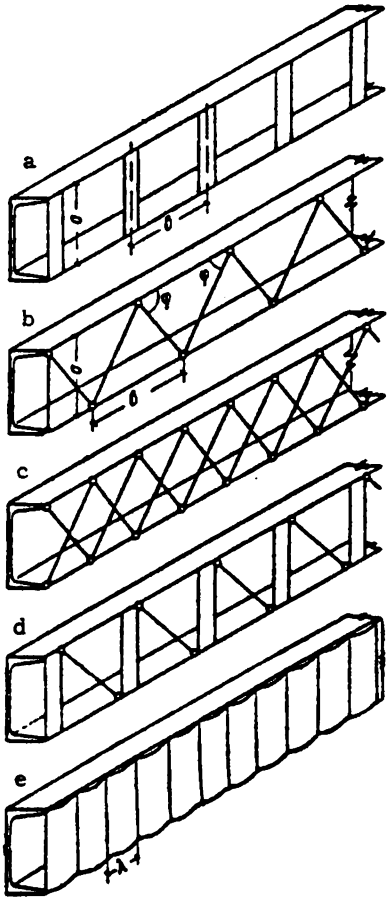


Figure 122

We return to the differential equations of equilibrium of the beam in principal coordinates (I.7.2). We rewrite these equations for the case of only a transverse load distribution and shear forces, applied along the lateral edges,

$$\left.\begin{aligned} EF\zeta'' &= T_K - T_L, \\ EJ_y\xi^{IV} &= q_x + T'_L x_L - T'_K x_K, \\ EJ_x\eta^{IV} &= q_y + T'_L y_L - T'_K y_K, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' &= m + T'_L\omega_L - T'_K\omega_K. \end{aligned}\right\} \qquad (3.1)$$

* An orthotropic plate was understood to be a plate exhibiting different properties in extension, shear and bending. In our case the orthotropy manifests itself by the plate being only able to sustain shear stresses and not tensile stresses.

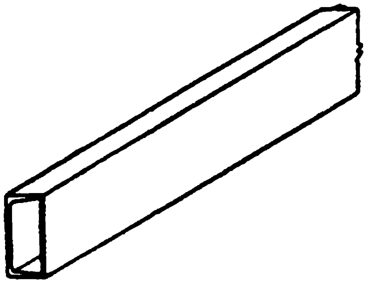


Figure 123

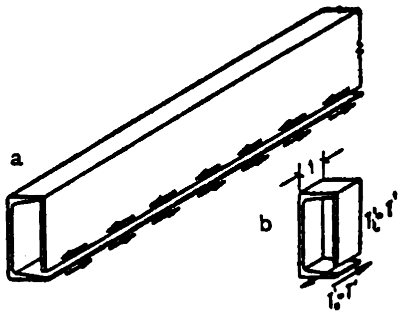


Figure 124

We cut from the beam (Figure 124a) along the section $z = \text{const}$, an elementary transverse strip of unit width (Figure 124b). On the cut we obviously have

$$T(z) = T_K(z) = T_L(z). \tag{3.2}$$

Let Ω, as shown in § 2, represent twice the area of the closed contour formed in the cross section of the contour line of the beam and the conditioning plate*. Using (3.2), we write the differential equations of equilibrium (3.1), corresponding to our case, in the following form

$$\left.\begin{aligned} EF\zeta'' &= 0, \\ EJ_y\xi^{IV} &= q_x, \\ EJ_x\eta^{IV} &= q_y, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' &= m + T'\Omega. \end{aligned}\right\} \tag{3.3}$$

Here, as before, T' denotes the derivatives of the shear force T with respect to the variable z.

By comparing the system of differential equations (3.3) with the system obtained earlier (I.7.3) we see that the presence of the conditioning elastic plate, whose action on the beam we replaced by the shear forces T, is not at all expressed in the first three equations of the system, which refer to extension (compression) and to bending in two planes. The fourth equation of the system, which refers to the torsion of the beam, differs from the one obtained earlier by the additinoal term $T'\Omega$.

For a shell of closed section, this term evidently represents the complementary torsional moment m_T due to the shear forces $T = T(z)$, uniformly distributed in the cross section $z = \text{const}$ on the contour line. Indeed, for the torsional moment H_T due to the shear forces T at an arbitrary section $z = \text{const}$ we obtain the value

$$H_T = \oint Th\,ds, \tag{3.4}$$

* We ought to emphasize that the plate we call "conditioning" is equivalent in its mechanical properties to transverse connections subjected to shear only.

As to the orthogonality conditions, these express the orthogonality of the diagrams of 1, x, y, ω, according to which the normal stresses vary. Therefore, since the transverse connections only sustain shear, they obviously will not affect the geometrical characteristics of the section of the beam. All the conditions of orthogonality remain valid and we are justified in using the equilibrium equations in the principal coordinates (I.7.2).

where $h = h(s)$ is the length of the perpendicular drawn to the tangent to the contour line. The integral in the equation is taken along the entire closed contour. Since $T = T(z)$ depends only on z and $h\,ds = d\omega$, we obtain from equation (3.4) for H_T the value

$$H_T = T\Omega, \tag{3.5}$$

where Ω is twice the area enclosed by the contour. The magnitude m_T of the torsional moment is obtained as the derivative of H_T:

$$m_T = H_T' = T'\Omega.$$

We now have to express the uniform shear force T the torsional deformation of a thin-walled closed section. To solve this problem we shall start from the elasticity equation

$$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial z} = \frac{T}{G\delta}. \tag{3.6}$$

Here, $u = u(z, s)$ and $v = v(z, s)$ are as before the displacements of a point on the cylindrical middle surface, directed along the generator and the tangent to the contour line, respectively; G is the shear [rigidity] modulus; δ is the thickness of the shell which can be also variable in the section $z = \text{const}$. Equation (3.6), for a beam-shell with a rigid cross section, takes the form

$$\frac{\partial u}{\partial s} = \frac{T}{G\delta} - h\theta'.$$

From these equations we find

$$u = \int_0^s \frac{T}{G\delta}\,ds - \theta'\omega(s) + U_0(z),$$

where $U_0(z)$ is an arbitrary integration function which represents the longitudinal displacement of the origin of the coordinate s and of the sectorial area $\omega(s)$. Since the longitudinal displacement $u = u(z, s)$ should satisfy the condition of continuity, in all points of a closed cross section, we obtain the equation

$$\oint \frac{T}{G\delta}\,ds - \theta'\Omega = 0,$$

where the integral is taken over the entire closed contour. Considering, in this equation, $T = T(z)$ not to depend on s, we obtain

$$\delta_T T = \Omega\theta'. \tag{3.7}$$

The coefficient δ_T represents here the relative longitudinal displacement of the ends of an elementary transverse strip, cut out of the shell, under the action of a unit shear force in this cut. In the case of a closed shell, we obtain for this coefficient the equation

$$\delta_T = \frac{1}{G}\oint \frac{ds}{\delta}. \tag{3.8}$$

In the case of the composite system we examine here, having also a closed cross section and consisting of a shell and a reduced orthotropic

plate, the coefficient δ_T can be computed by the equation

$$\delta_T = \frac{1}{G}\int\frac{ds}{\delta} + \delta_{pl}.$$

Here the integral in the first term is a definite integral taken over that part of the section which belongs to the thin-walled beam itself; δ_{pl} is the longitudinal relative displacement due to a force $T=1$, which occurs as a result of the elastic flexibility of the (reduced) orthotropic plate only.

Considering the thin-walled beam to be free of shear (as laid down in the initial geometrical hypotheses (§ 2, Chapter I), i.e. assuming in the last equation for the integral which extends only over the profile of the beam $G=\infty$, we obtain

$$\delta_T = \delta_{pl}.$$

Thus the relative longitudinal displacement of the ends of the transverse strip occurs as a result of the fact that only one elastic orthotropic plate is subjected to shear.

From equation (3.7) we obtain the following expression for the shear force $T(z)$:

$$T(z) = \frac{\Omega}{\delta_T}\theta'(z). \tag{3.9}$$

In this equation Ω and δ_T do not depend on the coordinate z. Therefore, the derivative of the shear force T with respect to z, appearing in the fourth equation of the system (3.3), can be given in the form

$$T' = \frac{\Omega}{\delta_T}\theta''(z)$$

and, consequently,

$$\Omega T' = \frac{\Omega^2}{\delta_T}\theta''(z). \tag{3.10}$$

We introduce the notation

$$\frac{\Omega^2}{\delta_T G} = \bar{J}_d. \tag{3.11}$$

Equation (3.10) will take the form

$$\Omega T' = G\bar{J}_d\theta''(z).$$

Using the last expression, we rewrite the differential equation for the torsion of the system (2.3) in the following form

$$EJ_\omega\theta^{IV} - G(J_d + \bar{J}_d)\theta'' = m. \tag{3.12}$$

We see that equation (3.12) is identical with equation (II.2.1) and, therefore, everything referring to the solution of equation (II.2.1) for an open section can be used for the solution of analogous problems, which refer to a thin-walled beam reinforced by transverse connections, and which were enumerated at the beginning of the present section. For this purpose it is necessary to replace the moment of inertia for a thin-walled beam in pure torsion, J_d, in all equations by the sum $J_d + \bar{J}_d$, where $\bar{J}_d$, as calculated by equation (3.11), can be called in analogy with J_d the torsional moment of inertia of a thin-walled beam with transverse

connections. In particular, for the characteristic parameter k, computed by equation (II.2.2), it is necessary to apply now the equation

$$k=l\sqrt{\frac{G(J_d+\bar{J}_d)}{EJ_\omega}}.$$

Instead of equation (I.5.7) for the torsional moment H_k which appears in pure torsion, it is necessary to use the equation

$$H_k=G(J_d+\bar{J}_d)\theta'.$$

We give the formulas for the coefficients δ_T and their corresponding moments of inertia $\bar{J}_d$ for the more common forms of transverse connections.

a) For the case shown in Figure 122e, where the connections are in the form of an elastic orthotropic plate subjected to shear only, we shall have the following equation for the coefficient δ_T

$$\delta_T=\frac{a}{G\delta},$$

where a, δ and G are, respectively, the width, thickness and rigidity modulus of the plate. The inertia moment $\bar{J}_d$ is calculated by equation (3.11).

b) For the case of the transverse connections having the form of transverse braces of a rectangular section uniformly spaced along the span (Figure 122a) and being subjected not only to shear but also to bending away from the plane of the cross section, we obtain the following expression for δ_T

$$\delta_T=\frac{ab}{G}\left(\frac{a^2G}{12EJ_{\text{br}}}+\frac{1.2}{F_{\text{br}}}\right), \tag{3.13}$$

where a is the length of the brace, b is the step (distance between the braces), E and G are the elastic moduli of the brace, F_{br} is the area of the brace cross section, and J_{br} the moment of inertia.

Equation (3.13) is obtained from equation (2.3) by introducing the factor b, as we transform discrete transverse connections into a conventional continuous elastic plate for which the displacements δ_T, due to a shear force $T=1$, will obviously be b times larger than for a continuous elastic plate whose cross section is equal to the longitudinal section of the brace.

The moment of inertia $\bar{J}_d$ is determined in this case by the equation

$$\bar{J}_d=\frac{\Omega^2}{\delta_T G}=\frac{\Omega^2}{ab}\cdot\frac{1}{\frac{a^2G}{12EJ}+\frac{1.2}{F}}. \tag{3.14}$$

c) For the case of the transverse connections having the form of diagonal braces and making an angle φ with the longitudinal edge (Figure 122b), the expression for δ_T becomes

$$\delta_T=\frac{a^2}{EF\sin^2\varphi\cos\varphi}, \tag{3.15}$$

where a is the depth of the girder, F is the area of the cross section and E is the modulus of longitudinal extension of the diagonal brace.

This equation is obtained in the following way (Figure 125). If the diagonal brace AB were lengthened so that the point A remained on its place but the point B moved by a distance B' to the position δ, a tensile force

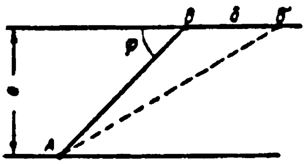


Figure 125

would have appeared in the beam

$$N = \frac{\cos\varphi \sin\varphi}{a} EF\delta.$$

The projection of this force on the direction of the displacement δ equals

$$N\cos\varphi = \frac{\cos^2\varphi \sin\varphi}{a} EF\delta.$$

For the shear force per unit length of the beam we obtain

$$S = \frac{N\cos\varphi}{a \operatorname{ctg}\varphi} = \frac{EF\cos\varphi \sin^2\varphi}{a^2}\delta,$$

whence equation (3.15) is obtained. $\bar{J}_d$ is computed by the equation

$$\bar{J}_d = \frac{E}{G}\frac{\Omega^2}{a^2} F \sin^2\varphi \cos\varphi. \qquad (3.16)$$

d) In the case of a cross lattice (Figure 122c) the previous equations (3.15) and (3.16) remain if the diagonals are subjected to extension only.

If the diagonals are subjected to extension and compression, the equations take the form

$$\delta_T = \frac{a^2}{2EF\sin^2\varphi\cos\varphi}, \quad \bar{J}_d = \frac{2\Omega^2 EF}{a^2 G}\sin^2\varphi\cos\varphi.$$

e) For the case of transverse connections, a combination of the previous ones (see, for example, Figure 122d), we obtain expressions for δ_T by adding the respective values obtained before.

The example of the truss girder, examined already many times over, will illustrate the application of this method of calculation. Figure 216 shows the graphs of the variation of the bimoments $B(z)$ and of the torsional moments $G(J_d + \bar{J}_d)\theta'(z)$ over the length of the span for a loaded truss girder, having the same load and boundary conditions as the example given earlier in § 2 for the case of the beam reinforced by 10 strips uniformly spaced along it.

The dimensions of the strips are the same as in § 2; length $a = 11.28$ cm, thicknes $\delta = 0.6$ cm, width $d = 10$ cm, double the area inside the closed contour formed at the connection of the strip, $\Omega = 113.48$ cm^2, the step of the strips $b = 32.22$ cm.

In this case we obtain:

$$J_d = 124.4\text{cm}^4, \quad J_d + \bar{J}_d = 130.6\text{cm}, \quad k = 31.20.$$

Here, as in § 5 of Chapter II, we took $G = 0.4E$.

If the same beam is reinforced by 10 diagonals according to the diagram shown in Figure 122b, we obtain, assuming the area of the cross section of the diagonal $F = 1$ cm^2, the step $b = 30$ cm and $\varphi = 20°36'$:

$$\bar{J}_d = 29.32\text{cm}^4, \quad k = 16.27.$$

The variation of the bimoment $B(z)$ and the torsional moment $G(J_d + \bar{J}_d)\theta'(z)$ along the span will have the same form as in Figure 126, and the corresponding maximum ordinates will be as follows.

For the bimoment in the middle of the span

$$B\left(\frac{l}{2}\right) = -21\,770\,\text{kg}\cdot\text{cm}^2,$$

For the torsional moment at the ends of the beam

$$G(J_d+\bar{J}_d)\theta'(l)=-G(J_d+\bar{J}_d)\theta'(0)=8418\ \text{kg·cm}.$$

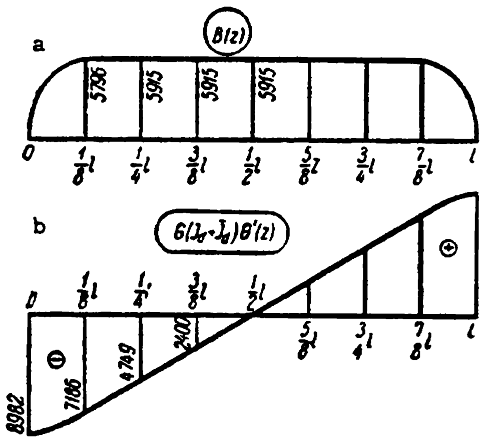


Figure 126

In order to determine the relative warping as a function of the form of the transverse connections, we introduce the quantity $G\theta'(z)$ for $z=$ for three cases:

1) For a beam not reinforced by transverse connections,

$$G\theta'(l)=1098\frac{\text{kg}}{\text{cm}^3};$$

2) For a beam reinforced by 10 uniformly spaced strips along the beam

$$G\theta'(l)=68.78\frac{\text{kg}}{\text{cm}^3};$$

3) For a beam reinforced by 10 diagonal braces

$$G\theta\ (l)=237.2\frac{\text{kg}}{\text{cm}^3}.$$

§ 4. Beams reinforced by diaphragms

1. We shall derive equations which account for the influence of stiffening plates on the behavior of a thin-walled beam. In deriving the equations we shall, as before, be interested only in the torsion of the beam associated with a warping of the section.

We shall assume that an elastic transverse plate of thickness z_1 is connected to the beam in the section situated at h from the initial section (Figure 127).

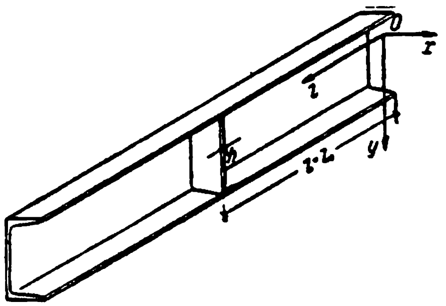


Figure 127

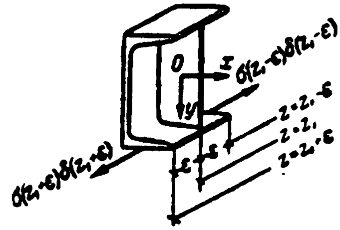


Figure 128

We shall solve the given contact problem by the variational method according to the principle of virtual displacements. Taking two cross sections $z=z_1-\varepsilon$ and $z=z_1+\varepsilon$, we isolate from the thin-walled beam an elementary strip which includes the plate, which we relate to the cross section $z=z_1$ (Figure 128), and we write for this strip the equation of virtual work. As the virtual displacements we take the longitudinal displacements $u\ (z,\ s)$. However, since we are interested only in the part of the displacements connected with the warping of the section, we shall use equation (2.5) for determining these displacements

$$u(z,\ s)=-\theta'(z)\,\omega(s). \tag{4.1}$$

The displacements $w(x,y)$ away from the plane of the plate, directed along the axis Oz and appearing in the pure torsion of this plate, will be considered admissible.

From the condition of compatibility of deformation it is obvious that the following conditions should be satisfied at the connection of the plate to the beam

$$u(z_1,\ s)=w(x,\ y). \tag{4.2}$$

By equating to zero the work of the external and internal forces in the corresponding virtual displacements, we obtain

$$-\int(\sigma_1\delta)\,u(z_1,\ s)\,ds+\int(\sigma_2\delta)\,u(z_1,\ s)\,ds+ \\ +\iint 2D(1-\mu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 dx\,dy=0. \tag{4.3}$$

The last term, a double integral evaluated over the whole surface of the plate, expresses the work of the external forces, i.e., of the torsional moments, in the corresponding torsional deformations. This term is obtained as follows.

It is known from the theory of plates that the torsional moments are determined by the equation

$$H_x=H_y=-D(1-\mu)\frac{\partial^2 w}{\partial x\,\partial y}, \tag{4.4}$$

where $D=\frac{Eh^3}{12(1-\mu^2)}$ is the cylindrical rigidity of the plate, h its thickness, μ Poisson's ratio. Each torsional moment H_x and H_y does work to produce

the torsional curvature $\frac{\partial^2 w}{\partial x\,\partial y}$. Since the torsional moments and the torsional deformation have different signs and the work of the internal forces is taken as negative, we obtain the last term of equation (4.3) by integrating the work done by the torsional moments over the whole area of the plate. The first two terms, expressed by integrals taken over the cross section of the thin-walled beam, refer to the work of the external forces with respect to our strip in the virtual longitudinal displacements $u(z_1, s)$. Such forces will be the longitudinal normal forces of the cross section $(\sigma\delta)$ for the isolated element of the beam. Here δ is the thickness of the thin-walled beam and σ the longitudinal normal stress which we denoted by $(\sigma_1\delta)$ for the section $z = z_1 - \varepsilon$ and by $(\sigma_2\delta)$ for the section $z = z_1 + \varepsilon$. We present these integrals in another form. Assuming $\delta\, ds = dF$ and using equation (4.1) we obtain

$$\int (\sigma\delta)\, u(z_1, s)\, ds = -\theta'(z_1) \int \sigma w(s)\, dF.$$

Since, by definition, $\int \sigma\omega\, dF = B$ represents the bimoment, we have

$$\int (\sigma\delta)\, u\, ds = -\theta'(z_1) B.$$

Consequently the first two terms can be replaced by

$$\theta'(z_1 - \varepsilon) B_1 - \theta'(z_1 + \varepsilon) B_2, \tag{4.5}$$

where

$$B_1 = \int \sigma_1 \omega\, dF,$$
$$B_2 = \int \sigma_2 \omega\, dF,$$

and the integrals are evaluated over the whole area of the cross section of the beam.

It is possible to state the following regarding the bimoments B_1 and B_2: for $\varepsilon \to 0$, i. e. for these two sections being infinitely close and in the absence of a plate, we have in the limit $B_2 - B_1 = 0$. With a plate B_2 will differ from B_1 by a certain value B, equal to the bimoment which describes the influence of the plate on the beam.

Thus, the two terms in the expression (4.3), written in the form of (4.5), can be given the following form in the limit for $\varepsilon \to 0$ and with a diaphragm in the section $z = z_1$

$$[\theta'(z_1 - \varepsilon) B_1 - \theta'(z_1 + \varepsilon) B_2]_{\varepsilon \to 0} = -\theta'(z_1) B, \tag{4.6}$$

where B stands for the bimoment simulating the influence of the diaphragm on the beam.

Considering (4.6), we can now write (for $\varepsilon \to 0$) equation (4.3) in the following form,

$$-\theta'(z_1) B + \iint 2D(1 - \mu) \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 dx\, dy = 0. \tag{4.7}$$

We shall turn now to the calculation of the double integral in equation (4.7). It is known from the theory of plates that in the case of pure torsion of a plate the deflection w is determined by the equation

$$w = Cxy, \tag{4.8}$$

where C is a constant determined by the relevant boundary conditions.

On the other hand we showed in sub-sec. 6 of § 7, Chapter II, that the bimoment due to an arbitrary balanced longitudinal load, whether distributed continuously or over the section consisting of concentrated forces, does not depend on the shear center and on the origin of the sectorial area. Since the effect of the plate on the beam is expressed by such a balanced system of forces, we may place the pole of the sectorial areas to suit the calculation of the sectorial area $\omega(s)$ in equation (4.1).

If, in particular, we place the pole of the sectorial areas at the center of the plate and let the axes of the coordinates pass through this center (Figure 128), we obtain for $\omega(s)$ the equation

$$\omega(s) = xy, \tag{4.9}$$

where x and y are the coordinates of the corresponding point on the contour line of the beam*.

We write equation (4.1) for the virtual longitudinal displacement in the following form

$$u(z, s) = -\theta'(z_1)\, xy. \tag{4.10}$$

Since the compatibility conditions should be satisfied on the intersection of the contour line with the plane of the plate, we find by comparing (4.10) with (4.8) that

$$C = -\theta'(z_1),$$

and, therefore, we have for the displacement w of the plate

$$w = -\theta'(z_1)\, xy. \tag{4.11}$$

Introducing (4.11) in expression (4.7) and noting that $\frac{\partial^2 w}{\partial x\, \partial y} = -\theta'(z_1)$, and $\iint dx\, dy$ expresses the area of the plate, we obtain after reduction by $\theta'(z_1)$

$$B = D(1-\mu)\,\Omega\theta'(z_1), \tag{4.12}$$

where, as usual, we have denoted by Ω twice the area of the plate.

Introducing the cylindrical rigidity D from equation (4.4) we can write equation (4.12) in another form

$$B = \frac{Eh^3\Omega}{12(1+\mu)}\,\theta'(z_1). \tag{4.13}$$

This is the equation of the bimoment which replaces the effect of the plate on the beam. As seen from this formula, *the bimoment is proportional to the relative warping of the beam in the cross section of the diaphragm. The proportionality coefficient $\frac{Eh^3\Omega}{12(1+\mu)}$ depends on the physical characteristics and the dimensions of the diaphragm.*

Thus, the effect on the behavior of a thin-walled beam of a transverse plate situated at the section $z = z_k$, leads to a complementary concentrated longitudinal bimoment in this section which simulates the action of the plate

* The system of coordinates for the plates is chosen so that formula (4.9) will be valid. Generally speaking, the coordinates x and y are not principal coordinates. Equation (4.9) is valid only within the limits of the cross section.

on the beam and which is determined by equation (4.13). Using the method of initial parameters, it is now easy to write the general design equations for the torsion of a thin-walled beam reinforced by transverse plates and subjected to any external load under arbitrary boundary conditions.

Thus, for example, by using Tables 7 and 9 of § 3, Chapter II, we obtain the following design equations for a beam subjected to the action of a torsional moment of magnitude $H = ql$, uniformly distributed over the whole length of the beam, and of a complementary concentrated force factor B_{z_k}, applied at the section $z = z_k$ to replace the effect of an elastic plate

$$\left.\begin{aligned}
\theta(z) &= \theta_0 + \theta_0' \frac{l}{k} \operatorname{sh} \frac{k}{l} z + \frac{B_0}{GJ_d}\left(1 - \operatorname{ch} \frac{k}{l} z\right) + \\
&\quad + \frac{H_0}{GJ_d}\left(z - \frac{l}{k} \operatorname{sh} \frac{k}{l} z\right) + \frac{B_{z_k}}{GJ_d}\left[1 - \operatorname{ch} \frac{k}{l}(z - z_k)\right] - \\
&\quad - \frac{m}{GJ_d}\left[\frac{z^2}{2} + \frac{l^2}{k^2}\left(1 - \operatorname{ch} \frac{k}{l} z\right)\right], \\
\theta'(z) &= \theta_0' \operatorname{ch} \frac{k}{l} z - \frac{B_0}{GJ_d} \frac{k}{l} \operatorname{sh} \frac{k}{l} z + \frac{H_0}{GJ_d}\left(1 - \operatorname{ch} \frac{k}{l} z\right) - \\
&\quad - \frac{B_{z_k}}{GJ_d} \frac{k}{l} \operatorname{sh} \frac{k}{l}(z - z_k) - \frac{m}{GJ_d}\left(z - \frac{l}{k} \operatorname{sh} \frac{k}{l} z\right), \\
B(z) &= -GJ_d \theta_0' \frac{l}{k} \operatorname{sh} \frac{k}{l} z + B_0 \operatorname{ch} \frac{k}{l} z + H_0 \frac{l}{k} \operatorname{sh} \frac{k}{l} z + \\
&\quad + B_{z_k} \operatorname{ch} \frac{k}{l}(z - z_k) + m \frac{l^2}{k^2}\left(1 - \operatorname{ch} \frac{k}{l} z\right), \\
H(z) &= H_0 - mz,
\end{aligned}\right\} \quad (4.14)$$

Here $B_{z_k} = -\bar{B} = -\frac{Eh^3\Omega}{12(1+\mu)} \theta'_{z_k}$ (sub-sec. 5, § 3, Chapter II).

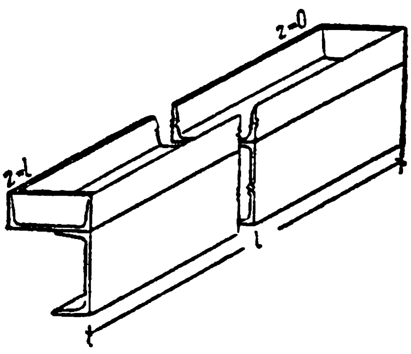


Figure 129

2. As an example we give the numerical calculation for the same truss girder which we repeatedly examined in this and the previous chapters (Figure 40b).

The geometrical dimensions of the girder, the modulus of elasticity and the load are left as before (§§ 1 and 5 Chapter II), i. e. $l = 300$ cm, $J_d = 6.17$ cm^4, $E = 2.1 \cdot 10^6$ kg/cm^2, $k = 6,782$, $m = -63.99$ kg. For the elastic plate we set: thickness $h = 1$ cm, Poisson's ratio $\mu = 0$. The plates are placed at the ends of the upper channel, as shown in Figure 129, and therefore $\Omega = 113.5$ cm^2 (§ 2). We shall regard the ends of the beam as independently hinged.

From the condition for hinged ends of the beam we have

$$\theta_0 = B_0 = 0,$$
$$\theta(l) = 0,$$
$$B(l) = \bar{B} = p\theta'(l).$$

Here, the last relation gives the bimoment at the section $z=l$, replacing the action of the elastic beam in this section. The expression $\frac{Eh^3\Omega}{12}$ is denoted by p, as follows from (4.13).

From the last two boundary conditions (with the help of equations (4.14) and taking into consideration that we also have a concentrated longitudinal bimoment at the initial section) we obtain

$$GJ_d\theta_0'\left[\frac{l}{k}\operatorname{sh}k+\frac{p}{GJ_d}(\operatorname{ch}k-1)\right]+H_0l\left(1-\frac{1}{k}\operatorname{sh}k\right)=$$
$$=ml^2\left[\frac{1}{2}-\frac{1}{k^2}(\operatorname{ch}k-1)\right],$$
$$GJ_d\theta_0'\left[\left(\frac{p}{GJ_d}\right)^2\frac{k}{l}\operatorname{sh}k+2\frac{p}{GJ_d}\frac{k}{l}\operatorname{sh}k+\frac{l}{k}\operatorname{sh}k\right]-$$
$$-H_0\left[\frac{p}{GJ_d}(\operatorname{ch}k-1)+\frac{l}{k}\operatorname{sh}k\right]=\frac{p}{GJ_d}ml\left(1-\frac{1}{k}\operatorname{sh}k\right)-\frac{ml^2}{k^2}(\operatorname{ch}k-1),$$

whence we obtain the following expressions for the initial parameters θ_0' and H_0

$$GJ_d\theta_0'=$$
$$=\frac{\frac{l}{2k}\operatorname{sh}k-\frac{l}{k^2}(\operatorname{ch}k-1)+p_1\left[\frac{1}{2}(\operatorname{ch}k+1)+\frac{2}{k^2}(\operatorname{ch}k-1)-\frac{2}{k}\operatorname{sh}k\right]}{\frac{l^2}{k}\operatorname{sh}k+p_1^2\left[k\operatorname{sh}k+2(1-\operatorname{ch}k)\right]+2pl\left(\operatorname{ch}k-\frac{\operatorname{sh}k}{k}\right)}ml^2 \quad (4.15)$$

$$H_0=\frac{\frac{l^2}{2k}\operatorname{sh}k+p_1^2\left(\frac{k}{2}\operatorname{sh}k+1-\operatorname{ch}k\right)+p_1l\left(\operatorname{ch}k-\frac{\operatorname{sh}k}{k}\right)}{\frac{l^2}{k}\operatorname{sh}k+p_1^2\left[k\operatorname{sh}k+2(1-\operatorname{ch}k)\right]+2p_1l\left(\operatorname{ch}k-\frac{\operatorname{sh}k}{k}\right)}ml,$$

where

$$p_1=\frac{p}{GJ_d}.$$

In our example

$$p_1=3.832\text{ cm},\quad GJ_d\theta_0'=-6235\text{ kg/cm},\quad H_0=-9596\text{ kg/cm}.$$

Figure 130 shows the variation along the beam of the quantities $GJ_d\theta'(z)$ and $B(z)$ for the values (4.15) of the parameters θ_0' and H_0. The functions shown are calculated for the values $z=0$, $\frac{1}{8}l$, $\frac{1}{4}l$, $\frac{3}{8}l$ and $\frac{1}{2}l$ and owing to symmetry apply to the second half of the beam as well.

From a comparison with the analogous graphs (Figure 121), calculated fro the same beam, subjected to the action of the same transverse load, though now not reinforced at the ends by plates but by braces, we note that the braces affect the state of stress of the beam more than the plates, although the bulk of material in the diaphragms is about 1.7 times that of the braces.

§ 5. Torsion of a beam embedded in an elastic medium

1. In many cases, the problem of designing structural elements leads to that of the strength and stability of thin-walled beams in an elastic medium. We shall develop below a general design method for thin-walled beams embedded in an elastic medium which undergo torsional deformations in addition to bending.

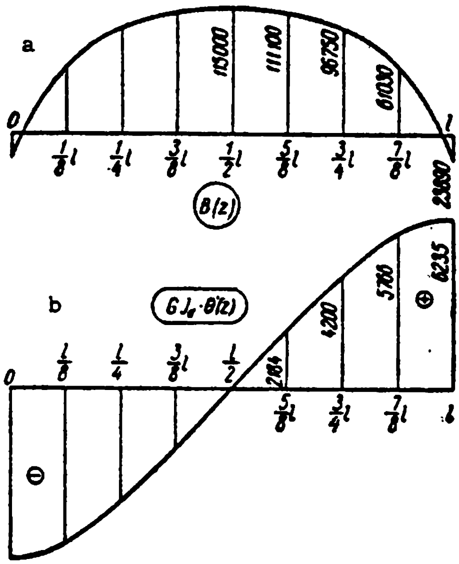


Figure 130

Suppose a beam having an arbitrary cross section and possessing a nondeformable contour is placed in a continuous elastic medium.

We shall assume, of this medium, that it elastically restrains any cross section from displacements in its plane. We shall imagine such a medium as continuously distributed elastic transverse connections along the beam, which prevent at each point z translational displacements in the plane Oxy and angular displacement about an axis parallel to the generator, in any longitudinal elementary strip, ds.

Let H and H' be infinitely close points at which the cross sections of the beam are elastically restrained from being displaced in the plane Oxy. Such a constraint is shown in Figure 131, where we denote by h_x and h_y the coordinates of the point H in the principal axes.

As the beam undergoes deformation, reactive forces parallel to the axes Ox and Oy and reactive torsional moments replacing the action of the elastic medium, appear in the elastic transverse connections. We shall take these forces and moments proportional to the respective (translational and rotational) displacements of the linear element HH' of the beam cross section in its plane.

Denoting the displacements of the point H along the principal axes of the section Ox and Oy by ξ_H and η_H respectively, and the rotation angle of the linear element HH' by θ_H, we may write the reactive forces and the

reactive moment of the elastic medium per unit length of the beam in the following form

$$\left.\begin{aligned}\bar{q}_x &= -K_\xi \xi_H,\\ \bar{q}_y &= -K_\eta \eta_H,\\ \bar{m} &= -K_\theta \theta_H + (h_x - a_x)\,\bar{q}_y - (h_y - a_y)\,\bar{q}_x,\end{aligned}\right\} \quad (5.1)$$

where K_ξ, K_η and K_θ represent the elastic constants of the medium. The first two of these coefficients have the dimensions kg/cm^2 and are numerically equal to the transverse forces which has to be applied to the elastic beam over a length of 1 cm in order to cause the point H to move 1 cm along the axes Ox or Oy. The third coefficient has the dimensions kg/cm and represents the moment per unit length of the beam corresponding to a unit rotation of the linear element HH'. All these coefficients depend only on the elastic characteristics of the medium and are easily determined in each particular case by the usual methods of structural mechanics. The minus sign on the right of equations (5.1) results from the fact that the reactive forces and moments of the elastic medium have negative values for positive linear and anglular displacements of the element HH'.

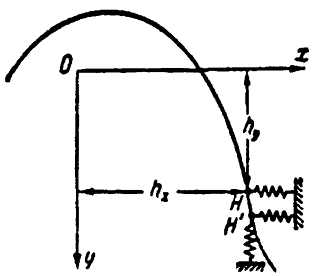


Figure 131

Starting from the hypothesis of the nondeformability of the contour of the cross section, we can express the displacements ξ_H, η_H and θ_H, referring to the linear element HH', by the displacements ξ, η of the shear center and the rotation angle θ of the whole section according to equations (I.3.1)

$$\left.\begin{aligned}\xi_H &= \xi - (h_y - a_y)\,\theta,\\ \eta_H &= \eta + (h_x - a_x)\,\theta,\\ \theta_H &= \theta.\end{aligned}\right\} \quad (5.2)$$

Introducing the equality (5.2) in equation (5.1), we obtain

$$\left.\begin{aligned}\bar{q}_x &= -K_\xi[\xi - (h_y - a_y)\,\theta],\\ \bar{q}_y &= -K_\eta[\eta + (h_x - a_x)\,\theta],\\ \bar{m} &= -K_\theta\theta + K_\xi(h_y - a_y)\,\xi - K_\eta(h_x - a_x)\,\eta -\\ &\quad -[K_\xi(h_y - a_y)^2 + K_\eta(h_x - a_x)^2]\,\theta.\end{aligned}\right\} \quad (5.3)$$

Equations (5.3) now express the reactions of the elastic medium as functions of the required basic displacements ξ, η and θ of the beam.

If the point H, at which the elastic connections are joined to the examined beam, coincides with the centroid of the section as, for example, in the case of metallic bridge constructions, $h_x = h_y = 0$, and equations (5.3) assume a simpler form

$$\begin{aligned}\bar{q}_x &= -K_\xi(\xi + a_y\theta),\\ \bar{q}_y &= -K_\eta(\eta - a_x\theta),\\ \bar{m} &= -K_\theta\theta - K_\xi a_y \xi + K_\eta a_x \eta - (K_\xi a_y^2 + K_\eta a_x^2)\,\theta.\end{aligned}$$

The reactive forces of the elastic medium obtained here with respect to the beam represent the additional external load. Combining this load with the given one we express the differential equilibrium equations (I.7.3) in the following form

$$\left.\begin{aligned} EJ_y\xi^{IV} - (\bar{q}_x + q_x) &= 0, \\ EJ_x\eta^{IV} - (\bar{q}_y + q_y) &= 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' - (\bar{m} + m) &= 0. \end{aligned}\right\} \tag{5.4}$$

Substituting here the reactive forces from equations (5.3) and transferring the load terms to the right-hand side, we have

$$\left.\begin{aligned} EJ_y\xi^{IV} + K_\xi[\xi - (h_y - a_y)\theta] &= q_x, \\ EJ_x\eta^{IV} + K_\eta[\eta + (h_x - a_x)\theta] &= q_y, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' + [K_\xi(h_y - a_y)^2 + K_\eta(h_x - a_x)^2 + K_\theta]\theta - & \\ - K_\xi(h_y - a_y)\xi + K_\eta(h_x - a_x)\eta &= m. \end{aligned}\right\} \tag{5.5}$$

Equations (5.5) for a given external load, determined by the functions $q_x(z)$, $q_y(z)$ and $m(z)$, refer to bending and torsion of a thin-walled beam embedded in an elastic medium with three bedding coefficients. These equations, for an arbitrary position of the contact point H of the beam with the elastic medium in the cross section, form a system of three ordinary linear differential equations with respect to the three unknown functions: ξ, η and θ.

For $h_x = a_x$ and $h_y = a_y$, i. e. for the case of the contact point H in the plane Oxy coinciding with the shear center A, the simultaneous system of differential equations (5.5) separates into three independent equations:

$$\left.\begin{aligned} EJ_y\xi^{IV} + K_\xi\xi &= q_x, \\ EJ_x\eta^{IV} + K_\eta\eta &= q_y, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' + K_\theta\theta &= m. \end{aligned}\right\} \tag{5.6}$$

The first two are the known equations of beams on an elastic foundation with the bedding coefficient K_ξ for bending in the plane Oxz and K_η for bending in the plane Oyz.

The third equation refers to beam torsion in an elastic medium with a bedding coefficient K_θ for torsion. These equations determine, together with the boundary conditions, the function $\theta(z)$, the torsional angle of the beam. For constant coefficients EJ_ω, GJ_d and K_θ the third equations (5.6) is easily integrated, for the general case, in terms of elementary transcendental functions, which are products of trigonometric and hyperbolic functions of different arguments.

Dividing the third equation (5.6) EJ_ω and retaining only its homogeneous part, since we intend to use the method of initial parameters, we may present it in the form

$$\theta^{IV} - 2r^2\theta'' + s^4\theta = 0, \tag{5.7}$$

where r^2 and s^4 are characteristics depending on the physical and geometrical properties of the beam and are determined by the equations

$$r^2 = \frac{GJ_d}{2EJ_\omega}, \quad s^4 = \frac{K_\theta}{EJ_\omega}. \tag{5.8}$$

Table 22

Case		Functions, their derivatives and integrals	$\Phi_1(z)$ (odd)	$\Phi_2(z)$ (even)	$\Phi_3(z)$ (odd)	$\Phi_4(z)$ (even)
1	$r<s$	$\Phi_j(z)$	$\operatorname{ch}\alpha z\cdot\sin\beta z$	$\operatorname{ch}\alpha z\cdot\cos\beta z$	$\operatorname{sh}\alpha z\cdot\cos\beta z$	$\operatorname{sh}\alpha z\cdot\sin\beta z$
		$\Phi'_j(z)$	$\alpha\Phi_4+\beta\Phi_2$	$\alpha\Phi_3-\beta\Phi_1$	$\alpha\Phi_2-\beta\Phi_4$	$\alpha\Phi_1+\beta\Phi_3$
		$\Phi''_j(z)$	$(\alpha^2-\beta^2)\Phi_1+2\alpha\beta\Phi_3$	$(\alpha^2-\beta^2)\Phi_2-2\alpha\beta\Phi_4$	$(\alpha^2-\beta^2)\Phi_3-2\alpha\beta\Phi_1$	$(\alpha^2-\beta^2)\Phi_4+2\alpha\beta\Phi_2$
		$\Phi'''_j(z)$	$\alpha(\alpha^2-3\beta^2)\Phi_4+$ $+\beta(3\alpha^2-\beta^2)\Phi_2$	$\alpha(\alpha^2-3\beta^2)\Phi_3-$ $-\beta(3\alpha^2-\beta^2)\Phi_1$	$\alpha(\alpha^2-3\beta^2)\Phi_2-$ $-\beta(3\alpha^2-\beta^2)\Phi_4$	$\alpha(\alpha^2-3\beta^2)\Phi_1+$ $+\beta(3\alpha^2-\beta^2)\Phi_3$
		$\int\Phi_j(z)\,dz$	$\dfrac{\alpha\Phi_4-\beta\Phi_2}{\alpha^2+\beta^2}$	$\dfrac{\alpha\Phi_3+\beta\Phi_1}{\alpha^2+\beta^2}$	$\dfrac{\alpha\Phi_2+\beta\Phi_4}{\alpha^2+\beta^2}$	$\dfrac{\alpha\Phi_1-\beta\Phi_3}{\alpha^2+\beta^2}$
2	$r=s$	$\Phi_j(z)$	$\operatorname{sh} rz$	$\operatorname{ch} rz$	$z\operatorname{ch} rz$	$z\operatorname{sh} rz$
		$\Phi'_j(z)$	$r\Phi_2$	$r\Phi_1$	$\Phi_2+r\Phi_4$	$\Phi_1+r\Phi_3$
		$\Phi''_j(z)$	$r^2\Phi_1$	$r^2\Phi_2$	$2r\Phi_1+r^2\Phi_3$	$2r\Phi_2+r^2\Phi_4$
		$\Phi'''_j(z)$	$r^3\Phi_2$	$r^3\Phi_1$	$3r^2\Phi_2+r^3\Phi_4$	$3r^2\Phi_1+r^3\Phi_3$
		$\int\Phi_j(z)\,dz$	$\dfrac{1}{r}\Phi_2$	$\dfrac{1}{r}\Phi_1$	$\dfrac{1}{r}\Phi_1-\dfrac{1}{r^2}\Phi_2$	$\dfrac{1}{r}\Phi_2-\dfrac{1}{r^2}\Phi_1$

3	$s < r$	$\Phi_j(z)$	$\operatorname{sh}\mu_1 z$	$\operatorname{ch}\mu_1 z$	$\operatorname{sh}\mu_2 z$	$\operatorname{ch}\mu_2 z$
		$\Phi_j'(z)$	$\mu_1\Phi_2$	$\mu_1\Phi_1$	$\mu_2\Phi_4$	$\mu_2\Phi_3$
		$\Phi_j''(z)$	$\mu_1^2\Phi_1$	$\mu_1^2\Phi_2$	$\mu_2^2\Phi_3$	$\mu_2^2\Phi_4$
		$\Phi'''(z)$	$\mu_1^3\Phi_2$	$\mu_1^3\Phi_1$	$\mu_2^3\Phi_4$	$\mu_2^3\Phi_3$
		$\int\Phi_j(z)\,dz$	$\frac{1}{\mu_1}\Phi_2$	$\frac{1}{\mu_1}\Phi_4$	$\frac{1}{\mu_2}\Phi_4$	$\frac{1}{\mu_2}\Phi_3$
4	$s = 0$	$\Phi_j(z)$	z	1	$\operatorname{sh}kz$	$\operatorname{ch}kz$
		$\Phi_j'(z)$	1	0	$k\Phi_4$	$k\Phi_3$
		$\Phi_j''(z)$	0	0	$k^2\Phi_3$	$k^2\Phi_4$
		$\Phi_j'''(z)$	0	0	$k^3\Phi_4$	$k^3\Phi_3$
		$\int\Phi_j(z)\,dz$	$\frac{z^2}{2}$	z	$\frac{1}{k}\Phi_4$	$\frac{1}{k}\Phi_3$

2. Equation (5.7) has the more general form of a fourth order differential equation, mentioned in § 3, Chapter II. For $r^2 \neq 0$, $s^4 = 0$ we obtain the equation for restrained torsion, studied in detail above. For $r^2 = 0$, $s^4 \neq 0$ we obtain an equation analogous to the first two equations of the system (5.6). For $r^2 = s^4 = 0$ we obtain an equation analogous to that of bending in a simple beam.

The characteristic equation which corresponds to equation (5.7) is

$$k^4 - 2r^2k^2 + s^4 = 0.$$

The roots of this equation are found from

$$k = \pm r \sqrt{1 \pm \sqrt{1 - \frac{s^4}{r^4}}} \tag{5.9}$$

and depend, therefore, on the ratio of s and r.

We shall examine some values of these ratios, noting for a start that physical considerations exclude negative values of the quantities s and r

a) $s > r$. All the four values of k are complex and can be given in the form

$$k = \pm \alpha \pm \beta i,$$

where α and β are real positive numbers determined by the equations

$$\alpha = \sqrt{\frac{s^2 + r^2}{2}}, \quad \beta = \sqrt{\frac{s^2 - r^2}{2}}. \tag{5.10}$$

The general solution of the homogeneous differential equation (5.7) can be written in the form

$$\theta(z) = C_1\Phi_1(z) + C_2\Phi_2(z) + C_3\Phi_3(z) + C_4\Phi_4(z). \tag{5.11}$$

The particular solutions $\Phi_1(z)$, $\Phi_2(z)$, $\Phi_3(z)$, $\Phi_4(z)$ are products of trigonometric and hyperbolic functions and are expressed by the equations

$$\left.\begin{aligned} \Phi_1(z) &= \operatorname{ch} \alpha z \cdot \sin \beta z, \\ \Phi_2(z) &= \operatorname{ch} \alpha z \cdot \cos \beta z, \\ \Phi_3(z) &= \operatorname{sh} \alpha z \cdot \cos \beta z, \\ \Phi_4(z) &= \operatorname{sh} \alpha z \cdot \sin \beta z \end{aligned}\right\} \tag{5.12}$$

(α and β are determined by equations (5.10)).

b) $s = r$ All the roots are real and equal in pairs: $k_1 = k_2 = r$, $k_3 = k_4 = -r$.

The general solution is given by equation (5.11); nevertheless, the particular solutions $\Phi_1(z)$, $\Phi_2(z)$, $\Phi_3(z)$ and $\Phi_4(z)$ will now be determined bv different equations, viz.

$$\Phi_1(z) = \operatorname{sh} rz, \qquad \Phi_2(z) = \operatorname{ch} rz,$$
$$\Phi_3(z) = z \operatorname{ch} rz, \qquad \Phi_4(z) = z \operatorname{sh} rz.$$

c) $s < r$. All the roots are real and unequal. They are given by the equations:

$$k_1 = -k_2 = \mu_1 = \sqrt{r^2 + \sqrt{r^4 - s^4}},$$
$$k_3 = -k_4 = \mu_2 = \sqrt{r^2 - \sqrt{r^4 - s^4}}.$$

The general solution of the homogeneous differential equation (5.7) is given, as before, by equation (5.11). The particular solutions are

$$\Phi_1(z) = \operatorname{sh}\mu_1 z, \quad \Phi_2(z) = \operatorname{ch}\mu_1 z,$$
$$\Phi_3(z) = \operatorname{sh}\mu_2 z, \quad \Phi_4(z) = \operatorname{ch}\mu_2 z.$$

d) Finally, as already shown, the case $s = 0$, $r \neq 0$ is that of restrained torsion previously dealt with in this book. In this case we proceed directly from the equation $\theta^{IV} - k^2\theta'' = 0$ and obtain for the particular solutions the equations

$$\left.\begin{aligned}\Phi_1(z) &= z, & \Phi_2(z) &= 1, \\ \Phi_3(z) &= \operatorname{sh} kz, & \Phi_4(z) &= \operatorname{ch} kz.\end{aligned}\right\} \tag{5.13*}$$

All the cases enumerated above, with their various expressions for the particular solutions, possess common basic design quantities

$$\theta, \theta', \quad B = -EJ_\omega\theta'', \quad H = -EJ_\omega\theta''' + GJ_d\theta' \tag{5.14}$$

and, consequently, it is possible to use in the solution of problems the matrix of the initial parameters, presented for that purpose with the influence coefficients in symbolical form in Table 6, § 3, Chapter II.

Thus all the conclusions reached in § 3 of Chapter II and elsewhere with respect to the design of beams can be extended also to the problems described in the present section** .

The method of obtaining the matrix of the initial parameters or, in other words, the method of calculating the influence coefficients, is presented in sufficient detail in § 3 of Chapter II for the case of restrained torsion and can be used in an analogous way for other cases. The first-, second- and third-order derivatives of the particular integrals $\Phi_1(z)$, $\Phi_2(z)$, $\Phi_3(z)$ and $\Phi_4(z)$, neeessary for this purpose (all expressed by the same functions) are given in Table 22 for the four cases investigated here. This table also provides for these cases the expressions $\int \Phi_i(z)\,dz$ in terms of the functions $\Phi_i(z)$.

Table 23

	θ_0	θ'_0	$\frac{1}{EJ_\omega}B_0$	$\frac{1}{EJ_\omega}H_0$
$\theta(z)$	$K_{\theta\theta}$	$K_{\theta\theta'}$	$K_{\theta B}$	$K_{\theta H}$
$\theta'(z)$	$K_{\theta'\theta}$	$K_{\theta'\theta'}$	$K_{\theta' B}$	$K_{\theta' H}$
$\frac{1}{EJ_\omega}B(z)$	$K_{B\theta}$	$K_{B\theta'}$	K_{BB}	K_{BH}
$\frac{1}{EJ_\omega}H(z)$	$K_{H\theta}$	$K_{H\theta'}$	K_{HB}	K_{HH}

* Equations (5.13) differ from equations (II.2.4) in that there we used dimensionless coordinates. We have also interchanged here the places of C_1 and C_2. This was done in order to facilitate the comparison with other cases and does not, of course, introduce any essential change.

** In § 3 some of the properties of the matrix of initial parameters were proved for the particular case of equation (II.3.1). This can be easily done also for equations of the more general form (5.7).

Table 24

Case / Influence coefficients	$s>r$	$s=r$	$s<r$
$K_{\theta\theta}=K_{HH}$	$\Phi_2-\frac{r^3}{2\alpha\beta}\Phi_4$	$\Phi_2-\frac{r}{2}\Phi_4$	$\frac{1}{\mu_1^2-\mu_2^2}(\mu_1^2\Phi_4-\mu_2^2\Phi_2)$
$K_{\theta'\theta}=K_{HB}$	$\frac{s^3}{2\alpha\beta}(\beta\Phi_3-\alpha\Phi_1)$	$\frac{r}{2}(\Phi_1-r\Phi_3)$	$\frac{\mu_1\mu_2}{\mu_1^2-\mu_2^2}(\mu_1\Phi_3-\mu_2\Phi_1)$
$K_{B\theta}=K_{H\theta'}$	$\frac{s^4}{2\alpha\beta}\Phi_4$	$\frac{r^3}{2}\Phi_4$	$\frac{\mu_1\mu_2}{\mu_1^2-\mu_2^2}(\Phi_3-\Phi_4)$
$K_{H\theta}$	$\frac{s^3}{2\alpha\beta}[\beta(3\alpha^2-\beta^2)\Phi_3+\alpha(3\beta^2-\alpha^2)\Phi_1]$	$\frac{r^3}{2}(3\Phi_1-r\Phi_3)$	$\frac{\mu_1\mu_2}{\mu_1^2-\mu_2^2}(\mu_1^3\Phi_3-\mu_2^3\Phi_1)$
$K_{\theta\theta'}=K_{BH}$	$\frac{1}{2\alpha\beta}(\beta\Phi_1+\alpha\Phi_3)$	$\frac{1}{2r}(\Phi_1+r\Phi_3)$	$\frac{1}{\mu_1^2-\mu_2^2}(\mu_1\Phi_1-\mu_2\Phi_3)$
$K_{\theta'\theta'}=K_{BB}$	$\Phi_2+\frac{r^3}{2\alpha\beta}\Phi_4$	$\Phi_2+\frac{r}{2}\Phi_4$	$\frac{1}{\mu_1^2-\mu_2^2}(\mu_1^2\Phi_2-\mu_2^2\Phi_4)$
$K_{B\theta'}$	$\frac{1}{2\alpha\beta}[\alpha(3\beta^2-\alpha^2)\Phi_1-\beta(3\alpha^2-\beta^2)\Phi_3]$	$-\frac{r}{2}(3\Phi_1+r\Phi_3)$	$\frac{1}{\mu_1^2-\mu_2^2}(\mu_2^3\Phi_3-\mu_1^3\Phi_1)$
$K_{\theta B}=K_{\theta' H}$	$-\frac{1}{2\alpha\beta}\Phi_4$	$-\frac{1}{2r}\Phi_4$	$\frac{1}{\mu_1^2-\mu_2^2}(\Phi_4-\Phi_2)$
$K_{\theta' B}$	$-\frac{1}{2\alpha\beta}(\alpha\Phi_1+\beta\Phi_3)$	$-\frac{1}{2r}(\Phi_1+r\Phi_3)$	$\frac{1}{\mu_1^2-\mu_2^2}(\mu_2\Phi_3-\mu_1\Phi_1)$
$K_{\theta H}$	$-\frac{1}{2\alpha\beta s^2}(\alpha\Phi_1-\beta\Phi_3)$	$\frac{1}{2r^3}(\Phi_1-r\Phi_3)$	$\frac{1}{\mu_1^2-\mu_2^2}\left(\frac{1}{\mu_2}\Phi_3-\frac{1}{\mu_1}\Phi_1\right)$

The matrix of the initial parameters, written symbolically, will have the same form for all the cases examined here. This matrix is given in Table 23. The influence coefficients for various cases will be different. They are given in Table 24 for the cases $s > r$, $s = r$ and $s < r$. This table does not include the case $r \neq 0$, $s = 0$ as it was examined in detail in § 3, Chapter II, and the matrices of Table 23 and Table 6 of § 3 differ from one another only in the factors of the statical quantities B and H; in Table 6 we have the factor $\frac{1}{GJ_d}$, in Table 23 the factor $\frac{1}{EJ_\omega}$.

§ 6. Joint action of a plate and thin-walled beams reinforcing it

1. The problem of the joint action of a plate and thin-walled beams is of great practical interest. For example, ship builders encounter it in the design of hulls and builders meet it in the design of roofs etc. There are not, at present, sufficiently simple and reliable methods for designing such structures. As a rule, the approximate methods inadequately describe the stresses and deformations in a plate stiffened by beams. The exact methods require very laborious calculations. For the purpose of reducing this gap a variational method has been devised, which will be presently described.

2. To solve the problem stated above, we shall start from the variational method osed in the design of rectangular plates (and prismatical shells) with fixed ribs, as developed in /46, 51/. This method is based on the idea of transforming the biharmonic equation of beam flexure into ordinary differential equations.

The method rests on the following assumptions: under the action of an external normal load the plates can undergo at any point only normal displacements; the deformation in width of the plate equals zero; the boundary values are constant on the longitudinal edges $x = 0$ and $x = a$ as well as on the transverse edges $z = 0$ and $z = l$; the external load, conforming to an arbitrary law over the width of the plate (in the direction Ox) can also vary over its length (in the direction Oz) in the same way in all the sections $x =$ const or $z =$ const. In other words, the load can be expressed as a product of two functions

$$p(x, z) = P_0(x)\, K(z). \tag{6.1}$$

We express the deflection of the plate also as a product of two functions

$$w(x, z) = W(z)\, \chi(x), \tag{6.2}$$

of which $\chi(x)$ satisfies the geometrical conditions on the longitudinal edges and the character of the load. The function $W(z)$, regarded as a generalized deflection of the plate, is the one which is sought.

It is possible to obtain a sufficiently exact solution, if we take for $\chi(x)$ the deflection of an elementary transverse strip of width $dz = 1$ (for corresponding geometrical support conditions) due to an auxiliary load which has the same law of variation as the given external load.

First choosing the function $\chi(x)$, we proceed according to Lagrange's principle [of virtual work] and equate to zero the total work done by all the external and internal forces on the transverse strip in its only admissible

displacement $\chi(x)$; we then find, following the author's variational method, an ordinary fourth order differential equation for the function $W(z)$:

$$AW^{IV} - 2BW'' + CW - G = 0. \tag{6.3}$$

The coefficients A, B, C for a plate restrained from bending at the longitudinal edges $x=0$, $x=a$ are calculated by the equations

$$A = D\int_0^a \chi^2\,dx,$$

$$B = D\int_0^a (\chi')^2\,dx,$$

$$C = D\int_0^a (\chi'')^2\,dx,$$

where $D = \frac{E\delta^3}{12}$ is the cylindrical rigidity.

The free term $G(z)$ in equation (6.3) is the work done by all the given loads (including the concentrated ones) per elementary strip in the virtual displacement $\chi(x)$:

$$G(z) = \int_0^a p(x, z)\,\chi(x)\,dx$$

(the integral in the sense of Stieltjes) or, with the concentrated parts of the load written separately

$$G(z) = \int_0^a p\chi\,dx + \sum p_k\chi(k) + \sum M_k\chi'(k), \tag{6.4}$$

where p_k and M_k are, respectively, the concentrated force and the concentrated moment distributed along the line $x = k$.

Having found the function $W(z)$ from equation (6.3) and the boundary conditions, all the internal forces and moments on the transverse edges are easily determined from this function and the corresponding equations.

The calculation for a plate can be also carried out by using the method of initial parameters. In this case the homogeneous equation corresponding to equation (6.3) will have the form

$$W^{IV} - 2r^2W'' + s^4W = 0, \tag{6.5}$$

where

$$r^2 = \frac{B}{A}, \quad s^4 = \frac{C}{A}. \tag{6.6}$$

The following are the basic design quantities: the generalized deflection $W(z)$, the generalized rotation angle $\varphi = \frac{dW(z)}{dz} = W'(z)$ and the generalized statical quantities corrdsponding to the above two geometrical quantities*:

$$\left.\begin{array}{r} \text{the generalized moment } M = -AW''(z), \\ \text{the generalized transverse force } Q = -AW'''(z) + 2BW'(z). \end{array}\right\} \tag{6.7}$$

* We take Poisson's ratio equal to zero in (6.7).

We can now write the matrix of the initial parameters in the following form for the examined case of a plate

Table 25

	w_0	φ_0	$\frac{1}{A} M_0$	$\frac{1}{A} Q_0$
$W(z)$	$K_{WW}(z)$	$K_{W\varphi}(z)$	$K_{WM}(z)$	$K_{WQ}(z)$
$\varphi(z)$	$K_{\varphi W}(z)$	$K_{\varphi\varphi}(z)$	$K_{\varphi M}(z)$	$K_{\varphi Q}(z)$
$\frac{1}{A} M(z)$	$K_{MW}(z)$	$K_{M\varphi}(z)$	$K_{MM}(z)$	$K_{MQ}(z)$
$\frac{1}{A} Q(z)$	$K_{QW}(z)$	$K_{Q\varphi}(z)$	$K_{QM}(z)$	$K_{QQ}(z)$

The influence coefficients K_{ij} are given in Table 24 in expanded form. We have only to replace the indexes W by θ, φ by ϑ, M by B, Q by H. This follows from a comparison of expressions (6.5) and (6.7) with the analogous expressions (5.7) and (5.14).

3. After these preliminary remarks we resume our investigation of the joint behavior of a plate and its reinforcing beam. Let a thin-walled beam, which we shall assume for simplicity to have two symmetry axes, be placed parallel to the longitudinal edges of the plate. The coordinate axes of the beam are parallel to the coordinate axes of the plate (Figure 132). The plate touches the beam along the intersection line $k-k$ of the middle surface of the plate with the plane of the beam web. The contact point on the cross section has the coordinate $x = x_k$.

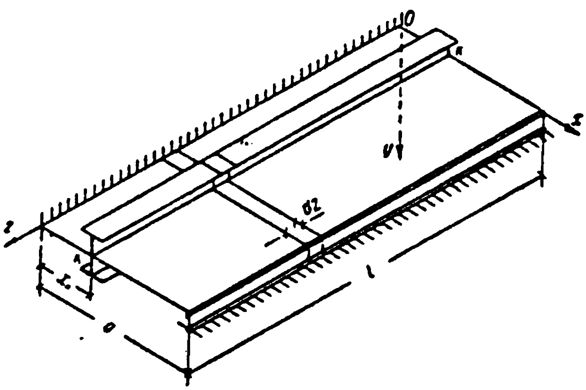


Figure 132

It follows from the condition of joint deformations along the contact line of the beam with the plate that in the isolated plate-strip plus beam enclosed between the sections $z =$ const and $z + dz =$ const, at the point

of contact,

1) the deflection of the plate is equal to that of the beam

$$W\chi(k) = \eta(k), \tag{6.8}$$

2) The rotation angle of the plate is equal to the torsion angle of the beam

$$W\chi'(k) = \theta(k). \tag{6.9}$$

In the thin-walled beam the deflection η and the torsion θ satisfy the differential equations

$$EJ_x\eta^{IV} = q_y,$$

$$EJ_\omega\theta^{IV} - GJ_d\theta'' = m.$$

Consequently, on the basis of (6.8), (6.9) and (6.2), the following relations will be satisfied at the contact

$$\left.\begin{aligned} EJ_x\chi_k W^{IV} &= q_{yk},\\ EJ_\omega\chi'_k W^{IV} - GJ_d\chi'_k W'' &= m_k, \end{aligned}\right\} \tag{6.10}$$

where $\chi_k = \chi(k)$, $\chi'_k = \chi'(k)$, $q_{yk} = q_y(k, z)$ and $m_k = m(k, z)$.

Substituting the action of the beam on the plate by a concentrated force $P_k = -q_{yk}$ and by a moment $M_k = -m_k$ (Figure 133) (if there are several beams, their separate effects should be added), we obtain the following expression for the free term (6.4) (provided that the concentrated factors in this equation are due to the influence of the beam)

$$G(z) = \int_0^a p\chi dx - \sum EJ_x\chi_k^2 W^{IV} - \sum[EJ_\omega(\chi'_k)^2 W^{IV} - GJ_d(\chi'_k)^2 W'']. \tag{6.11}$$

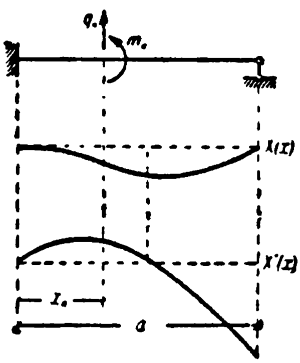


Figure 133

If the free term $G(z)$ expressed by equation (6.11) is now substituted in the general equation (6.3) and similar terms are combined, we obtain

$$\bar{A}W^{IV} - 2\bar{B}W'' + CW - \bar{G} = 0, \tag{6.12}$$

where

$$\left.\begin{aligned} \bar{A} &= A + \sum EJ_x\chi_k^2 + \sum EJ_\omega(\chi'_k)^2,\\ \bar{B} &= B + \frac{1}{2}\sum GJ_d(\chi'_k)^2,\\ \bar{G} &= \int_0^a p\chi\, dx. \end{aligned}\right\} \tag{6.13}$$

Equation (6.12) has the same form as equation (6.3) for a plate which is not reinforced by beams. According to equations (6.13) the physical and geometrical characteristics of the beams which reinforce the plate enter in the expressions for the coefficients $\bar{A}$ and $\bar{B}$.

In solving equation (6.12) by the method of initial parameters we transform the homogeneous part of the equation into the form (6.5).

The coefficients $\bar{r}^2$ and $\bar{s}^4$ are calculated from the equations:

$$\left.\begin{aligned}\bar{r}^2 &= \frac{\bar{B}}{\bar{A}} = \frac{B + \frac{1}{2}\sum GJ_d\,(\chi_k')^2}{A + \sum EJ_x\chi_k^2 + \sum EJ_\omega\,(\chi_k')^2}\,,\\ \bar{s}^4 &= \frac{C}{\bar{A}} = \frac{C}{A + \sum EJ_x\chi_k^2 + \sum EJ_\omega\,(\chi_k')^2}\,.\end{aligned}\right\} \qquad (6.14)$$

The remaining steps of calculation are the same as in the solution for a normal plate. The matrix of the initial parameters is given by Table 25. The basic design factors are calculated from equations (6.7) with the coefficients A and B replaced respectively by $\bar{A}$ and $\bar{B}$, which are calculated from (6.13). The roots of the characteristic equation are determined not by the quantities r^2 and s^4 as obtained from (6.6), but by the quantities $\bar{r}^2$ and $\bar{s}^4$ calculated from equations (6.14).

Chapter IV

THIN-WALLED BEAM-SHELLS OF CLOSED SECTION. ACCOUNT OF SHEAR DEFORMATIONS

§ 1. General variational method of reducing complex two-dimensional problems of shell theory to one-dimensional problems

1. The idea behind this method is the following. The required functions which depend on two variables and which satisfy a system of partial differential equations are represented as finite sums of products of pairs of functions, each pair consisting of the given (preselected) function of one variable and the required function of the second variable. Applying the principle of virtual displacements and introducing the system of the required functions whose number equals the number of degrees of freedom of isolated elementary strips, considered as a jointed frame, we arrive at a system of ordinary linear differential equations. This simplifies the solution of the problem to a considerable extent.

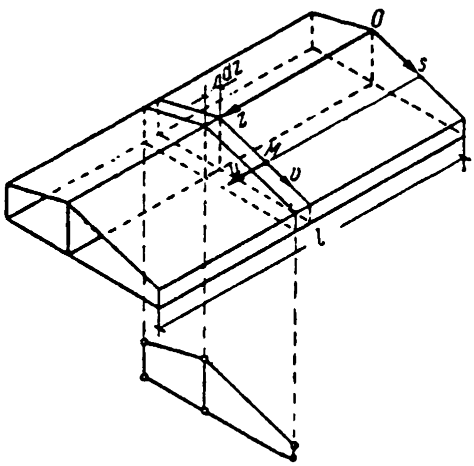


Figure 134

2. We shall examine a thin-walled beam whose cross section consists of a few closed contours [multicell beam], composed of narrow rec- - tangular plates (Figure 134). We assume that these plates are rigidly connected to one another along the junction lines so that they have no relative mobility along these lines.

The position of the point M on the middle prismatical surface of the thin-walled beam is determined, as shown above, tby two coordinates: the coordinate z which is the distance from an origin plane $z=0$ and the coordinate s which is the distance along the contour of the cross section of the beam from an initial generator $s=0$ (the positive senses of the coordinate axes are given in Figure 134).

Let the displacements of the point $M(z, s)$ be $u(z, s)$, for the longitudinal displacement in the direction of the generator, and $v(z, s)$ for the tangential displacement along the contour line of the thin-walled beam. The positive directions of the displacements $u(z, s)$ and $v(z, s)$ coincide, in accordance with the sign rules adopted before, with the positive directions of the coordinates z, s..

We shall represent the displacements $u(z, s)$ and $v(z, s)$ in the form of finite series

$$\left\{\begin{aligned} u(z, s) &= \sum_1^m U_i(z)\varphi_i(s) \qquad (i=1, 2, 3, \ldots, m), && (1.1)\\ v(z, s) &= \sum_1^n V_k(z)\psi_k(s) \qquad (k=1, 2, 3, \ldots, n), && (1.2)\end{aligned}\right.$$

where the functions $\varphi_i(s)$ and $\psi_k(s)$ of the single argument s are chosen to start with, so that their values will be known. The functions $U_i(z)$ and $V_k(z)$ (these depend only on the second variable, z) will be the unknown quantities required. The meaning of m and n will be clarified below.

We isolate from the thin-walled beam, by two sections $z=\text{const}$ and $z+dz=\text{const}$, an elementary transverse strip of width dz. We shall regard this strip as a plane beam system—a frame which consists of several closed contours.

We shall examine the strained state of this strip as determined only by the longitudinal displacements $u(z, s)$ (for $v(z, s)=0$).

In this case the two-dimensional plane contour line of the elementary frame around the prismatical surface of the thin-walled beam is transformed into a space curve, determined with respect to the initial section $z=\text{const}$ (for preassigned functions $\varphi_i(s)$ by the expression (1.1).

We make the assumption that in the examined case of deformation the rectilinear elements of the contour line of the beam remain straight on leaving the plane $z=\text{const}$. Such an assumption is equivalent to the hypothesis of plane sections, applied separately for each of the narrow rectangular plates forming the thin-walled beam. The position after deformation of the elementary strip with respect to the initial plane $z=\text{const}$ is then completely determined by the longitudinal displacements of its m nodal points. Consequently, the elementary strip can be considered as a beam system of m degrees of freedom with respect to the longitudinal displacements. The required functions $U_1(z)$, $U_2(z), \ldots, U_m(z)$ are taken in equation (1.1) as the longitudinal displacements of the m nodes of the elementary frame. The functions $\varphi_1(s)$, $\varphi_2(s), \ldots, \varphi_m(s)$ which correspond to these displacements satisfy all the necessary continuity conditions of the longitudinal displacements $u(z, s)$ on the contour of the cross section of the thin-walled beam. The functions $\varphi_i(s)$ have in this case a very simple and obvious geometrical expression: each of them differs from zero only on the straight segments of the contour which meet at the i-th node. Over each of these portions $\varphi_i(s)$ varies linearly, from 0 at the i-th nodal point to 1 at the

nodal point which lies at the other end of this segment. The function $\varphi_i(s)$ will vanish on all the remaining parts of the contour line (Figure 135). This method of constructing the functions $\varphi_i(s)$ is not the only one possible. As the functions $U_i(z)$ we can take any m independent function of the longitudinal displacements. A set of m independent functions $U_i(z)$ will correspond to m linearly independent functions $\varphi_i(s)$. Each of these will be continuous on the whole contour and represented by a linear diagram on its separate segments.

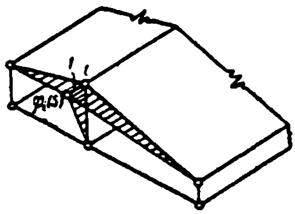


Figure 135

We first separate in the expansion (1.1) the longitudinal displacements, which refer to the elementary design of a thin-walled beam of multiply connected cross section according to Bernoulli's hypothesis, as applied to the section $z = \text{const}$. Of the m possible longitudinal displacements $U_i(z)$, the three functions $U_1(z)$, $U_2(z)$, $U_3(z)$ determine the displacements in which the prismatical surface of the examined hinged model moves as a rigid plane system. These will be the variation functions of the longitudinal displacements of the nodal points along the beam in extension or compression and in bending in the vertical and horizontal planes. The law of plane sections, applied to the section as a whole, requires special linear combinations of the Cartesian coordinates $x = x(s)$, $y = y(s)$ of a contour point to form the functions $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$.

The remaining terms of expansion (1.1) stand for the part of $u(z, s)$ which represents longitudinal displacements in which the cross sections become warped, i.e. do not remain plane. The warping of a thin-walled beam having a multiply connected cross section is thus determined by $m-3$ independent functions of the longitudinal displacements $U_4(z)$, $U_5(z)$, $\ldots$, $U_m(z)$.

The continuous linearly independent functions $\varphi_i(s)$ $(i = 1, 2, 3, \ldots, m)$ selected (by any method) and inserted in the expansion (1.1), represent the generalized coordinates of the warped elementary transverse strip of a thin-walled beam with respect to the plane of the initial cross section $z = \text{const}$.

The functions $U_i(z)$ $(i = 1, 2, 3, \ldots, m)$ which correspond to these generalized coordinates and depend on z only are the required generalized longitudinal displacements of the thin-walled beam.

Constructing another system of functions $\psi_k(s)$ $(k = 1, 2, 3, \ldots, n)$ for expansion (1.2), the deformed state of the elementary strip in its plane is examined, i.e. in the plane $z = \text{const}$.

Regarding, as before, this strip as a beam system and assuming the non-extensibility of its elements (the components of the beam) we are led to the conclusion that the contour displacements $v(z, s)$ can be expressed by the displacements $V_k(z)$ $(k = 1, 2, 3, \ldots, n)$ of a hinged beam system in its plane. The coefficients $V_k(z)$ in (1.2) are seen as independent functions which determine the displacements of the hinged beam system in its plane. The number n of the functions $V_k(z)$ equals the number of degrees of freedom of the beam system in its plane and is determined from the equation

$$n = 2m - c,$$

where m is the number of nodes and c the number of simple beams composing the multiply connected section.

We shall now describe a simpler method for setting up the system of functions $\psi_k(s)$. We choose n independent functions $V_k(z)$ for the displacements of the elementary beam system in the plane of the section $z = \text{const}$ and assign to each of these quantities in turn the value 1, setting the others equal to zero. From the expansion (1.2) we can then determine all the necessary functions $\psi_k(s)$ by examining the elementary displacements of the system thus obtained. Each of these functions will give the contour displacement of some point, corresponding to an elementary state $V_k^* = 1$ and $V_h^* = 0$ for $h \neq k$ (Figure 136). At the boundaries of each rectilinear part of the contour of the thin-walled beam the function $\psi_k(s)$ is constant (owing to the nonextensibility of the elements of the contour line); it thus does not depend on the coordinate s and hence describes the axial displacement of the hinged model.

In this way we can construct n linearly independent diagrams of the functions $\psi_k(s)$ for the arbitrarily chosen quantities $V_k(z)$.

In particular, we can choose these quantities so that three of them, $V_1(z)$, $V_2(z)$ and $V_3(z)$, will determine the displacements of the entire model as a rigid plane beam system. The remaining quantities $V_4(z)$, $V_5(z), \ldots, V_n(z)$ will refer in this case to such displacements of the system for which the relative position of its separate links changes, i. e. where the contour of the cross section is deformed.

Consequently, the deformation of the contour of a thin-walled beam is determined by $n-3$ independent variables, n being the number of the degrees of freedom of an elementary strip considered as a plane hinged beam system. The functions $\psi_k(s)$, corresponding to the n degrees of freedom of the hinged system in its plane, satisfy the condition of linear independence and the continuity condition for the displacements of the points of the contour of an elementary transverse frame (including the nodal points of the contour) since the hinged model remains everywhere continuous in each of the n possible elementary states $V_k^* = 1$. The chosen continuous linearly independent functions $\psi_k(s)$ $(k = 1, 2, 3, \ldots, n)$ represent the **generalized deformation coordinates of an elementary transverse strip of a thin-walled beam in the plane of this strip** (the plane of the section $z = \text{const}$). The quantities $V_k(z)$ $(k = 1, 2, 3, \ldots, n)$ which correspond to these functions and each of which depends only on the coordinate z, are the required **generalized transverse displacements of the thin-walled beam.**

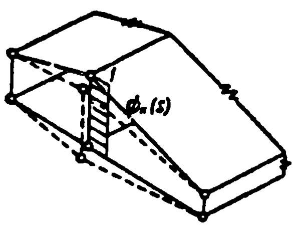


Figure 136

Consequently, the position in space of all the nodal points of the transverse elementary strip of a thin-walled beam after deformation are determined for the chosen generalized coordinates

$$\varphi_i(s) \quad (i = 1, 2, 3, \ldots, m) \text{ and } \psi_k(s) \quad (k = 1, 2, 3, \ldots, n)$$

by the required $m + n$ generalized displacements: the longitudinal displacements $U_i(z)$ $(i = 1, 2, 3, \ldots, m)$ and the transverse displacements $V_k(z)$ $(k = 1, 2, 3, \ldots, n)$.

3. Having established the generalized coordinates $\varphi_i(s)$ and $\psi_k(s)$ in (1.1) and (1.2), we can proceed to determine the generalized displacements $U_i(z)$ and $V_k(z)$ $(i=1, 2, 3, \ldots, m;\ k=1, 2, 3, \ldots, n)$.

Let $\sigma=\sigma(z, s)$ and $\tau=\tau(z, s)$ denote, respectively, the normal and tangential stresses which appear in the sections of a thin-walled beam. Assuming the stresses σ and τ to be uniformly distributed across the beam wall (the tangential stresses τ only produce shear forces), we shall consider these stresses as functions of the coordinates z, s only.

By virtue of Hooke's law

$$\left.\begin{aligned}\sigma(z, s)&=E\frac{\partial u}{\partial z},\\ \tau(z, s)&=G\left(\frac{\partial u}{\partial s}+\frac{\partial v}{\partial z}\right).\end{aligned}\right\} \tag{1.3}$$

Introducing here the expansions (1.1) and (1.2), we obtain

$$\sigma(z, s)=E\sum U_i'(z)\varphi_i(s) \qquad (i=1, 2, 3, \ldots, m), \tag{1.4}$$

$$\tau(z, s)=G\left[\sum U_i(z)\varphi_i'(s)+\sum V_k'(z)\psi_k(s)\right] \tag{1.5}$$

$$(i=1, 2, 3, \ldots, m;\ k=1, 2, 3, \ldots, n).$$

The elementary transverse strip isolated from the thin-walled beam will be subjected to external normal and shear forces in the sections $z=$ const and $z+dz=$const, and to the given surface forces (Figure 137). Let $p^*=p^*(z, s)$ and $q^*=q^*(z, s)$ denote the external forces, with respect to the given strip, which act respectively in direction of the shell generator and the contour line (a force is regarded as positive when directed toward increasing z and s).

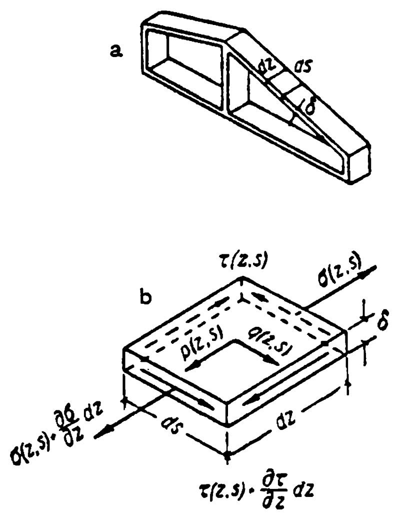


Figure 137

Referring these forces to a unit area of the middle surface of the thin-walled beam, we obtain for them the following expressions

$$p^*=\frac{\partial\sigma}{\partial z}\delta+p, \tag{1.6}$$

$$q^*=\frac{\partial\tau}{\partial z}\delta+q, \tag{1.7}$$

where the shell thickness $\delta=\delta(s)$ is assumed to be a given function (discontinuous in the general case) of the coordinate s only, and $p=p(z, s)$ and $q=q(z, s)$ represent the given external surface forces.

The integral equilibrium conditions of the elementary strip having $m+n$ degrees of freedom for the chosen displacements $\varphi_i(s)$ and $\psi_k(s)$ are found from the principle of virtual displacements as the following equations,

$$\int_F \frac{\partial\sigma}{\partial z}\varphi_j\,dF-\int_F \tau\varphi_j'\,dF+\int_L p\varphi_j\,ds=0 \qquad (j=1, 2, 3, \ldots, m), \tag{1.8}$$

$$\int_F \frac{\partial\tau}{\partial z}\psi_h dF - \sum V_k \int_l \frac{M_k M_h}{EJ} ds + \int_l q\psi_h ds = 0 \quad (h = 1, 2, 3, \ldots, n), \quad (1.9)$$

where $dF = \delta ds$ is the differential area element of the cross section of the shell.

In these equations and in the following the integrals are definite, taken either over the area of the cross section $\left(\text{for example} \int_F \frac{\partial\sigma}{\partial z} \varphi_j dF\right)$, or around the contour of the cross section $\left(\text{for example} \int_l p\varphi_j ds\right)$.

Equations (1.8) are the m equilibrium conditions of an elementary strip $dz = 1$ in a direction perpendicular to the plane $z =$ const.

Each of equations (1.8) shows the sum of the work of all the external and internal forces to equal zero in the elementary strip when deformed away from its plane.

In the j-th equation of this group we take the longitudinal displacements $U_j(z, s) = 1\varphi_j(s)$ as the virtual displacements of the points of the elementary strip, determined by the single j-th term of expansion (1.1) for $U_j^*(z) = 1$.

The last term of equations (1.8) refers to the work of the external forces p^* (1.6) acting on a strip of width $dz = 1$ and perpendicular to the cross section of the beam. The middle term expresses the work of the internal shear forces. For an element ds of the strip this work is determined as the product (of opposite sign) of the shear forces $\tau\delta ds$ and the shearing strain. In the present case of variation of the strained state, the latter equals

$$\frac{\partial U_j}{\partial s} = \varphi_j'(s).$$

Equations (1.9) furnish n equilibrium conditions for this strip in the plane $z =$ const. Each of equations (1.9) is obtained by equating to zero the sum of the work done by all external and internal forces of the elementary strip in the corresponding displacements, as the strip is deformed in its plane. We take in the h-th equation as virtual displacements the transverse tangential displacements $V_h(z, s) = 1\psi_h(s)$ of the strip, determined only by the h-th term of (1.2) for a generalized transverse displacement $V_h^*(z) = 1$. The last term of equations (1.9) refers to the work of the external contour forces q^* (1.7) acting in the plane of the strip. The middle term expresses the work of the internal forces on the bending deformations of the strip corresponding to the h-th elementary state of displacement of the hinged kinematical polygon in its plane. For an element ds of the strip, this work is given, in the case of bending, as the product (with the opposite sign) of the bending moment

$$M(z, s) = \sum V_k(z) M_k(s) \qquad (k = 1, 2, 3, \ldots, n)$$

and the relative angle of rotation $\frac{M_h(s)}{EJ} ds$ of two adjacent sections enclosing this element. Under the assumption that the nodes of the frame are free of external angular connections, i. e. that the bending moments $M_k = M_k(s)$ and $M_h = M_h(s)$ in each of the possible n possible states $(h, k = 1, 2, 3, \ldots, n)$ satisfy the equilibrium conditions at all nodes, the quantities $M_k(s)$ and $M_h(s)$ equal the bending moments of the elementary transverse frame corresponding to the elementary deformation states of this frame $V_k^*(z) = 1$

and $V_k^*(z)=1$. The moments $M_k(s)$ and $M_h(s)$ are found by the usual methods of structural mechanics, by performing the inverse transformation from the deformations of the frame system to the internal forces.

The quantity $J=J(s)$ represents the moment of inertia of the cross section of the isolated elementary strip of width $dz=1$. Without transverse connections the inertia moment is $J=\frac{\delta^3}{12}$. In the presence of additional sufficiently closely-spaced transverse connecting frames reinforcing the thin-walled beam, the moment of inertia J should be calculated from the average moment of inertia of these frames, i. e. the moment of inertia per unit length of the reinforced beam.

Introducing in (1.8) and (1.9) for $\sigma(z,s)$ and $\tau(z,s)$ their expressions (1.4) and (1.5), we obtain a system of $m+n$ linear differential equations for the required generalized m longitudinal displacements $U_i(z)$ $(i=1, 2, 3, \ldots, m)$ and the n transverse displacements $V_k(z)$ $(k=1, 2, 3, \ldots, n)$

This system can be written in the following form

$$\left.\begin{aligned}&\gamma\sum_i a_{ji}U_i''-\sum_i b_{ji}U_i-\sum_k c_{jk}V_k'+\frac{1}{G}p_j=0,\\&\sum_i c_{hi}U_i'+\sum_k r_{hk}V_k''-\gamma\sum_k s_{hk}V_k+\frac{1}{G}q_h=0\\&(i,j=1,2,3,\ldots,m;\quad h,k=1,2,3,\ldots,n),\end{aligned}\right\}\qquad(1.10)$$

where $\gamma=\frac{E}{G}$.

The coefficients of equation (1.10) are obtained from the equations

$$\left.\begin{aligned}a_{ji}&=\int_F\varphi_j(s)\,\varphi_i(s)\,dF,\\b_{ji}&=\int_F\varphi_j'(s)\,\varphi_i'(s)\,dF,\\c_{jk}&=\int_F\varphi_j'(s)\,\psi_k(s)\,dF,\\c_{hi}&=\int_F\psi_h(s)\,\varphi_i'(s)\,dF,\\r_{hk}&=\int_F\psi_h(s)\,\psi_k(s)\,dF,\\s_{hk}&=\frac{1}{E}\int_L\frac{M_h(s)\,M_k(s)}{EJ}\,ds,\end{aligned}\right\}\qquad(1.11)$$

where the integrals are taken over all the elements of the cross section of the thin-walled beam. These coefficients are symmetrical: $a_{ji}=a_{ij}$, $b_{ji}=b_{ij}$, $r_{hk}=r_{kh}$, $s_{hk}=s_{kh}$ and for $h=k$ $c_{jk}=c_{hi}$. This property is in accordance with the known reciprocal theorem of Betti concerning the work done in an elastic system.

Equations (1.11) have a general character and permit the calculation of the coefficients of equation (1.10) for a thin-walled beam with a cross section of arbitrary form by any approximation of the required displacements $u(z,s)$, $v(z,s)$ with respect to the variable s.

In selecting by the above method the functions $\varphi_i(s)$ $(i=1, 2, 3, \ldots, m)$ and $\psi_k(s)$ $(k=1, 2, 3, \ldots, n)$, the integrals in expressions (1.11) assume a very simple expression for each straight segment of the contour since the

functions $\varphi_i(s)$ on this segment depend linearly on the coordinate s while the derivatives $\varphi_i'(s)$, as well as the functions $\psi_k(s)$, have a constant value on the straight part of the contour.

The integrals for $dF=\delta\, ds$ have the same form in the first five equations (1.11) as in the sixth, the only difference being that instead of the quantity $\frac{1}{J}$ entering in the last equation we have in those integrals the thickness δ of the plates which compose the thin-walled beam.

All the coefficients of equation (1.10) can be calculated by the known methods of the theory of frames with the help of the diagrams of the functions $\varphi_i(s)$, $\varphi_i'(s)$, $\psi_k(s)$, $\varphi_j(s)$, $\varphi_j'(s)$, $\psi_h(s)$, $M_h(s)$, $M_k(s)$ constructed for the whole multiply connected contour.

The quantities $p_j(z)$ and $q_h(z)$ which enter in the free terms of equations (1.10) are known functions of z and are calculated for given surface forces $p(z, s)$ and $q(z, s)$ from the equations

$$p_j=\int_L p\varphi_j\, ds, \quad q_h=\int_L q\psi_h\, ds. \tag{1.12}$$

In accordance with the physical meaning which is apparent from the method by which they were obtained, the quantities $p_j(z)$ and $q_h(z)$ can be termed *generalized external forces.*

The differential equations (1.10) are derived for an arbitrary choice of the functions $\varphi_i(s)$ and $\psi_k(s)$ which determine the coefficients of these equations.

Since the functions $\varphi_i(s)$ and $\psi_k(s)$ are linearly independent and each of them may be given to any degree of accuracy, we can always choose for the required functions $U_i(z)$ and $V_k(z)$ such independent generalized longitudinal and transverse displacements for which the corresponding generalized coordinates $\varphi_i(s)$ and $\psi_k(s)$ are orthogonal over the entire cross section, so that for each group of the functions $\varphi_i(s)$ and $\psi_k(s)$ the following conditions are fulfilled

$$\left.\begin{aligned} a_{ji}&=\int_F \varphi_j\varphi_i\, dF=0 \quad \text{for} \quad j\neq i,\\ r_{hk}&=\int_F \psi_h\psi_k\, dF=0 \quad \text{for} \quad h\neq k. \end{aligned}\right\} \tag{1.13}$$

The system of differential equations (1.10) will have a simpler form on imposing the conditions (1.13):

$$\left.\begin{aligned} &\gamma a_{jj}U_j''-\sum_i b_{ji}U_i-\sum_k c_{jk}V_k'+\frac{1}{G}p_j=0 \quad (i,\ j=1, 2, 3, \ldots, m);\\ &\sum_i c_{hi}U_i'+r_{hh}V_h''-\gamma\sum_k s_{hk}V_k+\frac{1}{G}q_h=0 \quad (h,\ k=1, 2, 3, \ldots, n). \end{aligned}\right\} \tag{1.14}$$

The functions $\varphi_i(s)$ and $\psi_k(s)$ which satisfy the conditions of orthogonality (1.13) may be termed *principal generalized coordinates.* The choice of the orthogonal functions $\varphi_i(s)$ and $\psi_k(s)$ can be made by the combined graphic-analytic methods of structural mechanics.

4. The symmetrical system of linear differential equations with constant coefficients (1.14) or (1.10) can be transformed into a single equivalent differential equation.

In the present case this will be an equation of order $2(m+n)$. It follows that the required functions $U_i(z)$ and $V_k(z)$, which satisfy the system (1.14), will be determined up to $2(m+n)$ arbitrary constants. The

number of these constants is equal to twice the number of the spatial degrees of freedom of the elementary transverse beam strip of width $dz=1$. This is in complete agreement with the number of independent boundary conditions which may be given at the transverse end sections, $z=0$ and $z=l$ (l is the span of the beam measured along the generator). Having obtained these arbitrary constants, we may find solutions for the given beam for different boundary conditions with respect to the longitudinal and transverse displacements. These solutions will be completely determinate and unique. They will satisfy all the necessary geometrical conditions. After determining $U_i(z)$ and $V_k(z)$ by solving equations (1.10), it is possible to find the stresses $\sigma(z)$ and $\tau(z)$ at an arbitrary point of the section $z=\text{const}$ from equations (1.4) and (1.5) similarly, up to $2(m+n)$ arbitrary constants.

We introduce the generalized longitudinal and transverse forces in the section $z=\text{const}$ of the beam. Considering the virtual work done by the normal and shear forces $\sigma\delta$ and $\tau\delta$ of this cross section in each of the $m+n$ admissible displacements, we determine these forces in the same way as we did when using the law of sectorial areas in § 8 of Chapter I, viz.

$$\left.\begin{aligned} P_j(z)&=\int_F \sigma\varphi_j\,dF \qquad (i=1,2,3,\ldots,m);\\ Q_h(z)&=\int_F \tau\psi_h\,dF \qquad (h=1,2,3,\ldots,n).\end{aligned}\right\} \tag{1.15}$$

Considering the quantities $P_j(z)$ and $Q_h(z)$ as internal forces in the beam, we express them by the basic functions $U_i(z)$ and $V_k(z)$. From (1.4), (1.5), (1.11) and (1.15) we obtain

$$P_j(z)=E\sum_i a_{ji}U_i', \quad Q_h(z)=G\left(\sum_i c_{hi}U_i+\sum_k r_{hk}V_k'\right) \tag{1.16}$$
$$(j,i=1,2,3,\ldots,m;\ k,h=1,2,3,\ldots,n).$$

If we take the functions $\varphi_i(s)$ and $\psi_k(s)$ as orthogonal, the expressions (1.16) will assume the simpler form

$$P_j(z)=Ea_{jj}U_j', \quad Q_h(z)=G\left(\sum_i c_{hi}U_i+r_{hh}V_h'\right) \tag{1.17}$$
$$(i=1,2,3,\ldots,m;\ h=1,2,3,\ldots,n).$$

We can now rewrite equation (1.4) as

$$\sigma=\sum_j \frac{P_j}{a_{jj}}\varphi_j \qquad (j=1,2,3,\ldots,m),$$

or, expanded

$$\sigma=\frac{P_1(z)}{a_{11}}\varphi_1(s)+\frac{P_2(z)}{a_{22}}\varphi_2(s)+\frac{P_3(z)}{a_{33}}\varphi_3(s)+\ldots+\frac{P_m(z)}{a_{mm}}\varphi_m(s). \tag{1.18}$$

Equation (1.18), which holds true only when the functions $\varphi_i(s)$ are orthogonal, is a generalization of equation (I.8.5), obtained there from the law of sectorial areas,

$$\sigma=\frac{N}{F}-\frac{M_y}{J_y}x+\frac{M_x}{J_x}y+\frac{B}{J_\omega}\omega$$

for a beam with an open inflexible contour. Indeed, if we assume

$$\varphi_1(s)=1,\quad \varphi_2(s)=x(s),\quad \varphi_3(s)=y(s)\ \text{and}\ \varphi_4(s)=\omega(s), \tag{1.19}$$

we shall have

$$\begin{aligned} P_1&=\int_F \sigma 1\, dF=N, & a_{11}&=\int_F 1^2\, dF=F,\\ P_2&=\int_F \sigma x dF=-M_y, & a_{22}&=\int_F x^2\, dF=J_y,\\ P_3&=\int_F \sigma y dF=M_x, & a_{33}&=\int_F y^2 dF=J_x,\\ P_4&=\int_F \sigma\omega dF=B, & a_{44}&=\int_F \omega^2 dF=J_\omega. \end{aligned}$$

On the basis of this analogy we shall call the geometrical characteristics $a_{jj}=\int_F \varphi_j^2 dF$ $(j=4, 5, \ldots, m)$ the bimoments of inertia; the generalized longitudinal forces $P_4, P_5, \ldots, P_m$ connected with the warping of the section we shall term the longitudinal bimoments. In the same way it is possible to elucidate the physical meaning of the other set of generalized forces Q_k.

Assuming

$$\psi_2(s)=y'(s),\quad \psi_1(s)=x'(s),$$
$$\psi_3(s)=x(s)y'(s)-y(s)x'(s)$$

and using equations (1.19), we obtain on the basis of expression (1.17)

$$Q_1=Q_x,$$
$$Q_2=Q_y,$$
$$Q_3=H^*).$$

In contrast to the above forces (transverse forces and torsional moments) we shall call the generalized forces $Q_4, Q_5, \ldots, Q_n$ corresponding to the generalized displacements $V_4, V_5, \ldots, V_n$ transverse bimoments.

Given the general integral of the differential equation of the beam (1.14), we can now determine the stress and deformation state of the beam for the most diverse boundary conditions at the transverse edges, given in terms of forces, displacements, or mixed.

§ 2. Beam-shell with variable rectangular profile

1. We apply the theory of the previous section to the calculation of the strength of a thin-walled beam consisting of four plates which form a rectangle in cross section. The cross section of such a beam has two

* Equations (1.8.9) for the tangential stresses are not valid here, since in the theory of thin-walled beams of open section the tangential stresses are determined from the statical conditions based on the hypothesis of absence of shear deformations, while in the theory of thin-walled beams of closed section we determine the tangential stresses from Hooke's law.

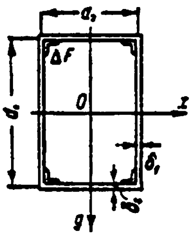


Figure 138

symmetry axes: the horizontal Ox and the vertical Oy (Figure 138).

The elementary strip of width $dz = 1$, isolated from the beam by two transverse sections $z = \text{const}$ and $z + dz = \text{const}$, has four degrees of freedom in motion out of the plane of the strip and four degrees of freedom in motion in this plane. Therefore, the displacements of any point of some plate in its plane can be given in the form of a finite series:

$$\left.\begin{aligned} u(z, s) &= U_1(z)\varphi_1(s) + U_2(z)\varphi_2(s) + U_3(z)\varphi_3(s) + U_4(z)\varphi_4(s), \\ v(z, s) &= V_1(z)\psi_1(s) + V_2(z)\psi_2(s) + V_3(z)\psi_3(s) + V_4(z)\psi_4(s). \end{aligned}\right\} \quad (2.1)$$

We denote, as usual, by $x(s)$ and $y(s)$ the coordinates with respect to the symmetry axes of an arbitrary point of the rectangular contour and by $h(s)$ the perpendicular from the center to a side of the rectangle; let $x'(s)$ and $y'(s)$ be the derivatives of these Cartesian coordinates with respect to s.

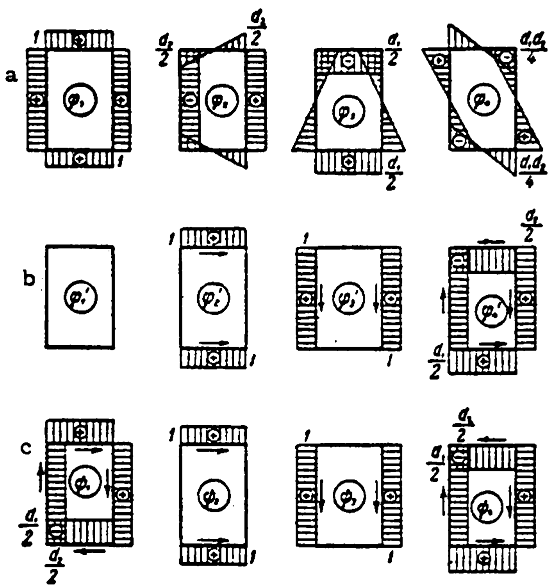


Figure 139

We determine the functions $\varphi_i(s)$ $(i = 1, 2, 3, 4)$ and $\psi_k(s)$ $(k = 1, 2, 3, 4)$ as follows

$$\varphi_1(s) = 1, \quad \varphi_2(s) = x(s), \quad \varphi_3(s) = y(s), \quad \varphi_4(s) = x(s)y(s); \quad (2.2)$$

$$\left.\begin{aligned} &\psi_1(s) = h(s), \quad \psi_2(s) = x'(s), \quad \psi_3(s) = y'(s), \\ &\psi_4(s) = x'(s)y(s) + x(s)y'(s). \end{aligned}\right\} \quad (2.3)$$

The diagrams of the functions $\varphi_i(s)$, of the derivatives $\varphi_i'(s)$ and of the functions $\psi_k(s)$ are shown in Figures 139a, b, c. The generalized

displacements corresponding to these generalized coordinates have a definite physical meaning, namely:

a) The generalized longitudinal displacements; $U_1(z)$ represent the translational displacement of the whole section $z = \text{const}$. $U_2(z)$ and $U_3(z)$ are the angles of rotation of the section $z = \text{const}$ with respect to the axes Oy and Ox respectively. $U_4(z)$ is the generalized warping of a section $z = \text{const}$.

b) The generalized transverse displacements; $V_1(z)$ represent the angle of rotation about the axis Oz of the section $z = \text{const}$ (considered as rigid), $V_2(z)$ and $V_3(z)$ are the translational displacements (deflections) of the section $z = \text{const}$ in the directions of the axes Ox and Oy; $V_4(z)$ is the generalized contour deformation of a rectangular section of the beam.

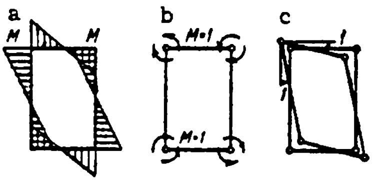


Figure 140

Let d_1, δ_1 and $F_1 = d_1\delta_1$ denote the width, the thickness and the area of the cross section of a vertical plate. d_2, δ_2 and $F_2 = d_2\delta_2$ denote the width, thickness and area of the cross section of a horizontal plate. Let l denote the length of the beam (Figure 138).

We assume, to be general, that the beam is reinforced by stringers placed on the ribs and operating together with the plates. Let ΔF be the area of the section of a stringer. The bending moments correspond to the generalized deformation $V_4(z)$ of a beam whose plates are rigidly connected with one another by the ribs. The diagrams of these moments $M(s)$ are shown in Figure 140a, for an elementary transverse strip*.

We use the method of forces in order to determine the value of the moment in the node of the examined strip. For this purpose we examine two states of our elementary frame which corresponding to a generalized unit displacement $V_4(z) = 1$. We obtain one state if we place hinges at the nodes of the frame and apply to these nodes unit bending moments (Figure 140b). The diagram of these moments will be obviously similar to the diagram of $M(s)$, but only at the nodes of the frame will the ordinates of the diagrams be equal to 1. The other elementary state corresponds to

* The form of this diagram is determined by the following considerations. The projection on the normal to this plate of all the external forces applied to an infinitely small element of the plate is expressed, as known from the theory of plates, for the case of zero external surface load, in terms of the bending and torsional moments, by

$$\frac{\partial^2 M_z}{\partial z^2} + \frac{\partial^2 M_s}{\partial s^2} - 2\frac{\partial^2 H}{\partial z \partial s} = 0.$$

Assuming that the longitudinal normal and tangential stresses are uniformly distributed across the plate, we obtain

$$M_z = H = 0.$$

Therefore

$$\frac{\partial^2 M_s}{\partial s^2} = 0.$$

The bending moment which refers to a longitudinal area varies linearly with respect to the coordinate s. It is obvious that the continuity condition of the transverse bending moments $M_s = M(z, s)$ along s and from $M(z, s)$ vanishing in the middle of each plate of our rectangular section lead to the diagram of $M(s)$ shown in Figure 140a.

the generalized transverse displacement ψ_4 (Figure 140c). We obtain from the condition of vanishing relative rotation of adjacent sections in the node of the frame

$$\delta_{11}M_1+\delta_{1\psi}=0.$$

In the present case the relative rotation of adjacent sections per node of the frame is calculated for the first state by the usual Mohr expression

$$\delta_{11}=\frac{1}{4}\oint\frac{M_1^2}{EJ}ds=\frac{2}{E}\left(\frac{d_1}{\delta_1^3}+\frac{d_2}{\delta_2^3}\right).$$

For the relative displacement of adjacent sections in the frame node we obtain the following equation for the state ψ_4 (as seen from Figure 140c):

$$\delta_{1\psi}=-2.$$

Thus, the ordinate of the diagram (Figure 140a) in the node of the frame, corresponding to a unit generalized displacement $V_4(z)=1$ is determined by equation

$$M=\frac{12}{\frac{d_1}{EJ_1}+\frac{d_2}{EJ_2}},$$

and the required bending moment in this node, corresponding to the displacement $V_4(z)$, equals

$$M(z)=\frac{12}{\frac{d_1}{EJ_1}+\frac{d_2}{EJ_2}}V_4(z),$$

where we have denoted by $J_1=\frac{\delta_1^3}{12}$, $J_2=\frac{\delta_2^3}{12}$ the moments of inertia per unit beam length with respect to a longitudinal section of the vertical and horizontal plates, respectively.

Writing out the general differential equations (1.10) and calculating their coefficients from equations (1.11) with the help of the diagrams shown in Figure 139, we obtain the following basic equations

$$\left.\begin{aligned}
&1)\quad EFU_1''+p_1=0,\\
&2)\quad EJ_yU_2''-2GF_2(U_2+V_2')+p_2=0,\\
&3)\quad 2GF_2(U_2'+V_2'')+q_2=0,\\
&4)\quad EJ_xU_3''-2GF_1(U_3+V_3')+p_3=0,\\
&5)\quad 2GF_1(U_3'+V_3'')+q_3=0,\\
&6)\quad aU_4''-b_1U_4-b_2V_1'-b_1V_4'+p_4=0,\\
&7)\quad b_2U_4'+b_1V_1''+b_2V_4''+q_1=0,\\
&8)\quad b_1U_4'+b_2V_1''+b_1V_4''-cV_4+q_4=0.
\end{aligned}\right\}\qquad(2.4)$$

The following notations for the geometrical characteristics and for the generalized rigidities were adopted here

$$F=2F_1+2F_2+4\Delta F,$$

$$J_x=d_1^2\left(\frac{F_1}{6}+\frac{F_2}{2}+\Delta F\right),$$

$$J_y=d_2^2\left(\frac{F_2}{6}+\frac{F_1}{2}+\Delta F\right);$$

$$\left.\begin{aligned} a &= \frac{1}{24} E d_1^2 d_2^2 (F_1 + F_2 + 6\Delta F), \\ b_1 &= \frac{1}{2} G (d_1^2 F_2 + d_2^2 F_1), \\ b_2 &= \frac{1}{2} G (-d_1^2 F_2 + d_2^2 F_1), \\ c &= \frac{96}{\frac{d_1}{EJ_1} + \frac{d_2}{EJ_2}}. \end{aligned}\right\} \tag{2.5}$$

The free terms are calculated from the general equations

$$\left.\begin{aligned} p_j &= \int p\varphi_j ds \qquad (j = 1,\ 2,\ 3,\ 4), \\ q_h &= \int q\psi_h ds \qquad (h = 1,\ 2,\ 3,\ 4). \end{aligned}\right\} \tag{2.6}$$

The longitudinal deformations of the beam in the case of axial extension or thrust are determined independently of the other 7 equations by the first equation of the system (2.4).

If we assume that the generalized longitudinal loads $p_2 = p_3 = 0$, then the second and third equations reduce to the single equation

$$EJ_y V_2^{IV} - q_2 + \frac{EJ_y}{2GF_2} q_2'' = 0, \tag{2.7}$$

the fourth and the fifth lead to the equation

$$EJ_x V_3^{IV} - q_3 + \frac{EJ_x}{2GF_1} q_3'' = 0. \tag{2.8}$$

The deformed states (the contour retaining its shape) under bending as a thin-walled beam in the horizontal and vertical planes are determined respectively by equations (2.7) and (2.8). The last terms of these equations represent the influence on the deflection of the beam of the shear deformation in the planes of the horizontal and vertical plates. If we neglect these deformations (i. e. if we assume that the cross sections of the bent beam remain not only plane but also normal to the curved axis of the beam) then equations (2.7) and (2.8) become the known equations from the elementary theory of bending.

We avail ourselves of this result and do not analyze the first five equations, but investigate instead the last three which, independently of the others, form a symmetrical system of three differential equations. They determine, together with the boundary conditions on the transverse edges, the deformed state of the beam, which for longitudinal displacements represents warping of the sections and for transverse displacements torsion and deformation of the contour.

1. The external forces in the last three equations of the system (2.4) are expressed by the free terms p_4, q_1 and q_4 as computed by the general equations (2.6). In accordance with the diagrams of the generalized coordinates $\varphi_4(s)$, $\psi_1(s)$ and $\psi_4(s)$ shown in Figures 139a and 193b, we may call p_4 the external longitudinal bimoment per unit length, q_4 the external transverse bimoment per unit length, and q_1 the external torsional moment per unit length.

Starting from the concept of virtual work and keeping in mind that in the case of torsion the elementary transverse strip of the beam has one degree of freedom for displacements of this strip out of its plane and two

degrees of freedom for displacements in its plane, we analogously obtain an equation for the internal generalized forces,

$$\left.\begin{aligned} B &= -\int \sigma\varphi_4 dF, \\ H &= \int \tau\psi_1 dF, \\ Q &= \int \tau\psi_4 dF. \end{aligned}\right\} \tag{2.9}$$

Here we call the generalized longitudinal force B the longitudinal bimoment. The generalized transverse force H is merely the torsional moment, and we shall call the new generalized force Q, corresponding to the deformation of the contour, the transverse bimoment. This force, similar to the longitudinal bimoment B and in contrast with the torsional moment H, is also statically equivalent to zero.

In the interest of making the outlined theory clearer and the notation more manageable, we shall denote in the following: the generalized warping $U_4(z)$ by $U(z)$, the torsion angle $V_1(z)$ by $\theta(z)$, and the generalized contour deformation $V_4(z)$ by $\varkappa(z)$. The generalized coordinates corresponding to these generalized displacements will be denoted by $\varphi_4(s)$, $\psi_1(s)$ and $\psi_4(s)$ instead of $\varphi(s)$, $\psi_\theta(s)$ and $\psi_\varkappa(s)$.

Applying (1.4) and (1.5), the equation for the normal and tangential stresses now become

$$\left.\begin{aligned} \sigma(z, s) &= EU'(z)\varphi(s), \\ \tau(z, s) &= G[U(z)\varphi'(s) + \theta'(z)\psi_\theta(s) + \varkappa'(z)\psi_\varkappa(s)]. \end{aligned}\right\} \tag{2.10}$$

Equations (2.9), after introducing in them the expressions (2.10) and calculating the definite integrals from the relations (2.5), will have the form

$$\left.\begin{aligned} B &= -aU', \\ H &= b_2U + b_1\theta' + b_2\varkappa', \\ Q &= b_1U + b_2\theta' + b_1\varkappa'. \end{aligned}\right\} \tag{2.11}$$

Finally, for the homogeneous case (for $p_4 = q_1 = q_4 = 0$) the above system of three differential equations is written in the new notation

$$\left.\begin{aligned} aU'' - b_1U - b_2\theta' - b_1\varkappa' &= 0, \\ b_2U' + b_1\theta'' + b_2\varkappa'' &= 0, \\ b_1U' + b_2\theta'' + b_1\varkappa'' - c\varkappa &= 0. \end{aligned}\right\} \tag{2.12}$$

3. We shall now deal with the integration of the system of homogeneous differential equations (2.12). We introduce a new function $f(z)$ and express by it and its derivatives the required displacements $U(z)$, $\theta(z)$ and $\varkappa(z)$, so that the first and third equations of the system (2.12) are identically satisfied for any form of the function $f(z)$. The following relatinos will fulfill this condition

$$\left.\begin{aligned} U &= f', \\ \theta &= -\frac{ab_1}{cb_2}f^{\text{IV}} + \frac{a}{b_2}f'' - \frac{b_1}{b_2}f, \\ \varkappa &= \frac{a}{c}f^{\text{IV}}, \end{aligned}\right\} \tag{2.13}$$

where f'', f'', f^{IV} are the derivatives of the indicated order with respect to the variable z. By introducing (2.13) in the second equation of the system (2.12) we obtain the final equation in the form

$$\frac{a}{c}\left(b_1^2-b_2^2\right)f^{VI}-ab_1f^{IV}+\left(b_1^2-b_2^2\right)f''=0,$$

or, alternatively,

$$f^{VI}-2r^2f^{IV}+s^4f''=0, \tag{2.14}$$

where r^2 and s^4 are quantities determined either from

$$r^2=\frac{b_1c}{2\left(b_1^2-b_2^2\right)}, \quad s^4=\frac{c}{a}, \tag{2.15}$$

or, if we use the relations (2.5), from

$$\left.\begin{aligned} r^2&=\frac{24}{\frac{d_1}{J_1}+\frac{d_2}{J_2}}\frac{E}{G}\left(\frac{1}{d_2^2F_1}+\frac{1}{d_1^2F_2}\right), \\ s^4&=\frac{2304}{\frac{d_1}{J_1}+\frac{d_2}{J_2}}\frac{1}{d_1^2d_2^2\left(F_1+F_2+6\Delta F\right)}. \end{aligned}\right\} \tag{2.16}$$

It is possible to consider the quantities r^2 and s^4 in equation (2.14) as generalized elastic characteristics. The generalized internal forces are also expressed by the function $f(z)$ and its derivatives. In order to obtain the corresponding equations it is necessary to substitute the expressions (2.13) in (2.11)

$$\left.\begin{aligned} B&=-af'', \\ H&=-\frac{a}{cb_2}\left(b_1^2-b_2^2\right)f^{V}+\frac{ab_1}{b_2}f'''-\frac{b_1^2-b_2^2}{b_2}f', \\ Q&=af''' \end{aligned}\right\} \tag{2.17}$$

The problem of torsion of a thin-walled beam of closed rectangular section is thereby reduced to the solution of a single differential equation (2,14) of the sixth order with constant coefficients. If we determine the functions $f(z)$ from these equations and the boundary conditions we can find, with the help of equations (2.13) and (2.17), the function for the basic generalized displacements and for the basic generalized internal forces. Incidentally, we may note that the first and third of equations (2.17) imply

$$Q=-B', \tag{2.18}$$

i. e. the generalized transverse force Q (the transverse bimoment) is equal to the derivative of the longitudinal force B (the longitudinal bimoment). We have obtained an analogous relation in § 8 of Chapter I for thin-walled beams of open, nondeformable contour. It was mentioned there that Zhuravskii's theorem corresponds to this assumption in the theory of strength of materials. It has to be noted that the equality (2.18) holds only for the case of $p_4=0$.

The general integral of equation (2.14) can be written in the form

$$f(z)=C_1\Phi_1+C_2\Phi_2+C_3\Phi_3+C_4\Phi_4+C_5z+C_6, \tag{2.19}$$

where C_1, C_2, . . . , C_6 are arbitrary constants to be determined from the boundary conditions, and Φ_1, Φ_2, Φ_3 and Φ_4 are products of hyperbolic

and trigonometric functions which are particular, linearly independent, solutions of equations (2.14) and are given by the expressions*:

$$\left.\begin{aligned}\Phi_1 &= \operatorname{ch}\alpha z \sin\beta z,\\ \Phi_2 &= \operatorname{ch}\alpha z \cos\beta z,\\ \Phi_3 &= \operatorname{sh}\alpha z \cos\beta z,\\ \Phi_4 &= \operatorname{sh}\alpha z \sin\beta z,\end{aligned}\right\} \tag{2.20}$$

where α and β are, respectively, the real part and the coefficient of the imaginary part of the four conjugate complex roots of the characteristic equation of the differential equation (2.14). The values of α and β are calculated from

$$\left.\begin{aligned}\alpha &= \sqrt{\frac{s^2+r^2}{2}},\\ \beta &= \sqrt{\frac{s^2-r^2}{2}},\end{aligned}\right\} \tag{2.21}$$

where s^2 and r^2 are the generalized elastic characteristics (2.15). The derivatives of various orders of the functions $\Phi_j(z)$ (2.20) are expressed by the same functions by the formulas of Table 26.

Table 26

Function Φ_j	Odd derivatives of Φ_j	Even derivatives of Φ_j
$\Phi_1 = \operatorname{ch}\alpha z \sin\beta z$	$A\Phi_4 + B\Phi_2$	$A\Phi_1 + B\Phi_3$
$\Phi_2 = \operatorname{ch}\alpha z \cos\beta z$	$A\Phi_3 - B\Phi_1$	$A\Phi_2 - B\Phi_4$
$\Phi_3 = \operatorname{sh}\alpha z \cos\beta z$	$A\Phi_2 - B\Phi_4$	$A\Phi_3 - B\Phi_1$
$\Phi_4 = \operatorname{sh}\alpha z \sin\beta z$	$A\Phi_1 + B\Phi_3$	$A\Phi_4 + B\Phi_2$

The coefficients A and B are expressed in terms of α and β for various derivatives in Table 27.

Table 27

Order of derivative	A	B
I	α	β
II	$\alpha^2 - \beta^2$	$2\alpha\beta$
III	$\alpha(\alpha^2 - 3\beta^2)$	$\beta(3\alpha^2 - \beta^2)$
IV	$\alpha^4 - 6\alpha^2\beta^2 + \beta^4$	$4\alpha\beta(\alpha^2 - \beta^2)$

Introducing the general integral of the function $f(z)$ (2.19) in the right-hand sides of equations (2.13) and (2.17) and using Table 26 and 27 for the derivatives of the functions $\Phi_j(z)$, we obtain the general integrals of the required generalized displacements and the fundamental generalized internal

* To be definite we consider here the case of complex roots.

forces as functions of six arbitrary constants $C_1, C_2, \ldots, C_6$. For convenient review they are arranged in Table 28.

For practical application, we may present the integrals of the basic generalized displacements and forces in another more convenient form, viz. as functions not of the arbitrary constants $C_1, C_2, \ldots, C_6$, but of the initial parameters which play the same role as the arbitrary constants of integration, but which have a definite mechanical meaning. The values of the fundamental generalized displacements U_0, θ_0, $\varkappa_0$ and fundamental generalized forces B_0, H_0 and Q_0 at the initial end $z=0$ are taken in our case as the initial parameters. Proceeding as explained in § 3, Chapter II, we obtain from the general integrals of Table 28, expressed as functions of the arbitrary constants $C_1, C_2, \ldots, C_6$ in the general integrals of the same fundamental quantities expressed as functions of the initial parameters U_0, θ_0, $\varkappa_0$, B_0, H_0 and Q_0, that is in the so-called matrix of initial parameters, given here in the form of Table 29.

Two new notations are used for compactness in Table 29

$$\gamma_1 = \frac{b_1}{b_1^2 - b_2^2}, \quad \gamma_2 = \frac{b_2}{b_1^2 - b_2^2}, \tag{2.22}$$

where b_1 and b_2 are determined by equations (2.5).

4. The solution of the boundary problem for a shell (or a thin-walled beam) of finite length leads, with the help of the integrals given in Table 29, to the determination of only three integration constants, since from the six initial parameters $\varkappa_0$, U_0, θ_0, H_0, B_0 and Q_0 three will be known quantities in each particular problem, determined from the conditions of constraint at the initial section $z=0$. The influence of the statical and kinematical factors, whether concentrated or distributed according to any law along the coordinate z, is considered following the rules of § 3, Chapter II, and in accordance with the matrix of initial parameters for thin-walled open sections. Since we have adopted for the shell a design model characterized by the elementary transverse strip possessing three degrees of freedom in torsion (one in deformation and two in the plane of the strip) there will be three independent boundary conditions at each end of the shell. Altogether six independent conditions must be thus prescribed for the two ends of the shell, which accords with the order of the fundamental differential equation (2.14), and thus with the number of arbitrary integration constants of this equation ($C_1, C_2, \ldots, C_6$ or $\varkappa_0$, U_0, θ_0, H_0, B_0 and Q_0).

The boundary conditions on any end (depending on the character of the problem) can be given either as displacements only, or as forces, or (in mixed problems) partly as displacements and partly as forces. Thus, for example, in the case of complete encastrement of one end of the shell we shall have boundary conditions of a purely geometrical character, i. e. the following conditions should be satisfied on this end: $U=\theta=\varkappa=0$. If this end is at the origin ($z=0$) we shall have $U_0=\theta_0=\varkappa_0=0$ and thus the three initial parameters will be determined independently of the conditions at the other end.

At the other end ($z=l$) the conditions can also be purely geometrical or purely statical. For example, if the end is free from longitudinal and transverse constraints and no forces are applied there, the conditions for $z=l$ will have the form

$$B(l)=H(l)=Q(l)=0.$$

	C_1	C_2
$U(z)$	$\alpha\Phi_4+\beta\Phi_2$	$\alpha\Phi_3-\beta\Phi_1$
$\theta(z)$	$-\frac{2r^2}{s^4}\frac{b_2}{b_1}(r^2\Phi_1+2\alpha\beta\Phi_3)$	$-\frac{2r^2}{s^4}\frac{b_2}{b_1}(r^2\Phi_2-2\alpha\beta\Phi_4)$
$\varkappa(z)$	$\frac{1}{s^4}[(\alpha^4-6\alpha^2\beta^2+\beta^4)\Phi_1+$ $+4\alpha\beta(\alpha^2-\beta^2)\Phi_3]$	$\frac{1}{s^4}[(\alpha^4-6\alpha^2\beta^2+\beta^4)\Phi_2-$ $-4\alpha\beta(\alpha^2-\beta^2)\Phi_4]$
$B(z)$	$-a(r^2\Phi_1+2\alpha\beta\Phi_3)$	$-a(r^2\Phi_2-2\alpha\beta\Phi_4)$
$H(z)$	—	—
$Q(z)$	$a[\alpha(\alpha^2-3\beta^2)\Phi_4+\beta(3\alpha^2-$ $-\beta^2)\Phi_2]$	$a[\alpha(\alpha^2-3\beta^2)\Phi_3-\beta(3\alpha^2-$ $-\beta^2)\Phi_1]$

	$\varkappa_0$	U_0	θ_0	H_0
$\varkappa(z)$	$\Phi_2+\frac{r^2}{2\alpha\beta}\Phi_1$	$-\frac{1}{2\alpha\beta}(\alpha\Phi_1+\beta\Phi_3)$	0	$-\frac{\gamma_2}{2\alpha\beta}(\alpha\Phi_1+\beta\Phi_3)$
$U(z)$	$\frac{s^2}{2\alpha\beta}(\alpha\Phi_1-\beta\Phi_3)$	$\Phi_2-\frac{r^2}{2\alpha\beta}\Phi_4$	0	$\gamma_2\left(-1+\Phi_2-\frac{r^2}{2\alpha\beta}\Phi_4\right)$
$\theta(z)$	$-\gamma_2\frac{as^4}{2\alpha\beta}\Phi_4$	$\gamma_2\frac{as^2}{2\alpha\beta}(\alpha\Phi_1-\beta\Phi_3)$	1	$\gamma_2^2\frac{as^2}{2\alpha\beta}(\alpha\Phi_1-\beta\Phi_3)+\gamma_1 z$
$H(z)$	0	0	0	1
$B(z)$	$-\frac{as^4}{2\alpha\beta}\Phi_1$	$\frac{as^2}{2\alpha\beta}(\alpha\Phi_4-\beta\Phi_3)$	0	$\gamma_2\frac{as^2}{2\alpha\beta}(\alpha\Phi_1-\beta\Phi_3)$
$Q(z)$	$\frac{as^4}{2\alpha\beta}(\alpha\Phi_1+\beta\Phi_3)$	$-\frac{as^4}{2\alpha\beta}\Phi_4$	0	$-\gamma_2\frac{as^4}{2\alpha\beta}\Phi_1$

Table 28

C_3	C_4	C_5	C_6
$\alpha\Phi_2 - \beta\Phi_4$	$\alpha\Phi_1 + \beta\Phi_3$	1	—
$-\frac{2r^2}{s^4}\frac{b_2}{b_1}(r^2\Phi_3 - 2\alpha\beta\Phi_1)$	$-\frac{2r^2}{s^4}\frac{b_2}{b_1}(r^2\Phi_4 + 2\alpha\beta\Phi_2)$	$-\frac{b_1}{b_2}z$	$-\frac{b_1}{b_2}$
$\frac{1}{s^4}[(\alpha^4 - 6\alpha^2\beta^2 + \beta^4)\Phi_3 - 4\alpha\beta(\alpha^2 - \beta^2)\Phi_1]$	$\frac{1}{s^4}[(\alpha^4 - 6\alpha^2\beta^2 + \beta^4)\Phi_4 + 4\alpha\beta(\alpha^2 - \beta^2)\Phi_2]$	—	—
$-a(r^2\Phi_3 - 2\alpha\beta\Phi_1)$	$-a(r^2\Phi_4 + 2\alpha\beta\Phi_2)$	—	—
—	—	$-\frac{b_1^2 - b_2^2}{b_2}$	—
$a[\alpha(\alpha^2 - 3\beta^2)\Phi_2 - \beta(3\alpha^2 - \beta^2)\Phi_4]$	$a[\alpha(\alpha^2 - 3\beta^2)\Phi_1 + \beta(3\alpha^2 - \beta^2)\Phi_3]$	—	—

Table 29

B_6	Q_6
$\frac{1}{a2\alpha\beta}\Phi_4$	$\frac{\alpha(\alpha^2 - 3\beta^2)\Phi_1 + \beta(3\alpha^2 - \beta^2)\Phi_3}{as^4 2\alpha\beta}$
$\frac{\alpha(\alpha^2 - 3\beta^2)\Phi_1 + \beta(\beta^2 - 3\alpha^2)\Phi_3}{as^2 2\alpha\beta}$	$\frac{1}{a2\alpha\beta}\Phi_4$
$\gamma_2\left(-1 + \Phi_3 - \frac{r^2}{2\alpha\beta}\Phi_4\right)$	$-\frac{\gamma_2}{2\alpha\beta}(\alpha\Phi_1 + \beta\Phi_3)$
0	0
$\Phi_3 - \frac{r^2}{2\alpha\beta}\Phi_4$	$-\frac{1}{2\alpha\beta}(\alpha\Phi_1 + \beta\Phi_3)$
$\frac{s^2}{2\alpha\beta}(\alpha\Phi_1 - \beta\Phi_3)$	$\Phi_3 + \frac{r^2}{2\alpha\beta}\Phi_4$

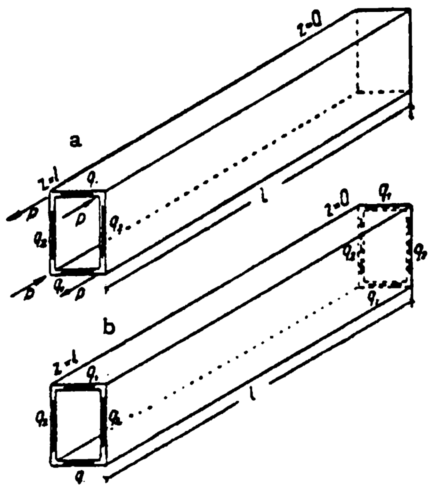


Figure 141

In the case of an antisymmetric system of four longitudinal forces equal in absolute value (P) and an antisymmetric system of two pairs of transverse forces, the members of each pair being equal in absolute value (q_1 and q_2) being applied to the end $z=l$ (Figure 141a), we shall have the following boundary conditions for $z=l$ on transforming these forces into generalized ones (one longitudinal and two transverse),

$$B(l)=d_1d_2P, \quad H(l)=d_1q_1+d_2q_2, \quad Q(l)=-d_1q_1+d_2q_2.$$

If only the longitudinal forces P are applied to the section $z=l$ and the transverse loads on this section vanish ($q_1=q_2=0$), the boundary conditions will have the form

$$\text{for } z=l \quad B(l)=d_1d_2P, \quad H(l)=Q(l)=0.$$

If, on the contrary, there are no longitudinal forces in the section $z=l$, and the transverse forces in this section are given by

$$q_1=d_2S, \quad q_2=d_1S,$$

where S is the uniform "flux intensity" of the given shear forces (per unit length of the contour line), the boundary conditions for $z=l$ will have the form

$$B(l)=0, \quad H(l)=2d_1d_2S, \quad Q(l)=0.$$

In this case the external load, given on the contour of the section $z=l$ only produces the torsional moment H determined as a product of double the area of the rectangle having the sides d_1 and d_2 and the shear force S, per unit length of the contour line.

In all the cases enumerated above, the three boundary conditions on the edge $z=l$ furnish three equations for determining the three remaining initial parameters, thus giving completely determinate solutions of the torsion problem in a thin-walled shell (or beam) of closed section, whose

one end $(z=0)$ is rigidly fixed and the other $(z=l)$ is subjected to one of the above boundary conditions.

We shall have a problem of pure torsion of a closed thin-walled rectangular section if we assume that both ends ($z=0$ and $z=l$) are free from longitudinal and transverse constraints and only the shear forces S are applied there, forming a closed circulation of constant intensity along the section contour (Figure 141b).

The boundary conditions in this case will have the form

$$\begin{aligned}&\text{for } z=0 \quad B_0=Q_0=0, \qquad H_0=2d_1d_2S;\\&\text{for } z=l \quad B(l)=Q(l)=0. \qquad H(l)=2d_1d_2S.\end{aligned}$$

Having determined the initial parameters from these conditions, we obtain for the basic kinematical and statical design quantities the following equations

$$\left.\begin{aligned}&U=-\gamma_2H_0,\\&\theta=\gamma_1H_0z+\theta_0,\\&H=H_0,\\&\varkappa=B=Q=0.\end{aligned}\right\} \tag{2.23}$$

The second equation of (2.23) can be written in a somewhat different form; by differentiating both sides once with respect to z we obtain

$$\theta'=\gamma_1H_0 \tag{2.24}$$

or

$$H_0=\frac{1}{\gamma_1}\theta'.$$

Inserting γ_1 from equation (2.22) and using the relations (2.5), we obtain

$$H_0=\frac{2G\Omega^2\delta_1\delta_2}{d_1\delta_2+d_2\delta_1}\theta', \tag{2.25}$$

where $\Omega=d_1d_2$ is the area of the rectangle having the sides d_1 and d_2.

This equation agrees with our earlier equation (III.3.5) $H_T=T\Omega$ derived for the case of a closed section. The shear force $T(z)$ is determined by equation (III.3.9) $T(z)=\frac{\Omega}{\delta_T}\theta'(z)$, where $\delta_T=\oint\frac{ds}{G\delta}$. For a closed rectangular section of a beam consisting of plates of width and thickness a_1, δ_1 and d_2, δ_2 respectively, we obtain the following value of δ_T if we assume that the shear [rigidity] modulus is the same for all the plates

$$\delta_T=\frac{2}{G\delta}\left(\frac{d_1}{\delta_1}+\frac{d_2}{\delta_2}\right)$$

and, consequently, $T=\frac{G\Omega\delta_1\delta_2}{2(d_1\delta_2+d_2\delta_1)}\theta'(z)$, whence we finally obtain

$$H_T=\frac{G\delta_1\delta_2\Omega^2}{2(d_1\delta_2+d_2\delta_1)}\theta'(z),$$

which fully agrees with equation (2.25) since in this case $\Omega=d_1d_2$ while in § 3 of Chapter III, $\Omega=2d_1d_2$.

If we assume in equation (2.25) $\delta_1=\delta_2=\delta$, we obtain the well-known Bredt equation for the problem of pure torsion of a closed thin-walled

rectangular section with walls of equal thickness

$$H_\theta = \frac{2G\Omega^2\delta}{d_1 + d_2}\theta', \tag{2.26}$$

where the quantity

$$J_d = \frac{2\Omega^2\delta}{d_1 + d_2} \tag{2.27}$$

is the moment of inertia for pure Saint-Venant torsion.

The quantity θ_0 which remains undetermined refers to the regular displacement of the whole beam as a rigid body, and thus does not influence the state of stress in the beam and for that reason can be set equal to zero. We shall not dwell on other forms of boundary conditions, referring the reader who wishes to pursue this further to the author's work /51/.

§ 3. Design of a shell of variable rectangular section without allowance for shear

The integrals of the fundamental forces and displacements given in the matrix of Table 29 were obtained under sufficiently general assumptions, taking into consideration not only the bending deformation of the beam cross section but also its shear deformation. Having obtained these integrals we could by passing to the limit find solutions for problems which are particular cases of the more general and exact theory developed here. Thus, taking limits in the integrals of Table 29 for $GF_1 \to \infty$ and $GF_2 \to \infty$ in accordance with the assumed absence of shear deformation, we would obtain the fundamental design equations for this problem.

For simplicity we shall start in this case not from Table 29 but from the system of differential equations (2.12). Since we assume that there is no shear deformation, it is obvious that we have to pass the limits of $GF_1 \to 0$ and $GF_2 \to \infty$ in the system (2.12). It is seen from equations (2.5) that in this case $b_1 \to \infty$ and $b_2 \to \infty$. Besides, since the torsion of the beam is connected with a shear deformation, it is obvious that in the absence of shear deformation the torsion angle $\theta(z) \equiv 0$ and, consequently, the generalized displacements $\psi_1(s) = \psi_\theta(s)$ corresponding to the rotation of the cross section also vanish.

The second equation is eliminated from the system (2.12) since it expresses the work done by all forces applied to the elementary transverse strip in the generalized displacement $\psi_\theta(s)$. The remaining equations of this system have the following form

$$\left.\begin{aligned} aU'' - b_1U - b_1x' &= 0, \\ b_1U' + b_1x'' - cx &= 0. \end{aligned}\right\} \tag{3.1}$$

Since a is a constant and $b_1 \to \infty$ we obtain from the first equation

$$U = -x'. \tag{3.2}$$

Differentiating the first equation (3.1) with respect to z and adding it to the second we obtain

$$aU''' - cx = 0.$$

Using expression (3.2) we write the last equation in the following form

$$x^{IV}+\frac{c}{a}x=0.$$

When $GF_1\to\infty$ and $GF_2\to\infty$ it follows from equations (2.15), (2.16) and (2.21), that $r^2=0$ and $\alpha^2=\beta^2=\frac{s^2}{2}$. Assuming $k^2=\frac{s^2}{2}$, we obtain

$$x^{IV}+4k^4x=0. \tag{3.3}$$

It is easy to see that equation (3.3) fully agrees with equation (2.14) if we assume $x=f''$ and if we write equation (2.14) in accordance with the conditions of our case, i. e., if we set $r^2=0$.

According to expressions (3.2), (2.13) and (2.17) we obtain for the basic stresses and displacements the following equations

$$U=-x',\quad B=ax'',\quad Q=-ax'''. \tag{3.4}$$

The hyperbolic-trigonometric functions (2.20) are written, in the case of equation (3.3), in the following form

$$\left.\begin{aligned}\Phi_1&=\operatorname{ch}kz\sin kz,\\ \Phi_2&=\operatorname{ch}kz\cos kz,\\ \Phi_3&=\operatorname{sh}kz\cos kz,\\ \Phi_4&=\operatorname{sh}kz\sin kz.\end{aligned}\right\} \tag{3.5}$$

The matrix of the fundamental integrals expressed by the initial parameters x_0, U_0, B_0 and Q_0 has the following form

Table 30

	x_0	U_0	B_0	Q_0
$x(z)$	Φ_2	$-\frac{1}{2k}(\Phi_1+\Phi_3)$	$\frac{1}{as^2}\Phi_4$	$-\frac{k}{as^4}(\Phi_1-\Phi_3)$
$U(z)$	$k(\Phi_1-\Phi_3)$	Φ_2	$-\frac{k}{as^2}(\Phi_1+\Phi_3)$	$\frac{1}{as^2}\Phi_4$
$B(z)$	$-as^2\Phi_4$	$ak(\Phi_1-\Phi_3)$	Φ_2	$-\frac{1}{2k}(\Phi_1+\Phi_3)$
$Q(z)$	$as^2k(\Phi_1+\Phi_3)$	$-as^2\Phi_4$	$k(\Phi_1-\Phi_3)$	Φ_2

The matrix of Table 30 is identical with the matrix of the initial parameters for a beam on elastic foundations with the only difference that, in the latter case, the deflection, the inclination of the tangent to the curved axis, the bending moment and the transverse force are taken as the basic design quantities. We may reach the same results yet in another way starting from the general eight-term differential equations. This method of calculation is in a sense a particular case of the use of the general variational method we developed for hipped systems, taking into consideration

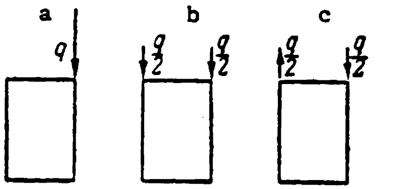


Figure 142

only the deformations of the contour of the cross section of the system and neglecting the shear deformations of one plate with respect to the other /51/. In order to prove this we examine the same box-like section (Figure 141) under the action of a transverse load in the plane of one of the vertical plates (Figure 142a); for simplicity we shall not take the stringers into consideration.

Resolving the load into symmetrical and antisymmetrical components (Figure 142b, c) and applying the theory of hipped systems to the design of the present section, i. e. starting from the general eight-term differential equation, we obtain a single differential equation for the symmetrical loading and a system of two simulatenous differential equations for the antisymmetrical load.

a) For the symmetrical load

$$\left(\frac{F_1}{6}+\frac{F_2}{2}\right)\frac{d^2\sigma}{dz^2}+\frac{q(z)}{2d_1}=0; \tag{3.6}$$

b) For the antisymmetrical load

$$\left.\begin{aligned}\frac{1}{6}(F_1+F_2)\frac{d^2\sigma}{dz^2}+\frac{8}{d_1d_2}G+\frac{q(z)}{2d_1}=0,\\ -\frac{8}{d_1d_2}\sigma+\frac{1}{6}\left(\frac{d_1}{J_1}+\frac{d_2}{J_2}\right)\frac{d^2G}{dz^2}=0,\end{aligned}\right\} \tag{3.7}$$

where σ is the longitudinal normal stress and G is the transverse bending moment for one corner of the cross section (for the other corners these stresses and moments have either the same value or differ in sign only).

Equation (3.6) determines the longitudinal normal stresses σ for bending of the beam in its vertical plane. This is a problem in the usual elementary theory of bending.

Equations (3.7) refer to beam torsion by the antisymmetrical forces $\frac{1}{2}q(z)$ which vary along the beam according to an arbitrary function. By simple elimination of $\sigma(z)$ or $G(z)$ the system of differential equations (3.7) can be reduced to one of the following equations

$$\left.\begin{aligned}\frac{d^4G}{dz^4}+4k^4G+A=0,\\ \frac{d^4\sigma}{dz^4}+4k^4\sigma+B=0,\end{aligned}\right\} \tag{3.8}$$

where $4k^4=s^4$, A and B are terms depending on the external load and determined by the equations

$$A=\frac{144J_1J_2}{d_1d_2(F_1+F_2)(d_1J_2+d_2J_1)}\cdot\frac{q(z)}{d_1},$$

$$B=\frac{3}{F_1+F_2}\cdot\frac{q''(z)}{d_1}.$$

Any one of the equations (3.8) evidently leads to the equation of a beam on an elastic foundation. The homogeneous parts of these equations

are of the form

$$x^{IV}+4k^4x=0,$$

which can be solved for the particular case to which the matric of Table 30 refers. We conclude that, starting from the eight-term equations, we finally arrive at the matrix of Table 30.

Example 1. As an example, we examine the same box-like shell of length l subjected to a vertical, uniformly distributed load q applied in the plane of one of the verrical boundaries (Figure 142a). In this case the second of equations (3.8) will be homogeneous

$$\frac{d^4\sigma}{dz^4}+4k^4\sigma=0.$$

The general integral of this equation is

$$\sigma=C_1\operatorname{sh}kz\cos kz+C_2\operatorname{ch}kz\cos kz+C_3\operatorname{ch}kz\sin kz+C_4\operatorname{sh}kz\sin kz,\quad(3.9)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Introducing expression (3.9) in the first equations (3.7) and determining the function G, we obtain

$$G=-\frac{k^2}{24}d_1d_2(F_1+F_2)(C_1\operatorname{ch}kz\sin kz+C_2\operatorname{sh}kz\sin kz-$$
$$-C_3\operatorname{sh}kz\cos kz-C_4\operatorname{ch}kz\cos kz)-\frac{qd_2}{16}.$$

We find the arbitrary constants C_1, C_2, C_3 and C_4 from the boundary conditions.

We shall examine the case of hinged ends of the beam with diaphragms rigid in their planes for $q=\text{const}$. Placing the origin of the coordinates z in the middle of the span and noting that the plane $z=0$ is a symmetry plane with respect to the boundary conditions and the external load, we obtain

$$C_1=C_3=0.$$

The stresses $\sigma(z)$ and the moments $G(z)$ are expressed, in our case, by even functions with respect to z,

$$\sigma=C_2\operatorname{ch}kz\cos kz+C_4\operatorname{sh}kz\sin kz,$$
$$G=-\frac{k^2d_1d_2(F_1+F_2)}{24}(C_2\operatorname{sh}kz\sin kz-C_4\operatorname{ch}kz\cos kz)-\frac{qd_2}{16}.\quad(3.10)$$

Assuming $\sigma=0$ and $G=0$ for $z=\frac{l}{2}$ (in agreement with the conditions of longitudinal mobility of the beam and nondeformability of the contour at the supported sections), we find

$$\operatorname{ch}\frac{kl}{2}\cos\frac{kl}{2}C_2+\operatorname{sh}\frac{kl}{2}\sin\frac{kl}{2}C_4=0,$$
$$\operatorname{sh}\frac{kl}{2}\sin\frac{kl}{2}C_2-\operatorname{ch}\frac{kl}{2}\cos\frac{kl}{2}C_4=\frac{3q}{2d_1k^2(F_1+F_2)}.$$

We obtain from these equations the values of the arbitrary constants C_2 and C_4,

$$C_2=\frac{3q}{2d_1k^2(F_1+F_2)}\cdot\frac{\operatorname{sh}\frac{kl}{2}\sin\frac{kl}{2}}{\sin^2\frac{kl}{2}-\operatorname{ch}^2\frac{kl}{2}},\quad(3.11)$$

$$C_4 = -\frac{3q}{2d_1k^4(F_1+F_2)}\cdot\frac{\operatorname{ch}\frac{kl}{2}\cos\frac{kl}{2}}{\sin^2\frac{kl}{2}-\operatorname{ch}^2\frac{kl}{2}}. \tag{3.11}$$

The stresses σ and moments G in any section are determined by equations (3.10) and (3.11).

Figures 143a and 143b show the diagrams of the maximum stresses σ for the case of a symmetrical and antisymmetrical load of a beam having a cross section of the following dimensions: $d_1 = 120$ cm, $d_2 = 70$ cm, $\delta_1 = 1.0$ cm, $\delta_2 = 1.6$ cm. The beam span is $l = 10$ m. The load is $q = 100$ kg/m.

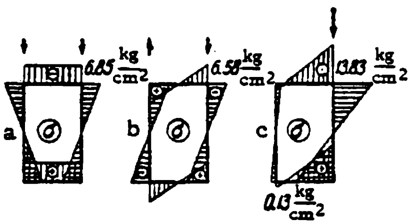


Figure 143

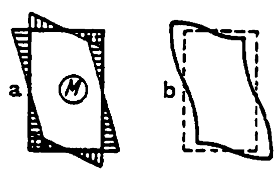


Figure 144

The stresses due to the symmetrical load vary along the beam according to a quadratic law, as follows from equation (3.6). The stresses due to the antisymmetrical load vary as the product of hyperbolic and trigonometric functions, as evident from equation (3.10). In bending and torsion the stresses σ vanish at the ends of the beam.

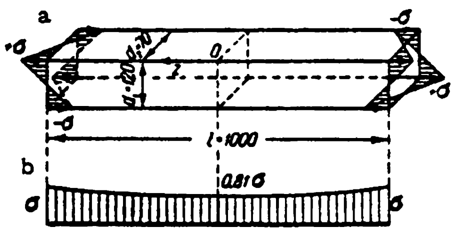


Figure 145

Figure 143c shows the diagram of the total stressed due to a unilateral load $q = 100$ kg/m. The stresses in the antisymmetrical and, consequently, also in the total-stress state do not follow a linear distribution. As seen from a comparison of the diagrams shown in Figures 143a and 143c, the departure from this distribution reaches very large values. In our case, this departure occurs as a result of the deformation of the section contour.

Figure 144a shows the diagram of maximum transverse bending moments in the section $z=0$. Figure 144b shows the bending deformations of the cross section corresponding to these moments.

Example 2. Let a rectangular box-like beam be loaded at the ends by longitudinal forces distributed over the section according to the σ-diagram shown in Figure 145a. These forces form, for the given section of the beam, a balanced system, i.e. a system for which the resultant and the moments with respect to an arbitrary axis are equal to zero.

According to Saint-Venant's principle, such a load does not cause stresses* in sections sufficiently far from its point of application.

* More precisely: causes very small stresses.

This principle does not hold in the case of the thin-walled beam examined here. It is easy to see that by designing the beam for the balanced longitudinal load applied at the ends. Placing the origin of the coordinates in the middle of the cross section and determining (from the equilibrium conditions of an infinitely small element of the beam) the tangential stresses τ we find

$$\sigma(z) = C_2 \operatorname{ch} kz \cos kz + C_4 \operatorname{sh} kz \sin kz,$$
$$\tau(z) = A\,[C_2 (\operatorname{sh} kz \cos kz - \operatorname{ch} kz \sin kz) + $$
$$+ C_4 (\operatorname{ch} kz \sin kz + \operatorname{sh} kz \cos kz)],$$

where A is a proportionality coefficient.

The arbitrary constants are determined from the boundary conditions

$$\text{for} \quad z = \frac{l}{2} \quad \sigma = 1, \quad \tau = 0.$$

Figure 145b shows the graph of the stresses calculated for a length $l = 10$ m. The stresses σ in the middle section (farthest from the ends of the beam) reach very large values. At each point these stresses are 81% of those at the ends. An increase in the length l of the beam reduces the stresses σ in the middle. For very long beams the stresses σ due to a balanced longitudinal load practically vanish in the middle section. In this case the stresses have a loacl character and, consequently, accord with Saint-Venant's principle.

In general, Saint-Venant's principle is not applicable to thin-walled beams and shells (this was already noted before).

§ 4. Design of a beam-shell of rigid rectangular section, allowing for shear deformations

We shall examine the problem of torsion of a thin-walled beam (or shell) with a rigid closed rectangular contour. The general design method for such a beam, subjected to restrained torsion (taking into account shear deformation), appears as a particular case of the general variational method we developed. Taking the contour of the beam to be nondeformable, we may obviously assume

$$J_1 = J_2 \to \infty.$$

This is equivalent to $c \to \infty$ (see equation (2.5)).

Since $x = 0$, only two of the three kinematical factors (2.13) remain; these are U and θ, which will be determined now from equations

$$U = f', \quad \theta = -\frac{1}{b_2}(b_1 f - a f''). \tag{4.1}$$

It is apparent from equations (2.16) that for $c \to \infty$, $r^2 \to \infty$ and $s^4 \to \infty$ but the ratio $\frac{s^4}{2r^2}$ remains finite. Therefore, by dividing the final equation (2.14) by $(-2r^2)$ and passing to the limit for $c \to \infty$ we obtain a final equation for the given particular case. This equation will have the form

$$f^{IV} - \frac{s^4}{2r^2} f'' = 0,$$

or

$$f^{IV}-\frac{k^2}{l^2}f''=0, \tag{4.2}$$

where k^2 denotes the dimensionless elastic characteristic defined here as

$$k^2=\frac{s^4}{2r^3}l^2=\frac{48G}{E}\frac{\delta_1\delta_2 l^2}{(d_1\delta_2+d_2\delta_1)(F_1+F_2+6\Delta F)}.$$

Equation (4.2) is absolutely identical with equation (II.3.1) for the torsion of a beam of open nondeformable section. They differ only in the elastic characteristics k^2 which have different values. The general integral of equation (4.2) will have the same form as the integral for the torsion angle θ (II.2.4) in the homogeneous equation (II.3.1), i. e.,

$$\text{for } \overline{\theta}(z)=0:\ f=C_1+C_2 z+C_3 \operatorname{sh}\frac{k}{l}z+C_4 \operatorname{ch}\frac{k}{l}z. \tag{4.3}$$

Since the final equation in the given particular case is of order four, we have yet to find, beside the integrals of the two kinematical factors U and θ, those for the two statical factors, in order to obtain the matrix of the initial parameters. Of the three fundamental statical quantities B, H and Q appearing in the general case we choose the pair B and H, (for $z=0$, $Q=0$, as follows from the general equations (2.9)). In our case (for $c\to\infty$) these statical factors will be calculated from the equations

$$\left.\begin{aligned} B&=-af'',\\ H&=-\frac{1}{b_2}[(b_1^2-b_2^2)f'-ab_2f''']. \end{aligned}\right\} \tag{4.4}$$

Introducing in equations (4.1) and (4.4) the value of the general integral f according to (4.3), we obtain the expressions for the four fundamental design quantities U, θ, B and H as functions of the four arbitrary integration constants C_1, C_2, C_3 and C_4. Replacing the arbitrary constants C_1, C_2, C_3 and C_4 by the initial parameters U_0, θ_0, B_0 and H_0 in the same way in § 3 of Chapter II for a thin-walled beam of open section, we obtain for the given particular case, i. e. that of torsion of a thin-walled beam with a rigid rectangular section, the matrix of the initial parameters (Table 31).

Table 31

	θ_0	U_0	B_0	H_0
$\theta(z)$	1	$-\frac{b_2}{b_1}\frac{l}{k}\operatorname{sh}\frac{k}{l}z$	$-\frac{b_2}{b_1}\frac{l^2}{ak^2}\left(1-\operatorname{ch}\frac{k}{l}z\right)$	$\frac{l^3}{ak^3}\left(\frac{k}{l}z-\frac{b_2^2}{b_1^2}\operatorname{sh}\frac{k}{l}z\right)$
$U(z)$	0	$\operatorname{ch}\frac{k}{l}z$	$-\frac{l}{ak}\operatorname{sh}\frac{k}{l}z$	$-\frac{b_2}{b_1}\frac{l^2}{ak^2}\left(1-\operatorname{ch}\frac{k}{l}z\right)$
$B(z)$	0	$-\frac{ak}{l}\operatorname{sh}\frac{k}{l}z$	$\operatorname{ch}\frac{k}{l}z$	$-\frac{b_2}{b_1}\frac{l}{k}\operatorname{sh}\frac{k}{l}z$
$H(z)$	0	0	0	1

It should be noted that the coefficient b_2 can be positive, negative or zero, depending on the dimensions of the contour. Therefore, the influence coefficients of the matrix in Table 31 that contain the factor b_2 can be positive or negative.

We examine in more detail the case of $b_2 = 0$. As seen from (2.5), for $b_2 = 0$ $d_1^2F_2 = d_2^2F_1$, which corresponds to a square contour for equal thickness of the plates of the beam. The system of the fundamental differential equations (2.12) assumes, for $x = 0$ and $b_2 = 0$, the following form

$$\left.\begin{aligned} aU'' - b_1 U = 0, \\ b_1\theta'' = 0. \end{aligned}\right\} \tag{4.5}$$

In our problem the last equation of (2.12) does not appear, since it expresses the work done by all the external and internal forces of the elementary transverse strip in the admissible displacements $\psi_2(s)$, corresponding to the deformation of the beam contour. This work vanishes for $c \to \infty$

Both the equations (4.5), like the system (2.12), are given for the case of no external distributed load. The longitudinal and transverse loads are given by various of concentrated force factors which are applied at the various sections of the beam.

Equations (4.5) can be solved independently since only a longitudinal displacement U, corresponding to warping, enters in the first equation and only the angle of rotation $\theta(z)$ of the cross section enters in the second one. Since the system of differential equations is resolved, for $b_2 = 0$, into several independent equations, the following important conclusion follows: *the state of warping $U(z)$ and the state of torsion $\theta(z)$ do not influence one another in the case of a beam of rigid contour when $b_2 = 0$.* The longitudinal bimoment load does not twist the beam in the same way as the torsional moments applied to the beam do not cause warping $\varphi(s)$ of its cross sections.

It follows from the second equation (4.5) that in the absence of a torsional moment distribution along it, the beam is in a condition of pure Saint-Venant torsion, since the torsion angle $\theta(z)$ varies linearly with the coordinate z. It does not follow, however, that the cross sections of such a beam cannot warp. A state of warping $\varphi(s)$, described by the first equation of the system (4.5), can set in due to longitudinal bimoment loads, but it will not depend on the torsion angle $\theta(z)$.

The matrix of Table 31 is entirely analogous to the matrix of Table 3 of Chapter II. This results from the analogy, noted above, between equations (II.4.2) and (II.3.1).

It follows from this analogy that the design methods for the torsion of thin-walled beams of open section, as developed in the previous sections, can be also applied to the torsion of thin-walled beams of closed nondeformable contour examined here.

§ 5. Space structures with rigid contours having a single symmetry axis

1. The design problems of thin-walled space structures with a rigid nondeformable contour, whether open or closed, and possessing a single symmetry axis in their cross section, are of great practical interest. Such problems occur, for example, in the design of a ship's hull, or a multi-cell symmetrical caisson, of an airplane wing, etc. The general variational method, described in § 4, can be successfully applied here.

In solving such a problem, we limit ourselves to examining the joint action of bending out of the plane of symmetry and of restrained torsion, and leave out the more elementary problem of longitudinal extension and bending in the plane of symmetry. We shall assume that the displacements $u(z, s)$ and $v(z, s)$ are given in the form of a four-term series as in equation (2.1), and the functions $\varphi_i(s)$ and $\psi_k(s)$ are determined by equations (2.2) and (2.3). On examining only the joint action of bending out of the plane of symmetry and of restrained torsion, we shall then have for the displacements the two term expressions

$$\left.\begin{aligned} u(z, s) &= U_1(z)\varphi_1(s) + U_2(z)\varphi_2(s), \\ v(z, s) &= V_1(z)\psi_1(s) + V_2(z)\psi_2(s), \end{aligned}\right\} \tag{5.1}$$

where the functions $\varphi_i(s)$ and $\psi_k(s)$ denote

$$\left.\begin{aligned} &\varphi_1(s) = x(s), & &\varphi_2(s) = x(s)y(s), \\ &\psi_1(s) = x'(s) = \cos\alpha(s), & &\psi_2(s) = h(s), \end{aligned}\right\} \tag{5.2}$$

and $\alpha(s)$ is the angle between the tangent at the given point and the axis Ox; $h(s)$ is the perpendicular drawn from the torsion center to the tangent.

In equations (5.1) the required generalized displacements have the following physical meaning:

$U_1(z)$ is the angle of rotation of the section $z =$ const with respect to the axis Oy;

$U_2(z)$ is the generalized warping of the section $z =$ const;

$V_1(z)$ is the translational displacement of the section $z =$ const in the direction of the axis Ox;

$V_2(z) = \theta(z)$ is the angle of rotation the section $z =$ const as a rigid body about the axis Oz.

The coefficients of the system of differential equations (1.10), calculated by equations (1.11) for the given values of the functions φ_i and ψ_k (5.2), will have the following form

$$a_{11} = \int_F \varphi_1^2(s)\,dF = \int_F x^2\,dF,$$

$$a_{12} = a_{21} = \int_F \varphi_1\varphi_2\,dF = \int_F x^2y\,dF,$$

$$a_{22} = \int_F \varphi_2^2\,dF = \int_F x^2y^2\,dF;$$

$$b_{11} = \int_F (\varphi_1')^2\,dF = \int_F (x')^2\,dF,$$

$$b_{12} = b_{21} = \int_F \varphi_1'\varphi_2'\,dF = \int_F x'(x'y + xy')\,dF,$$

$$b_{22}=\int_F (\varphi_2')^2\,dF=\int_F (x'y+xy')^2\,dF;$$

$$c_{11}=b_{11}=\int_F \varphi_1'\psi_1\,dF=\int_F (x')^2\,dF,$$

$$c_{12}=\int_F \varphi_1'\psi_2\,dF=\int_F x'h\,dF,$$

$$c_{21}=b_{12}=\int_F \varphi_2'\psi_1\,dF=\int_F (x'y+xy')\,x'\,dF,$$

$$c_{22}=\int_F \varphi_2'\psi_2\,dF=\int_F (x'y+xy')\,h\,dF;$$

$$\bar{c}_{11}=b_{11}=\int_F \psi_1\varphi_1'\,dF=\int_F (x')^2\,dF,$$

$$\bar{c}_{12}=b_{12}=\int_F \psi_1\varphi_2'\,dF=\int_F x'\,(x'y+xy')\,dF,$$

$$\bar{c}_{21}=c_{12}=\int_F \psi_2\varphi_1'\,dF=\int_F hx'\,dF,$$

$$\bar{c}_{22}=c_{22}=\int_F \psi_2\varphi_2'\,dF=\int_F h\,(x'y+xy')\,dF;$$

$$r_{11}=b_{11}=\int_F \psi_1^2\,dF=\int_F (x')^2\,dF,$$

$$r_{12}=r_{21}=c_{12}=\int_F \psi_1\psi_2\,dF=\int_F x'h\,dF,$$

$$r_{22}=\int_F \psi_2^2\,dF=\int_F h^2\,dF.$$

Of the 20 coefficients clearly no more than 9 are distinct: a_{11}, a_{12}, a_{22}, b_{11}, b_{12}, b_{22}, c_{12}, c_{22}, r_{22}. The rest are repetitions.

One of these coefficients can be equated to zero, as we can see by the following considerations.

The angle $\theta(z)$ is the angle of rotation of the whole section about the axis Oz passing through an arbitrarily chosen point of the symmetry axis Oy of the beam section. We first choose the centroid of the section as its torsion center and denote by $h(s)$ the length of the perpendicular drawn from the centroid to the contour line. If we now choose an arbitrary point a_y on the axis Oy as the torsion center of the section, the length of the perpendicular $h_1(s)$ from this point to the profile line is expressed by $h(s)$ as follows

$$h_1(s)=h(s)+a_yx'.$$

On orthogonalizing the functions $h_1(s)$ and $x'(s)$, the coefficient c_{12} vanishes,

$$c_{12}=\int_F x'\,(h+a_yx')\,dF=0.$$

Accordingly we find the coordinate of the required torsion center,

$$a_y=-\frac{\int_F x'h\,dF}{\int_F (x')^2\,dF}$$

After these remarks we can now present the system of differential equations in the form of Table 32.

Table 32

$U_1(z)$	$U_2(z)$	$V_1(z)$	$V_2(z)=\theta(z)$	Load	Right-hand side
$Ea_{11}D^2-Gb_{11}$	$Ea_{12}D^2-Gb_{12}$	$-Gb_{11}D$	—	p_1	0
$Ea_{12}D^2-Gb_{12}$	$Ea_{22}D^2-Gb_{22}$	$-Gb_{12}D$	$-Gc_{22}D$	p_2	0
$Gb_{11}D$	$Gb_{12}D$	$Gb_{11}D^2$	—	q_1	0
—	$Gc_{22}D$	—	$Gr_{22}D^2$	q_2	0

The symbol D denotes, as before, differentiation with respect to the variable z of the function at the head of the table. $(h+a_y x')$ should now be taken instead of h to calculate the coefficients c_{22} and r_{22}.

Indicating by primes the derivatives with respect to z, we find from the last two equations of the system (Table 32),

$$\left.\begin{aligned} U_2' &= -\frac{r_{22}}{c_{22}}\theta'' - \frac{1}{Gc_{22}}q_2, \\ U_1' &= -V_1'' + \frac{b_{12}r_{22}}{b_{11}c_{22}}\theta'' + \frac{b_{12}}{Gb_{11}c_{22}}q_2 - \frac{1}{Gb_{11}}q_1. \end{aligned}\right\} \tag{5.3}$$

Differentiating the first equation of the system (Table 32) with respect to z, adding it to the third equation and substituting in the result, instead of U_1' and U_2', their expressions from equations (5.3), we obtain

$$-Ea_{11}V_1^{IV} + E\frac{a_{11}b_{12}-a_{12}b_{11}}{b_{11}}\cdot\frac{r_{22}}{c_{22}}\theta^{IV} + \bar{q}_1 = 0, \tag{5.4}$$

where

$$\bar{q}_1 = \frac{E}{G}\left(\frac{a_{11}b_{12}}{b_{11}c_{22}} - \frac{a_{12}}{c_{22}}\right)q_2'' - \frac{E}{G}\frac{a_{11}}{b_{11}}q_1'' + p_1' + q_1.$$

The second equation of the system (Table 32) furnishes, after differentiating once with respect to z and using relations (5.3)

$$-Ea_{12}V_1^{IV} + E\frac{a_{12}b_{12}-a_{22}b_{11}}{b_{11}}\cdot\frac{r_{22}}{c_{22}}\theta^{IV} + \\ + G\frac{(b_{11}b_{22}-b_{12}^2)\,r_{22}-b_{11}c_{22}^2}{b_{11}c_{22}}\theta'' + \bar{q}_2 = 0, \tag{5.5}$$

where

$$\bar{q}_2 = \frac{E}{G}\left(\frac{a_{12}b_{12}}{b_{11}c_{22}} - \frac{a_{22}}{c_{22}}\right)q_2'' - \frac{E}{G}\cdot\frac{a_{12}}{b_{11}}q_1'' + \left(\frac{b_{22}}{c_{22}} - \frac{b_{12}^2}{b_{11}c_{22}}\right)q_2 + \frac{b_{12}}{b_{11}}q_1 + p_2'.$$

Multiplying equation (5.4) by a_{12}, equation (5.5) by $(-a_{11})$ and adding

them, we obtain

$$E(a_{11}a_{22}-a_{12}^2)\frac{r_{22}}{c_{22}}\theta^{IV}-G\left[(b_{11}b_{22}-b_{12}^2)\frac{a_{11}r_{22}}{b_{11}c_{22}}-a_{11}c_{22}\right]\theta''+$$
$$+a_{12}\bar{q}_1-a_{11}\bar{q}_2=0. \quad (5.6)$$

Thus, we can replace the system of four equations of Table 32 by another system of four equations (5.3), (5.5) and (5.6), of which equation (5.6) is identical in form with the investigated equation of restrained torsion in thin-walled open sections. Consequently, all the solutions we have found before can be applied here. After determining the torsion angle θ and introducing it in equation (5.4) we find for the deflection V_1 the well-known bending equation. Inserting then the found values of V_1 and θ in (5.3), we obtain first-order differential equations from which the remaining functions U_1 and U_2 are readily determined.

§ 6. Experimental verification

1. The theoretical studies made on the basis of the method developed by the author have shown that the longitudinal normal stresses in thin-walled beams and shells with closed variable section contours which which are caused by a balanced longitudinal load or bimoment, do not have the character of local stresses and fall-off very slowly with the distance from the point of application of this load. The rate of fall-off depends on the ratio of the wall thickness δ to the length d of the closed contour, and on the contour rigidity. By increasing the ratio $\frac{\delta}{d}$ we increase the rate of fall-off of the stresses, though not sufficiently fast to be able to consider these stresses as local. Similarly, in the bending of the elements of a transverse closed frame, the rate of fall-off of the longitudinal normal stresses σ due to a longitudinal bimoment load increases when the average linear rigidities (EJ_1 and EJ_2 in the case of a rectangular contour) are increased. Only in the limiting case, i. e. in the case of a shell with a rigid closed contour, can the complementary stresses associated with the departure from the law of plane sections be regarded as local. This fact has great importance in many practical cases.

This problem, connected with Saint-Venant's principle, can be easily studied by varying in equations (2.12) the ratio $\frac{\delta}{d}$ as well as the quantities J_1 and J_2 which represent, for a ribbed shell, the reduced average moments of inertia (per unit length) of the longitudinal sections of the shell. It was shown earlier (§ 8 Chapter II) that in the case of thin-walled beams (as in that of shells) of open section the stresses due to a bimoment extend, even without any deformation of the cross section, over a considerable length of the beam and do not have a local character. Experimental studies were conducted in 1952 in the Department of Construction Engineering of the Institute of Mechanics of the Academy of Sciences of the USSR with the purpose of verifying the theory and the design method for beams and shells of closed sectional contour*.

* The experiments were conducted at the Institute of mechanics of the Academy of Sciences by candidate of technical sciences N.D. Levitskaya

Two closed prismatical shells with square cross sections of identical dimensions and length $l = 1$ m, were tested by a bimoment load applied at one end (Figure 146a).

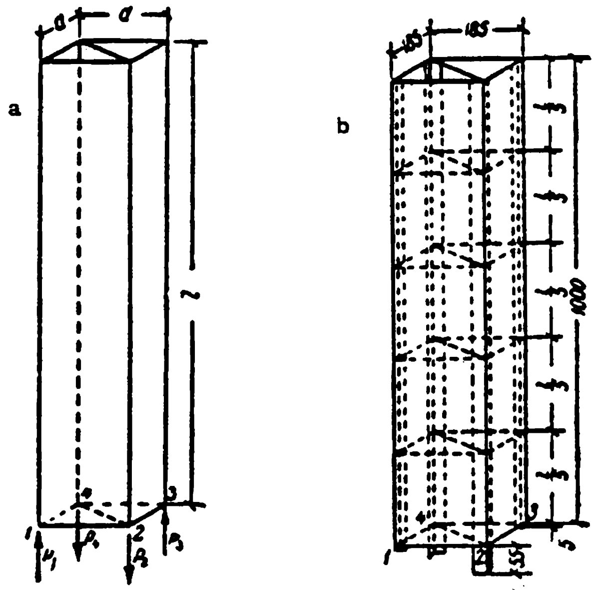
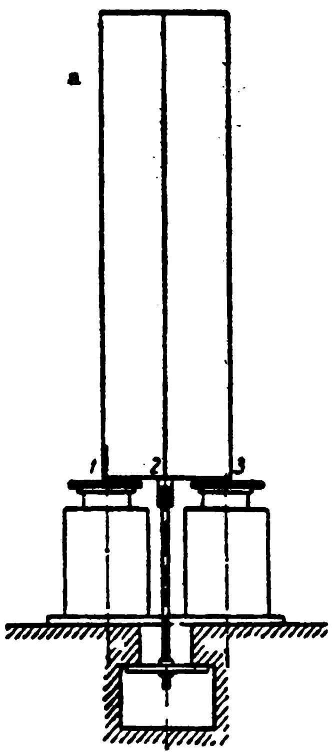


Figure 146

The shells were made of steel sheets of $\delta = 1$ mm thickness and the ribs reinforced by longitudinal stringers made of No 2.5 (25×25×4) angles riveted to the shell (ø 2 mm, step 20 mm, two row joints). Each end face was reinforced diagonally in order to secure rigidity. One of the shells was reinforced by diagonals welded (cross to cross) to the stringers inside the shell at four sections: $z = \frac{l}{5}, \frac{2l}{5}, \frac{3l}{5}$ and $\frac{4l}{5}$ in order to be able to assume the contour sufficiently rigid (practically nondeformable in the plane of the cross section). At one end of the shell the stringers were machined flush with the end face. Two stringers (1 and 3, Figure 146b) were made to protrude 5 mm from the end face and two others (2 and 4) 55 mm, so that the bimoment load could be applied.

The [Young's] modulus of elasticity of the shell material was $E = 1.95 \cdot 10^6$ kg/cm^2, as determined in the laboratory of testing materials at TsNIPS. The tested samples had been cut from the same sheet of metal the shell was made of.

The loading of the test specimen was effected at one end face with the help of two welded hydraulic jacks. The shell was placed in a vertical position over the slot in the test floor in such a way that the end diagonals were in the same vertical plane with the slot. The latter was covered with a massive plate on which the hydraulic jacks were placed one opposite the other on different sides of the slot. The jack pistons were also covered by a common plate. The test piece was placed on it in such a way that the ends of the stringers 1–3 (Figure 147a) which protruded from its lower end were above the centers of the jack pistons. Traction anchors, firmly fixed



In the slot of the test floor, were attached to the two other ends of the stringers (2 and 4) which protruded from the end face by 55 mm.

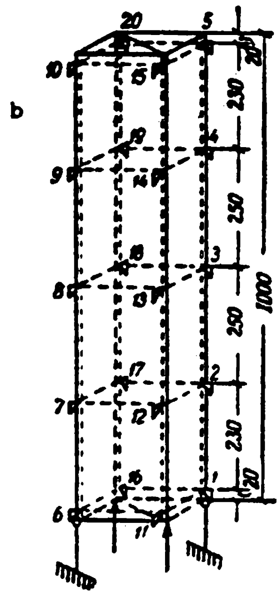


Figure 147

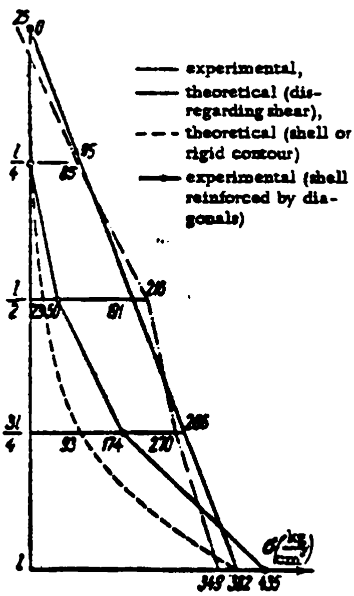


Figure 148

The longitudinal deformations were measured with a mechanical strain gauge having a 2 cm base placed on the contours of five sections ($z = 0, \frac{l}{4}, \frac{l}{2}, \frac{3l}{4}$ and l). Four instruments (the number of edges) were placed at each section. Altogether 20 strain gauges were attached to the structure (Figure 147b). At the sections $z = 0$ and $z = l$ the strain gauges were placed at a distance of 2 cm from the ends. The shells were tested under a bimoment load according to the diagram shown in Figure 146a, with $P = 1{,}000$ kg. The load was increasing in steps of 100 kg up to 1,000 kg and then decreased to zero. Altogether six cycles of loading and unloading were performed. The results of the experiments (the average of six measurements) are plotted, showing the distribution of the longitudinal normal stresses in kg/cm^2 along the shell (Figure 148) and over its cross section (Figure 149).

On these diagrams the theoretical distributions of the stresses along the beam are shown. The design of the shell with a deformable contour (neglecting shear deformations) is carried out with the help of the matrix of Table 30.

The boundary conditions in this case have the form:

$$\text{for } z=0 \qquad x_0 = B_0 = 0,$$
$$\text{for } z=l \qquad x(l)=0, \qquad B(l)=Pd^2.$$

The generalized rigidities and the characteristics have the values

$$a=\frac{1}{12}Ed^4(F+3\Delta F)=149\,400\cdot 10^6\,\text{kg}\cdot\text{cm}^4,$$
$$c=48\frac{EJ}{d}=421.6\,\text{kg},$$
$$s^4=\frac{c}{a}=0.2822\cdot 10^{-8}\,\text{cm}^{-4},$$
$$k=\sqrt{\frac{s^2}{2}}=0.00515\,\text{cm}^{-1}.$$

The initial parameters, determined from the boundary conditions at the end $z=l$, have the values

$$U_0=3.801\cdot 10^{-8}P,$$
$$Q_0=-3.406\,P.$$

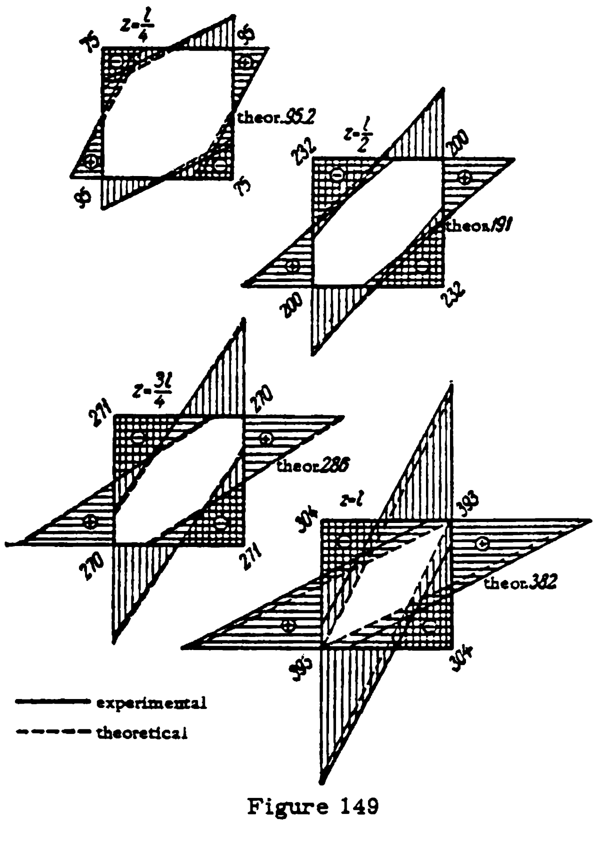


Figure 149

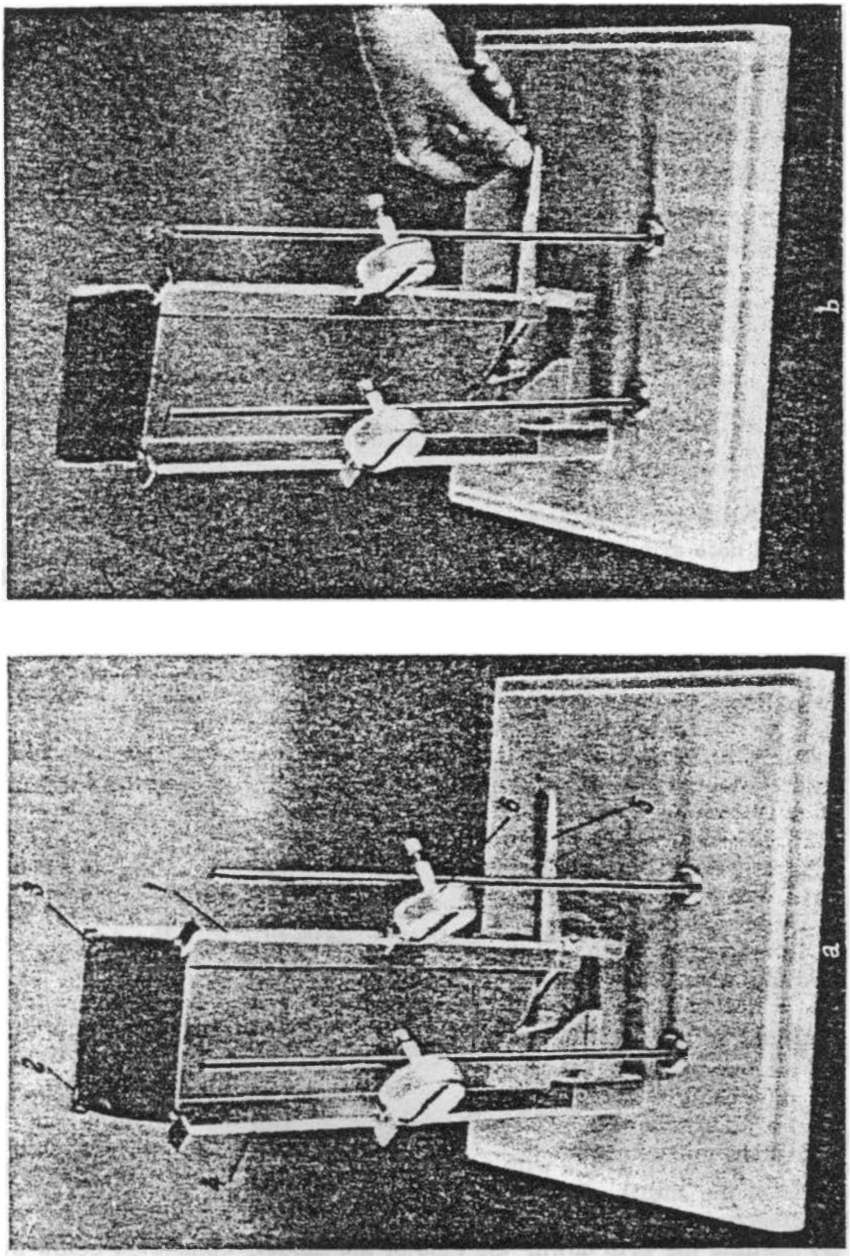


Figure 150

The graph (Figure 148) shows the longitudinal normal stresses $\sigma = -\frac{E}{a}\varphi B$ proportional to the bimoment $B(z)$ (see (2.10) and (2.11) and computed for the sections where the gauges were placed.

The design of a shell with a rigid, nondeformable contour of the cross section was carried out with the help of the matrix of Table 31 in which b_3 must equal 0 for a square contour. The boundary conditions have in this case the form

$$\begin{aligned}&\text{for } z=0 \quad B_0 = H_0 = 0,\\ &\text{for } z=l \quad \theta(l)=0 \ \text{(that is, } \theta_0 = 0), \ B(l) = Pd^2.\end{aligned}$$

The rigidities and the characteristic have the values

$$a = \frac{1}{12}Ed^4(F + 3\Delta F) = 149\,400\cdot 10^6\,\text{kg}\cdot\text{cm}^4,$$

$$b_1 = Gd^2F = 474\,900\cdot 10^3\,\text{kg}\cdot\text{cm}^2,$$

$$k = \sqrt{\frac{b_1}{a}} = 5.637\cdot 10^{-2}\,\text{cm}^{-1}.$$

The initial parameter U_{0}, determined from the boundary condition at the end $z = l$, equals

$$U_0 = 2.896\cdot 10^{-10} P.$$

An analysis of the graphs and a comparison of the theoretical and experimental data allow the following conclusions to be drawn:

1) In thin prismatical shell structures of closed section without additional transverse connections the contour deformation is a decisive factor which influences the state of stress. The shear deformations play a secondary role and can be neglected in the design.

2) The presence of transverse connections, which prevent the contour deformations, favors a more rapid fall-off of the longitudinal stresses with length.

3) Our statement that the Saint-Venant principle has a limited range of validity is obviously confirmed, not only for thin-walled beams of open nondeformable contour but also for thin-walled beams and shells of closed contour, if not reinforced by additional rigid ribs.

2) In contrast with the thin-walled open-contour beam, the section warping in a cylindrical or prismatical shell of closed contour without additional transverse connections is associated as a rule with the deformation of the shell contour in its cross section. The magnitude of this deformation depends to a large extent on the thickness of the shell.

Figure 150 shows a model of a shell consisting of a closed prism (square in plan) with the ratio of length to width $\frac{l}{d} = 4$. The ribs of the shell are reinforced by angles.

The model is placed vertically. Two diagonally opposed ribs (3 and 4) are fixed to the frame at the lower end. Two others (1 and 2) are connected by a transverse brace with a hinged loading arm fixed in the middle (in the axis of the shell). The model is fitted at the upper end with a detachable diagonal rigid plate (not appearing in the figure) (Figure 150a).

By pushing the arm down, the ribs of the model, connected below by a brace, are extended. The ribs connected to the frame are compressed, i. e. a self-balanced system of longitudinal forces (bimoments) is transmitted to the lower end. Upon removing the upper diagonal, this load

causes a deformation of the model contour: The square cross sections of the model are transformed into rhombi; this can be clearly seen (Figure 150b).

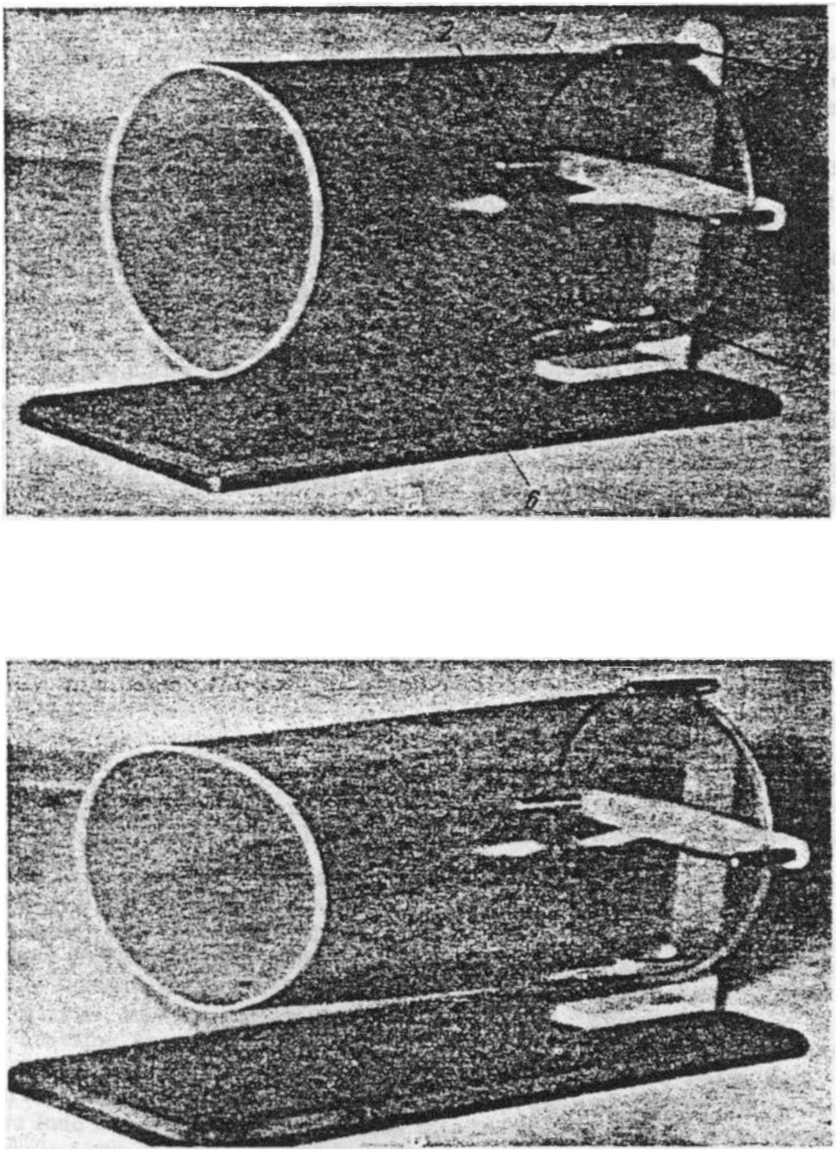


Figure 151

The longitudinal deformation, i. e. the warping (the extension of some diagonal ribs and the compression of others) is carried far from the lower end along the height of the model and is shown by the indicators, 6, which can be placed in any section along the shell.

In a model with a brace at the upper diagonal, i. e. with a more rigid contour, all the described phenomena (contour deformation and warping of the model cross sections) due to a bimoment load fall-off rapidly with increasing height.

3. Figure 151 shows a model consisting of a closed cylindrical shell with a length-diameter ratio of $\frac{l}{d} = 4$ (Figure 151a). One end of the shell is free, at the other the shell has a diaphragm. Four bolts (1, 2, 3, 4) at tips of the vertical and horizontal diameters are rigidly connected to the shell at four points of this end. Two of these bolts, placed in the vertical plane, are connected to a bar 5, which is rigidly connected to the frame 6. The two others, placed in the horizontal plane, are connected to the transverse brace 7 which has in the center a pivot coinciding with the geometrical axis of the shell. The pivot of the brace, which passes freely through the above-mentioned bar, has a thread at its end. By turning the crank mounted on it we impart a translational movement to the transverse brace and at the same time transmit extension forces at two points on the horizontal diameter of the end face of the shell. The two other points, situated on the vertical diameter, will be in compression. Thus, a self-balancing system of longitudinal forces (bimoments) is transmitted to the end face of the shell.

When the model is exposed to a bimoment load, applied at the end, its transverse sections are warped as well as bent (Figure 151b). This bending deformation does not decrease with the distance from the application point of the longitudinal bimoment load, as is the case with Geckeler's boundary effect but increases, reaching in our model its maximum values at the other (free) end of the shell.

The phenomenon described here, which is connected with the warping of the transverse sections and with the bending deformation, has a fundamental importance in the author's general "semi-momentless" [membrane state] theory of cylindrical and prismatical shells of intermediate length.

Chapter V

SPATIAL STABILITY OF THIN-WALLED BEAMS LOADED AT THE ENDS BY LONGITUDINAL FORCES AND MOMENTS

§ 1. Differential equations of beam stability

1. We have shown in the foregoing that the cross sections of thin-walled beams as a rule do not remain plane after deformation. These beams behave as thin-walled space systems undergoing longitudinal deformations, not only because of bending but also because of torsion. The complementary sectorial stresses attain in many cases very large values, a result of the fact that beams with an open nondeformable contour possess a relatively small torsional rigidity. It follows that torsion associated with sectorial stresses in the cross sections can also play an important part in the stability of thin-walled beams.

In the present chapter a general linear theory is developed of the spatial stability and the critical states of beams of thin-walled and solid sections (whether open or closed) subjected at the ends to longitudinal forces and moments. This theory differs from Euler's classical theory of lateral buckling in that, resting on laws which allows for warping of the cross section, it provides a broader perspective for the study of the loss of stability and the critical states of a beam, embracing spatial equilibrium states in flexural torsion. Euler's theory, based on the law of plane sections and considering only equilibrium in pure flexure, is a very special case of this general theory.

The linear theory of an elastic system in its critical state distinguishes two kinds of stability problems, depending on the character of the load.

Stability problems of the first kind include problems of the theory of bifurcation in which other forms of equilibrium, qualitatively different from the fundamental precritical shape, become possible (bifurcation) for an elastic system under critical conditions. Euler's theory of lateral buckling is a classical example of this kind of problem. In this theory the fundamental, precritical equilibrium shape of a beam will be a straight line in the case of central compression. When the compressive load reaches its critical value this form can change into a flexural equilibrium form, characterized by deflections of the beam in the plane of least rigidity. A second instance of a stability problem of the first kind is found in the author's theory of spatial stability of the beam under central compression. This theory, which considers a generalization of Euler's problem, is characterized by the fact that in the critical state of a beam under central compression a spatial flexural-torsional equilibrium shape becomes possible, in addition to the fundamental straight equilibrium

shape and the Euler bending. In many cases, this shape gives a smaller value of the critical thrust than that given by Euler's theory.

Stability problems of the second kind include problems in which the deformation of the system increases with the increase of the external load. This deformation, characteristic of the elastic equilibrium shape at buckling, shows only qualitative and not quantitive variations. Problems of this kind are the compression and bending of a beam in the theory of lateral buckling and certain problems of the general theory presented here of flexural torsion in beams due to eccentric longitudinal compression or tension. The fundamental pre-critical equilibrium shape will also occur in these problems at the moment of buckling when the external load reaches its critical value.

The loss of stability of an elastic system in problems of the first and second kind is characterized, according to the linear theory, by the fact that the deformation of the system in the post-critical state can reach infinite values for arbitrarily small load increments. This deformation, taken as a variation of the possible equilibrium shapes for the given system in its critical state, (with the additional statical condition of no variation of the external load) is described in stability problems of the first and second kind by linear homogeneous differential equations and homogeneous boundary conditions.

This mathematical criterion serves as the basis for the general theory of stability treated in the present chapter, including the critical states of a beam in stability problems of the second kind.

2. We shall examine the stability problem of a weightless beam under a longitudinal terminal compressive force P. In order to obtain the general solution, we shall consider the thrust to be applied at an arbitrary point (e_x, e_y) of the cross section (Figure 152).

We shall either assume that the beam has at the ends rigid plates to restrain motion in their planes and that the longitudinal force is transmitted to the beam through these plates, or that the longitudinal forces pass through the zero sectorial points. In such cases there is no external bimoment load and the supported sections of the beam, plane before the deformation, will remain plane after it. The action of the longitudinal force manifests itself in the transmission of an axial force and bending moments to the beam. The beam, in problems of the first kind, will be under conditions of axial compression or extension and of pure bending before buckling occurs. Normal stresses, determined by the law of plane sections, will appear in the cross sections of the beam.

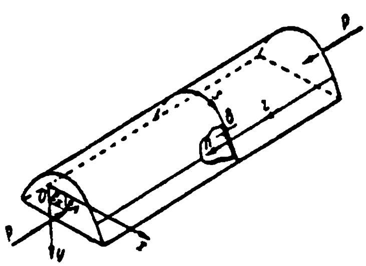


Figure 152

If P is the compressive force and M_x and M_y the bending moments, which for the case of eccentric compression are the product of the force P and the corresponding arms (eccentricities e_x and e_y), we obtain for the stresses n

$$n = -\frac{P}{F} + \frac{M_x}{J_x}y - \frac{M_y}{J_y}x. \quad (1.1)$$

For constant P M_x, M_y these stresses only depend on the position of

the point on the section contour (the arc s). The normal stresses n remain constant along the beam. Tangential stresses τ do not appear in the present case of longitudinal load.

The exact solution in the theory of elasticity of the spatial problem of a beam of cylindrical or prismatical form in case the external load consists only of normal stresses n applied at the end faces of the beam and distributed according the plane law, is essentially given by equation (1.1). The lateral surface of the beam should be taken as unloaded. In such a problem, of the six components of the stress tensor only the longitudinal normal stresses n, linear in the coordinates x, y of the plane of the cross section, would be different from zero.

Under these conditions, all equations of the spatial problem in the linear theory of elasticity, i. e. the continuity [compatibility] differential equations for the strains written in terms of the components of the stress tensor (the Beltrami-Michell equations), and the equations which express the given statical conditions at points of the surface, are identically satisfied. The strained state for small displacements of the beam is characterized, according to the classical linear theory of elasticity, by an axial contraction (under compression) and by the deflections ξ_0 and η_0 in the principal planes. These deflections are connected with the moments M_x, M_y by the relations

$$\xi_0'' = \frac{M_y}{EJ_y}, \quad \eta_0'' = -\frac{M_x}{EJ_x}.$$

If, in determining the bending moments in the pre-critical state, we also take into consideration the moments of the longitudinal force obtained by multiplying this force by the deflections, the relation between the deflections and the load will in this case be rather more complicated, to wit, nonlinear. However, if this load does not exceed its critical value then, in the case of stability of the first kind, the deformation of a beam under an eccentric compression will represent, according to the linear theory, pure bending in the plane of the bending moment.

We can consider the quantity P, entering in equation (1.1), to be a parameter of the external load. When this parameter changes (for a given position of the force P in the cross section), the bending moments M_x, M_y and the state of stress of the beam change as well. If the compression force P does not exceed a certain limit, only the normal stresses n determined by equation (1.1) appear in the beam in the case of stability of the first kind. The basic equilibrium shape in bending, characterized by longitudinal (axial) extensions and deflections in the principal planes, is determined by these stresses and the corresponding extensions. There is no torsional deformation in the precritical state for the described mode of transmission of the load P. For a certain value of the force P the basic equilibrium shape becomes unstable. This state of the beam is called c r i t i c a l. It is characterized by the fact that besides the basic equilibrium shape of the beam another one is also possible. The transition of the beam from one equilibrium state to another is associated, in the general case, with the appearance of additional stresses and deformations. Let $\sigma(z, s)$, $\tau(z, s)$ and $H(z)$ be the additional stresses and the moments which appear in the beam with a change in the basic equilibrium shape and $\xi(z)$, $\eta(z)$ and $\theta(z)$ be the deflections of the axes of the shear center and the

torsion angle determined by the additional displacements of the cross section of the beam in the plane of the section $z = \text{const}$. The additional stresses σ, τ and the moments H should in the sought deformed state of the beam be in equilibrium with the given normal stresses n.

Considering the displacements ξ, η and θ to be very small, we can write the equilibrium conditions for the beam in the deformed state in the form of equations (I.7.3). The free terms of these equations depend on the external transverse load. In our case, this part in this load consists of the forces, continuously distributed over the middle surface, which are the projection of the given internal forces on the fixed axes Ox and Oy in the deformed state. With the transition of the beam from one equilibrium state to another, the elementary strip ds undergoes additional deflections $\xi_s(z)$ and $\eta_s(z)$ in the horizontal and vertical planes. We obtained before for these deflections the expressions (I.3.4),

$$\left.\begin{aligned} \xi_s &= \xi - (y - a_y)\theta, \\ \eta_s &= \eta + (x - a_x)\theta. \end{aligned}\right\} \tag{1.2}$$

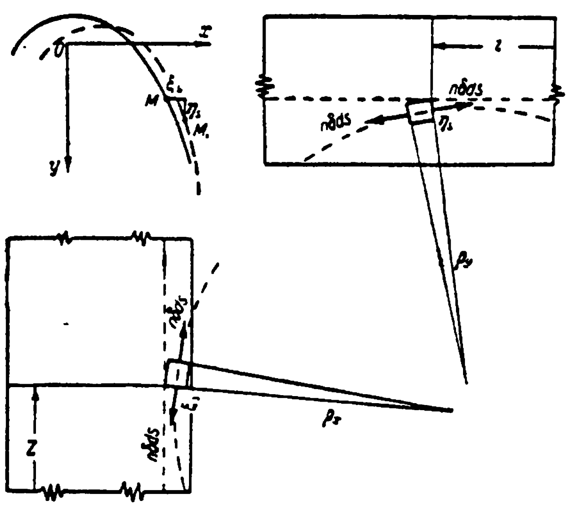


Figure 153

The normal forces $n\delta\, ds$, acting in the cross sections of the isolated strip, are projected, as a result of the additional bending deformations, on the directions of the axes Ox and Oy. Denoting the magnitudes of these projections by p_x and p_y and considering that in the examined case the normal stresses n do not vary along the beam, we obtain from Figure 153

$$\left.\begin{aligned} p_x\, dz\, ds &= n\delta\, ds\, \frac{dz}{\rho_x}, \\ p_y\, dz\, ds &= n\delta\, ds\, \frac{dz}{\rho_y}, \end{aligned}\right\} \tag{1.3}$$

where ρ_x and ρ_y are the radii of curvature of the projections of $\xi_s(z)$ and $\eta_s(z)$ on the coordinate planes Oxz and Oyz of the space curve into which the generator of the middle cylindrical surface is transformed on buckling.

Replacing in equation (1.3) the curvatures $\frac{1}{\rho_x}$ and $\frac{1}{\rho_y}$ by their approximate expressions according to the equations

$$\frac{1}{\rho_x} = \xi_s'', \quad \frac{1}{\rho_y} = \eta_s''$$

and dropping the factor $dz\,ds$, we obtain

$$p_x = n\delta\xi_s'', \quad p_y = n\delta\eta_s''.$$

Introducing here ξ_s and η_s from equations (1.2) we find

$$\begin{aligned} p_x &= n\delta\xi'' - n\delta(y - a_y)\theta'', \\ p_y &= n\delta\eta'' + n\delta(x - a_x)\theta''. \end{aligned}$$

These equations were derived under the assumption that the variations of the displacements ξ_s, η_s and θ are small. Their products and squares are taken to be negligible as compared with the first-order quantities.

Knowing the magnitude of the reduced surface loads, we can determine with their help the loads q_x, q_y and the torsional moment m per unit length,

$$\left.\begin{aligned} q_x &= \xi'' \int_L n\delta\,ds - \theta'' \int_L n\delta(y - a_y)\,ds, \\ q_y &= \eta'' \int_L n\delta\,ds + \theta'' \int_L n\delta(x - a_x)\,ds, \\ m &= -\xi'' \int_L n\delta(y - a_y)\,ds + \eta'' \int_L n\delta(x - a_x)\,ds + \\ &\quad + \theta'' \int_L n\delta[(x - a_x)^2 + (y - a_y)^2]\,ds. \end{aligned}\right\} \tag{1.4}$$

Substituting in the right-hand sides of equations (1.4) the value of n from equation (1.1) and keeping in mind that $\delta ds = dF$ and that, in the principal axes, $\int_F x\,dF = \int_F y\,dF = \int_F xy\,dF = 0$, we obtain by integration

$$\left.\begin{aligned} q_x &= -P\xi'' - (a_yP + M_x)\theta'', \\ q_y &= -P\eta'' + (a_xP - M_y)\theta'', \\ m &= -(a_yP + M_x)\xi'' + (a_xP - M_y)\eta'' + \\ &\quad + (-r^2P + 2\beta_yM_x - 2\beta_xM_y)\theta''. \end{aligned}\right\} \tag{1.5}$$

In these equations a_x and a_y are the coordinates of the shear center; r, β_x and β_y are geometrical characteristics with the dimension of length and calculated, in the most general case, by the equations

$$r^2 = \frac{J_x + J_y}{F} + a_x^2 + a_y^2, \quad \beta_x = \frac{U_y}{2J_y} - a_x, \quad \beta_y = \frac{U_x}{2J_x} - a_y, \tag{1.6}$$

where J_x and J_y are the moments of inertia, F is the area of the section, and U_x and U_y new geometrical characteristics

$$U_x = \int_F y\rho^2\,dF, \quad U_y = \int_F x\rho^2\,dF, \tag{1.7}$$

ρ is the distance from the centroid of the section to a point on the contour with the coordinate s,

$$\rho^2 = x^2 + y^2. \tag{1.8}$$

By virtue of (1.8), equations (1.7) can be written in the following form

$$\left.\begin{aligned} U_x &= \int_F y^3 dF + \int_F x^2 y dF, \\ U_y &= \int_F x^3 dF + \int_F y^2 x dF. \end{aligned}\right\} \quad (1.9)$$

The terms on the right-hand side of (1.9) give the third-order axial moments and products of inertia of the cross section area.

These moments can be calculated analytically, but also graphically, starting from the theory of the string polygon and considering the separate factors of the integrands as elementary fictitious forces. It should be remembered that the functions $x = x(s)$ and $y = y(s)$ in the integrands are the coordinates of a contour point s in the principal axes system.

Thus, if we are given the cross section of a thin-walled beam we can express by equations (1.5) the components of the reduced loads per unit length q_x, q_y and m. Introducing these components in equation (I.7.3) we obtain

$$\left.\begin{aligned} EJ_y\xi^{IV} + P\xi'' + (M_x + a_yP)\theta'' &= 0, \\ EJ_x\eta^{IV} + P\eta'' + (M_y - a_xP)\theta'' &= 0, \\ (M_x + a_yP)\xi'' + (M_y - a_xP)\eta'' + & \\ + EJ_\omega\theta^{IV} + (r^2P + 2\beta_xM_y - 2\beta_yM_x - GJ_d)\theta'' &= 0. \end{aligned}\right\} \quad (1.10)$$

Equations (1.10) form a system of linear differential equations. In these equations the required functions are the displacements $\xi = \xi(z)$, $\eta = \eta(z)$ and the torsion angle $\theta = \theta(z)$ associated with buckling.

The coefficients of the differential equations (1.10) depend not only on the geometrical and elastic characteristics of the beam but also on the quantities P, M_x and M_y which characterize the external load under eccentric compression (its magnitude and point of application in the cross section). Since in designing a beam for stability P, M_x and M_y are given to within a single parameter (for example, the value of the force P), the coefficients of equations (1.10) will be also determined to within this parameter.

3. The differential equations (1.10) are linear, homogeneous and have constant coefficients. Their integration in the most general case can be performed without undue difficulty. Setting, according to the general theory,

$$\xi = Ae^{kz}, \quad \eta = Be^{kz}, \quad \theta = Ce^{kz},$$

where A, B and C are arbitrary constant quantities, substituting these values in the differential equations (1.10) and reducing by the common factor e^{kz}, we obtain

$$\left.\begin{aligned} (EJ_yk^4 + Pk^2)A + (M_x + a_yP)k^2C &= 0, \\ (EJ_xk^4 + Pk^2)B + (M_y - a_xP)k^2C &= 0, \\ (M_x + a_yP)k^2A + (M_y - a_xP)k^2B + & \\ + [EJ_\omega k^4 + (r^2P + 2\beta_xM_y - 2\beta_yM_x - GJ_d)k^2]C &= 0. \end{aligned}\right\} \quad (1.11)$$

Since the required solution for A, B, C must not vanish, the determinant of the system of homogeneous equations (1.11) must equal zero. This

condition yields the characteristic equation for k which can be written in the form

$$\Delta = k^6 \begin{vmatrix} EJ_y k^2 + P & 0 & M_x + a_y P \\ 0 & EJ_x k^2 + P & M_y - a_x P \\ M_x + a_y P & M_y - a_x P & EJ_\omega k^2 + (r^2 P + \\ & & + 2\beta_x M_y - 2\beta_y M_x - GJ_d) \end{vmatrix} = 0. \quad (1.12)$$

In addition to the six zero roots, equation (1.12) gives for the other six roots nonvanishing values which depend on the parameter of the external load. The general solution of the system of differential equations (1.10) will thus consist of twelve particular solutions. Each of these solutions is determined up to an arbitrary constant factor. It is easy to show that the number of particular solutions of the system (1.10) fully agrees with the statical and kinematical conditions at the ends of a beam under torsion and bending in two planes. Integrating each of equations (1.10) twice with respect to z and expressing, in agreement with the statical meaning of the indicated quadratures, the arbitrary constants by the transverse forces, moments, and bimoments at the initial supported section $z = 0$, we obtain

$$\left.\begin{aligned} EJ_y\xi'' + P\xi + (M_x + a_y P)\,\theta = \bar{M}_y - \bar{Q}_x z, \\ EJ_x\eta'' + P\eta + (M_y - a_x P)\,\theta = -\bar{M}_x - \bar{Q}_y z, \\ (M_x + a_y P)\,\xi + (M_y - a_x P)\,\eta + \\ + EJ_\omega\theta'' + (r^2 P + 2\beta_x M_y - 2\beta_y M_x - GJ_d)\,\theta = -\bar{B} - \bar{H}z. \end{aligned}\right\} \quad (1.13)$$

The statical quantities on the right-hand side of the equations, which play the role of the initial parameters, are components of the external load applied at the section $z = 0$; $\bar{Q}_x$, $\bar{Q}_y$ are the transverse forces, $\bar{H}$ the torsional moment, $\bar{M}_x$, $\bar{M}_y$ the bending moments and $\bar{B}$ the bimoment. Equations (1.13) express all the necessary conditions of elastic equilibrium of an initially compressed and bent cantilever subjected to the action of a load at the free end. The required functions in equations (1.13) are the deflections $\xi = \xi(z)$, $\eta = \eta(z)$ and the torsion angle $\theta = \theta(z)$, which appear in the initially compressed and bent beam due to an external additional load applied at the free end. If the given load, having the components P, M_x, M_y does not exceed a critical value, the homogeneous equations (1.13) for the additional boundary conditions with respect to the deflections ξ, η and the torsion angle θ will have a determinate unique solution. These equations (1.13) form, with respect to the functions ξ, η, θ a complete system of linear inhomogeneous second-order equations with a symmetric matrix and with constant coefficients which depend on a number of statical, physical and geometrical characteristics with respect to the initially compressed and bent elastic beam. The total integrals of the system (1.13) can be expressed as the sum of the integrals of the homogeneous equations and of any particular solutions of the inhomogeneous system (1.13). Since the right-hand members of equations (1.13) are linear functions of z, the particular integrals can be also written in the form of linear functions of z. Denoting these particular integrals by $\bar{\xi}$, $\bar{\eta}$, $\bar{\theta}$ we obtain for them the following equations

$$\left.\begin{aligned} P\bar{\xi} + (M_x + a_y P)\,\bar{\theta} = \bar{M}_y - \bar{Q}_x z, \\ P\bar{\eta} + (M_y - a_x P)\,\bar{\theta} = -\bar{M}_x - \bar{Q}_y z, \\ (M_x + a_y P)\,\bar{\xi} + (M_y - a_x P)\,\bar{\eta} + \overline{GJ}_d\bar{\theta} = -\bar{B} - Hz, \end{aligned}\right\} \quad (1.14)$$

where we denote by $\overline{GJ}_d$ a sort of reduced rigidity for pure torsion, determined by the equation

$$\overline{GJ}_d = -GJ_d + r^2P + 2\beta_x M_y - 2\beta_y M_x.$$

We find from these equations

$$\left.\begin{aligned}
\bar{\xi} &= \frac{1}{\Delta}\{[P\overline{GJ}_d - (M_y - a_xP)^2](\tilde{M}_y - \tilde{Q}_x z) - \\
&\quad - (M_y - a_xP)(M_x + a_yP)(\tilde{M}_x + \tilde{Q}_y z) + \\
&\qquad + P(M_x + a_yP)(\tilde{B} + \tilde{H}z)\},\\
\bar{\eta} &= \frac{1}{\Delta}\{(M_y - a_xP)(M_x + a_yP)(\tilde{M}_y - \tilde{Q}_x z) - \\
&\quad - [P\overline{GJ}_d - (M_x + a_yP)^2](\tilde{M}_x + \tilde{Q}_y z) + \\
&\qquad + P(M_y - a_xP)(\tilde{B} + \tilde{H}z)\}.\\
\bar{\theta} &= \frac{P}{\Delta}\{-(M_x + a_yP)(\tilde{M}_y - \tilde{Q}_x z) + \\
&\quad + (M_y - a_xP)(\tilde{M}_x + \tilde{Q}_y z) - P(\tilde{B} + \tilde{H}z)\},
\end{aligned}\right\} \tag{1.15}$$

where Δ is the determinant of the system (1.14)

$$\Delta = P[P\overline{GJ}_d - (M_x + a_yP)^2 - (M_y - a_xP)^2].$$

We shall represent the general integrals of the inhomogeneous equations (1.13) in trigonometric form

$$\left.\begin{aligned}
\xi &= -\frac{M_x + a_yP}{EJ_y}\sum_{i=1,2,3}\frac{1}{\lambda_y^2 - \lambda_i^2}(A_i \sin\lambda_i z + B_i\cos\lambda_i z) + \bar{\xi},\\
\eta &= -\frac{M_y - a_xP}{EJ_x}\sum_{i=1,2,3}\frac{1}{\lambda_x^2 - \lambda_i^2}(A_i \sin\lambda_i z + B_i\cos\lambda_i z) + \bar{\eta},\\
\theta &= \sum(A_i\sin\lambda_i z + B_i\cos\lambda_i z) + \bar{\theta},
\end{aligned}\right\} \tag{1.16}$$

where A_i, B_i $(i = 1, 2, 3)$ are arbitrary integration constants; λ_x, λ_y are quantities having the dimension^{-1} and defined as

$$\lambda_x^2 = \frac{P}{EJ_x}, \quad \lambda_y^2 = \frac{P}{EJ_y};$$

λ_i^2 are the roots of the characteristic equation

$$\begin{vmatrix}
P - EJ_y\lambda^2 & 0 & M_x + a_yP\\
0 & P - EJ_x\lambda^2 & M_y - a_xP\\
M_x + a_yP & M_y - a_xP & -\overline{GJ}_d - EJ_\omega\lambda^2
\end{vmatrix} = 0. \tag{1.17}$$

Since the determinant of (1.17) is symmetric, all the roots λ_1^2, λ_2^2, λ_3^2 will be real. In the case of a negative root λ_i will have an imaginary value. The trigonometric functions referring to these roots will then become hyperbolic functions. Equations (1.16) and (1.15) represent the general solution of the problem of the equilibrium of an initially stressed beam taking into consideration the flexural-torsional deformations which appear due to a load applied at the initial sections $z = 0$ and which have, in the general case, six components. In order to determine the arbitrary constants, it is necessary to assign the boundary conditions. The number of these conditions should be 12 here. Depending on the form of restraint at the ends

of the beam, these conditions, prescribed for the functions and their derivatives, can be either purely statical, purely geometrical or of a mixed type. Developing the boundary conditions in some concrete problem with the help of the integrals (1.16), we shall always have a complete system of linear homogeneous equations which enable to determine all the constants of integration. For a beam subjected to bending loads these equations will have in the precritical states determinate unique solution. Thus, using the general equations (1.16) we can determine the flexural-torsional deformation of an initially compressed or stretched and bent beam which is subjected to the action of any load and, in particular, to the action of bending moments applied at the end. When the eccentrically applied compression or tension reaches its critical value, the solution for the displacements ξ, η, θ given in stability problems of the first kind will become indefinite, and in stability problems of the second kind will tend to infinity. In both cases, the value of the critical load is found from the transcendental characteristic equation obtained by equating to zero the determinant of the system of homogeneous equations in the given boundary-value problem. We obtain, for the critical load, an infinite number of values, all real. In the general case of flexural torsion to each of these values of the critical load will correspond (up to a constant factor) a separate equilibrium shape. We shall have an infinite number of equilibrium shapes, and their totality will represent for the problem at hand a complete system of eigenfunctions. The critical values of the load are expressed by the eigenvalues of these functions. In practice we are interested in the mode of instability and the corresponding eigenvalue for which the critical load is minimal.

4. The stability equations (1.10) have been derived for thin-walled open sections. These equations will be also valid for beams which are designed taking into account longitudinal bending moments. It is only necessary to remember that in this case, when computing the geometrical characteristics (including the quantities U_x and U_y), the contour integrals will be replaced by double integrals taken over the area of the whole cross section, as equations (II.14.3) and (II,14.4) show. The coordinates of the shear center should be determined from (II.14.5) and the moment of inertia J_d from the theory of pure torsion.

Equations (1.10) will be also valid for closed sections with a rigid invariable contour. Indeed, the differential equations of bending (IV.2.7) and (IV.2.8) for closed sections are completely identical with the corresponding equations of the theory of open sections, since in the design of beams according to the classical theory of lateral buckling the shear deformations are equated to zero. The differential equation (IV.4.2) which refers to the warping of the cross section of the beam also recalls the torsion equation (II.3.1) in the theory of thin-walled beams. The difference consists only in the values of the dimensionless elastic characteristic k determined by equation (II.2.2)

$$k^2 = l^2 \frac{GJ_d}{EJ_\omega}.$$

The bimoment of inertia J_ω and the torsional moment of inertia J_d entering in this characteristic (in the theory of open-contour beams where we use the law of sectorial warping) are calculated from equations

$$J_d = \frac{1}{3} \sum d\delta; \quad J_\omega = \int_F \omega^2 dF.$$

In the theory of beam-shells of closed contour the axial instead of the sectorial warping of the section is taken as the function $\omega = xy$. The geometrical characteristic J_ω is calculated from

$$J_\omega = \int_F x^2 y^2 dF,$$

and the characteristic J_d is determined from Bredt's equation according to the theory of pure torsion. For the thin-walled box-like rectangular contour that we examined in Chapter IV, with a wall thickness δ and the sides of the rectangle d_1 and d_2, we shall have the following equations for the geometrical characteristics

$$\left.\begin{aligned} &F = 2\delta(d_1 + d_2), \quad J_x = \frac{1}{6}\delta d_1^2 (d_1 + 3d_2), \quad J_y = \frac{1}{6}\delta d_2^2 (d_2 + 3d_1), \\ &J_d = \frac{4}{F}\delta^2 d_1^2 d_2^2, \quad J_\omega = \frac{1}{48} F d_1^2 d_2^2, \quad r^2 = \frac{J_x + J_y}{F}, \\ &a_x = a_y = \beta_x = \beta_y = 0, \end{aligned}\right\} \tag{1.18}$$

where the torsional moment of inertia J_d is determined by equation (IV.2.27).

§ 2. Integration of the stability equations for the cases of hinged or fixed ends

The stability problem of a beam whose end sections are fixed to restrain displacements (translational ξ, η and rotational θ) in the plane of the cross section and are free of normal stresses (Figure 154a) has a particularly simple solution. The boundary conditions for such a beam are

$$\left.\begin{aligned} &\text{for } z = 0 \quad \xi = \eta = \theta = 0, \quad \xi'' = \eta'' = \theta'' = 0; \\ &\text{for } z = l \quad \xi = \eta = \theta = 0, \quad \xi'' = \eta'' = \theta'' = 0, \end{aligned}\right\} \tag{2.1}$$

where l is the length of the beam. For the boundary conditions (2.1) the eigenfunctions of equations (1.10) will be

$$\xi = A \sin\frac{n\pi z}{l}, \quad \eta = B \sin\frac{n\pi z}{l}, \quad \theta = C \sin\frac{n\pi z}{l}, \tag{2.2}$$

where A, B and C are constant coefficients, and n is an arbitrary positive integer ($n = 1, 2, 3, \ldots$).

Inserting the values (2.2) in equations (1.10) and using the notation $\lambda = \frac{n\pi}{l}$, we obtain for the unknowns A, B and C, (dropping the common factor $\lambda^2 \sin\lambda z$) the following system of homogeneous equations

$$\left.\begin{aligned} &(EJ_y\lambda^2 - P)A - (M_x + a_y P)C = 0, \\ &(EJ_x\lambda^2 - P)B - (M_y - a_x P)C = 0, \\ &-(M_x + a_y P)A - (M_y - a_x P)B + \\ &\quad + [EJ_\omega\lambda^2 - (r^2 P + 2\beta_x M_y - 2\beta_y M_x - GJ_d)]C = 0. \end{aligned}\right\} \tag{2.3}$$

Since the coefficients A, B and C must be different from zero (otherwise we obtain for ξ, η and θ trivial zero solutions), the determinant of

the system (2.3) vanishes

$$\begin{vmatrix} EJ_y\lambda^2 - P & 0 & -(M_x + a_yP) \\ 0 & EJ_x\lambda^2 - P & -(M_y - a_xP) \\ -(M_x + a_yP) & -(M_y - a_xP) & EJ_\omega\lambda^2 - (r^2P + 2\beta_xM_y - \\ & & - 2\beta_yM_x - GJ_d) \end{vmatrix} = 0. \quad (2.4)$$

Equation (2.4) is quite general and allows the determination of the critical values of the load parameter for an arbitrary open contour.

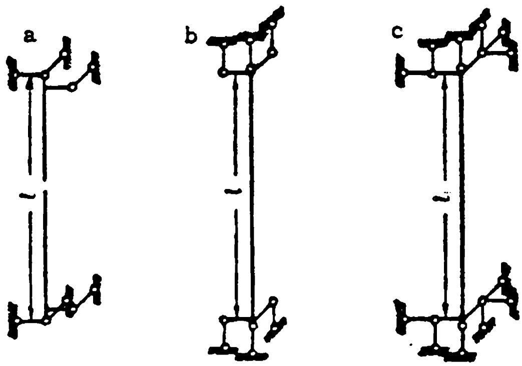


Figure 154

We obtain an analogous solution for a beam whose end sections are fixed so as to restrain rotation about the x- and y-axes and are free from torsional moments and shear forces (Figure 154b).

In this case the boundary conditions are

$$\left.\begin{array}{ll} \text{for } z = 0 & \xi' = \eta' = \theta' = 0, \quad \xi''' = \eta''' = \theta''' = 0; \\ \text{for } z = l & \xi' = \eta' = \theta' = 0, \quad \xi''' = \eta''' = \theta''' = 0. \end{array}\right\} \quad (2.5)$$

The functions ξ, η and θ which satisfy the boundary conditions (2.5) and the differential equations (1.10), will have the form

$$\xi = A\cos\frac{n\pi z}{l}, \quad \eta = B\cos\frac{n\pi z}{l}, \quad \theta = C\cos\frac{n\pi z}{l}.$$

The determinantal equation which determines the critical load for the examined boundary conditions will be the same as equation (2.4). It follows that the critical loads will have the same values as in the case of the boundary conditions (2.1).

We shall now examine a beam with rigidly built-in ends (Figure 154c). The boundary conditions will be

$$\left.\begin{array}{ll} \text{for } z = 0 & \xi = \eta = \theta = 0, \quad \xi' = \eta' = \theta' = 0; \\ \text{for } z = l & \xi = \eta = \theta = 0, \quad \xi' = \eta' = \theta' = 0. \end{array}\right\} \quad (2.6)$$

For the boundary conditions (2.6) the functions ξ, η and θ can be put

in the form

$$\left.\begin{aligned} \xi &= A(1-\cos 2\lambda z), \\ \eta &= B(1-\cos 2\lambda z), \\ \theta &= C(1-\cos 2\lambda z). \end{aligned}\right\} \quad (2.7)$$

A, B and C are (as before) arbitrary coefficients, and n is any positive integer.

Inserting (2.7) in the differential equations (1.10) and dividing by $4\lambda^2 \cos 2\lambda z$, we obtain for the constants A, B and C a homogeneous system of algebraic equations. Equating again the determinant of this system to zero, we obtain for the critical force the equation

$$\begin{vmatrix} EJ_y(2\lambda)^2 - P & 0 & -(M_x + a_y P) \\ 0 & EJ_x(2\lambda)^2 - P & -(M_y - a_x P) \\ -(M_x + a_y P) & -(M_y - a_x P) & EJ_\omega(2\lambda)^2 - (r^2 P + 2\beta_x M_y - \\ & & - 2\beta_y M_x - GJ_d). \end{vmatrix} = 0.$$

§ 3. Axial compression. Study of the roots of the characteristic equation. Generalization of Euler's theory

If the compressive force passes through the centroid of the beam cross section, the beam is in condition of axial compression. In this case, setting in equations (1.10) the moments M_x and M_y equal to zero, we obtain

$$\left.\begin{aligned} EJ_y \xi^{IV} + P\xi'' + a_y P\theta'' &= 0, \\ EJ_x \eta^{IV} + P\eta'' - a_x P\theta'' &= 0, \\ a_y P\xi'' - a_x P\eta'' + EJ_\omega \theta^{IV} + (r^2 P - GJ_d)\,\theta'' &= 0. \end{aligned}\right\} \quad (3.1)$$

The differential equations (3.1) and the conditions of constraint at the beam ends determine all the forms of stability loss under axial compression of a weightless beam. It follows from equations (3.1) that if the coordinates of the shear center a_x and a_y are not equal to zero, i.e. if the shear center does not coincide with the centroid of the section, a beam in axial compression cannot lose stability by Euler buckling.

For the boundary conditions (2.1) or (2.5) the critical forces are determined by equation (2.4). Assuming in this equation $M_x = M_y = 0$ we obtain a characteristic equation for the case of axial compression.

This equation can be written in the form

$$\begin{vmatrix} P_y - P & 0 & -a_y P \\ 0 & P_x - P & a_x P \\ -a_y P & a_x P & r^2(P_\omega - P) \end{vmatrix} = 0, \quad (3.2)$$

where

$$P_x = EJ_x\lambda^2, \quad P_y = EJ_y\lambda^2 \quad (3.3)$$

are the Euler critical forces;

$$P_\omega = \frac{EJ_\omega\lambda^2 + GJ_d}{r^2} \quad (3.4)$$

is the critical force for a purely torsional form of buckling.

Expanding the determinant (3.2) we obtain

$$(P_x-P)(P_y-P)(P_\omega-P)r^2-a_y^2P^2(P_x-P)-a_x^2P^2(P_y-P)=0$$

or, arranging in descending powers of P,

$$(a_x^2+a_y^2-r^2)P^3+[(P_x+P_y+P_\omega)r^2-a_y^2P_x-a_x^2P_y]P^2-$$
$$-r^2(P_xP_y+P_xP_\omega+P_yP_\omega)P+P_xP_yP_\omega r^2=0. \qquad (3.5)$$

We shall study the behavior of the roots in this general case. Let the roots of this cubic equation be P_1, P_2, P_3. Since we fix here λ_n (i. e. we examine not only definite boundary conditions and the length l but also a definite form of stability loss) P_x, P_y and P_ω are entirely determined. We shall now show how the roots P_1, P_2 and P_3 are disposed with respect to the Euler critical forces P_x and P_y.

To be definite, let $P_x<P_y$. We can always arrange this by a proper choice the x- and y-axes. We shall denote the expression on the left-hand side of equation (3.5) by $f(P)$.

For small values of P the sign of the function will coincide with the sign of the free term; hence

$$f(P)>0.$$

For $P=P_x$ $f(P_x)=-a_x^2P_x^2(P_y-P_x)<0$. It is therefore evident that the function $f(P)$ changes its sign in the interval $0<P\leqslant P_x$ and so one root of the equation must lie in this interval. Let it be P_1. Thus $P_1<P_x$.

For $P=P_y$ $f(P_y)=a_y^2P_y^2(P_y-P_x)>0$. Thus also the function $f(P)$ changes its sign in the interval $P_x<P\leqslant P_y$ and in this interval lies the second root of the equation. Let it be P_2. Thus $P_x<P_2<P_y$.

For sufficiently large P the sign of the function $f(P)$ will coincide with the sign of the coefficient of the highest order term (and this is negative),

$$f(P_\infty)<0.$$

Consequently, the third root of the equation, P_3, will lie in the interval $P_y<P<\infty$, since it is there that the function $f(P)$ changes its sign.

As to the root P_ω, it is easy to show that if

$$P_\omega>P_y, \quad \text{then } f(P_\omega)>0$$

and the function $f(P)$ does not change its sign in the interval $P_y<P<P_\omega$. Consequently, $P_3>P_\omega$. Now, if

$$P_\omega<P_x, \quad \text{then } f(P_\omega)<0$$

and the function $f(P)$ also does not change its sign in the interval $P_\omega<P<P_x$. Consequently, $P_1<P_\omega<P_x$.

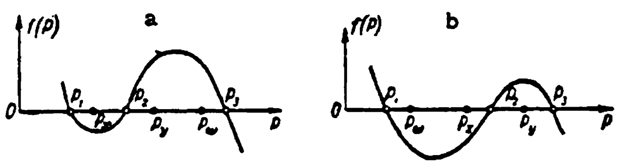


Figure 155

Depending on the relation between the forces P_x, P_y and P_ω, as

determined by equations (3.3) and (3.4), we shall obtain different inequalities for these roots. These inequalities are

$$\left.\begin{aligned}&\text{for } P_x < P_y < P_\omega\\ &\quad P_1 \leqslant P_x \leqslant P_2 \leqslant P_y \leqslant P_\omega \leqslant P_3 \text{ (Figure 155a)},\\ &\text{for } P_\omega < P_x < P_y\\ &\quad P_1 \leqslant P_\omega \leqslant P_x \leqslant P_2 \leqslant P_y \leqslant P_3 \text{ (Figure 155b)}.\end{aligned}\right\} \tag{3.6}$$

Summing up, we note that all the roots of the characteristic equation are real. The root P_1 is the smallest; the root P_2 is situated somewhere between P_x and P_y, and the root P_3 is the largest. The lowest critical force P_1, being a design quantity, is in the general case less than the critical force P_x determined by the usual theory of longitudinal bending. This means that for an asymmetrical contour with a shear center not coinciding with the centroid ($a_x \neq 0$ and $a_y \neq 0$), Euler buckling through bending is impossible. The natural buckling for such a beam is through flexural torsion, for which the critical force P_1 will have a smaller value than the force obtained from the usual theory of lateral buckling.

We can arrive at this result also in another way, analyzing the phenomenon of buckling from a purely physical point of view. As a matter of fact, in calculating the beam for stability according to the theory of lateral buckling, we take into account buckling by pure bending only. According to this theory, the cross sections of the beam can undergo only translational displacements. Torsion angles are equal to zero. Such an assumption is equivalent to the introduction of angle connections along the whole of the beam which prevent the rotation of the beam sections about the axis (Figure 156). The critical forces are increased as a result of introducing these connections.

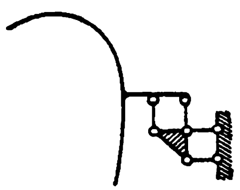


Figure 156

If the cross sections of the beam have two symmetry axes, the shear center will coincide with the centroid, i. e. $a_x = a_y = 0$.

The system of differential equations (3.1) falls into three independent differential equations

$$EJ_y\xi^{\text{IV}} + P\xi'' = 0,$$
$$EJ_x\eta^{\text{IV}} + P\eta'' = 0,$$
$$EJ_\omega\theta^{\text{IV}} + (r^2P - GJ_d)\,\theta'' = 0.$$

The characteristic equation (3.2) assumes the form

$$(P_x - P)(P_y - P)(P_\omega - P)\,r^2 = 0.$$

Whence we obtain the following expressions for the three critical forces P_1, P_2 and P_3 (for fixed λ),

$$P_1 = P_x, \quad P_2 = P_y, \quad P_3 = P_\omega. \tag{3.7}$$

The first two critical forces coincide with the Euler forces. The third corresponds to a torsional mode of buckling with a torsion center coinciding with the centroid of the section.

In this particular case (for $a_x = a_y = 0$) inequality (3.6) changes into equality (3.7). From this analysis it follows that buckling by bending in the principal planes, accounted for by the usual Euler theory, is possible for a beam free from angular connections only in case the shear center coincides with the centroid of the section. In addition to the critical forces, determined by the theory of lateral buckling, P_ω forces are also possible, corresponding to a buckling of the beam by twisting about a longitudinal axis. These forces can be smaller than the Euler forces.

In the case of a beam with a single axis of symmetry the expression (3.2) is simplified. Suppose, to be definite, that the x-axis is the symmetry axis. As a result of symmetry we then have $a_y = 0$, and equation (3.2) is separated into two equations:

$$P_y - P = 0,$$
$$\begin{vmatrix} P_x - P & a_x P \\ a_x P & r^2 (P_\omega - P) \end{vmatrix} = 0$$

We obtain from the first equation the Euler force

$$P = P_y.$$

The second equation determines the two other critical forces. This equation can be written in expanded form thus

$$r^2 (P_x - P)(P_\omega - P) - a_x^2 P^2 = 0$$

or, arranging the powers of P,

$$(r^2 - a_x^2) P^2 - r^2 (P_x + P_\omega) P + r^2 P_x P_\omega = 0. \tag{3.8}$$

The following equation will be used for the determination of the two other critical forces,

$$P = \frac{r^2 (P_x + P_\omega) \pm \sqrt{r^4 (P_x + P_\omega)^2 - 4 r^2 P_x P_\omega (r^2 - a_x^2)}}{2 (r^2 - a_x^2)}. \tag{3.9}$$

For $a_x = 0$ equation (3.9) gives

$$P_1 = P_x; \quad P_2 = P_\omega$$

§ 4. Analysis of beam forms after buckling. Centers of rotation

It follows from equations (2.2) that, for the boundary conditions (2.1), the quantity $\lambda_n = \frac{n\pi}{l}$ determines the form of buckling along the beam. As we saw above, the quantity n in the argument of the trigonometric functions can take any positive integral values $n = 1, 2, 3, \ldots$

To each value of n there will correspond (up to a constant factor) a sinusoidal form of the buckled beam. Thus, for example, for $n = 1$ the displacements ξ, η and θ vary as a sinusoidal function with one half-wave along the beam, which has stationary hinges at the ends. For $n = 2$ these displacements vary as a sinusoidal function with two half-waves, etc. (Figure 157).

By describing the quantity λ_n or (which amounts to the same) the number of half-waves n, we fix a definite mode of buckling. For a given number of half-waves of the sinusoidal function, the characteristic

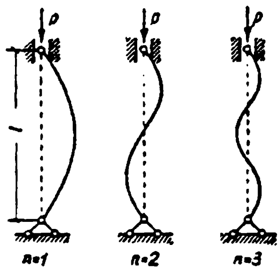


Figure 157

equation (3.2) yields, as we saw above, three critical forces, corresponding to three degrees of freedom in the plane of the cross section of the space beam. The lowest values of these three critical forces are obtained for $n=1$, i. e. the case where the beam buckles assuming the shape of a single sinusoidal half-wave. The parameter λ_n in equation (3.2) has the following value for $n=1$

$$\lambda=\frac{\pi}{l}.$$

To each of the three critical forces, determined by equation (3.2) for a given λ_n, will correspond buckling in the plane of the cross section. This form is characterized by the cross section rotating about a certain point in its plane which, for small values of ξ, η and θ, can be taken as the instantaneous center of rotation. The coordinates of this center can be obtained from the condition of its being stationary and by using the equations (1.2):

$$\left.\begin{aligned}\xi_c&=\xi-(c_y-a_y)\,\theta=0,\\ \eta_c&=\eta+(c_x-a_x)\,\theta=0,\end{aligned}\right\} \tag{4.1}$$

where c_x and c_y are the coordinates of the center of rotation. ξ, η and θ will have the following form under boundary conditions corresponding to hinged ends

$$\xi=A\sin\lambda z,\quad \eta=B\sin\lambda z,\quad \theta=C\sin\lambda z. \tag{4.2}$$

The relation of the constant coefficients A, B and C to the case of axial compression can be found from the homogeneous equations (2.3) for

$$M_x=M_y=0:\quad \frac{A}{C}=\frac{a_yP}{P_y-P},\quad \frac{B}{C}=-\frac{a_xP}{P_x-P}. \tag{4.3}$$

We obtain from equations (4.1)–(4.3) the following expressions for the coordinates of the centers of rotation,

$$c_x=\frac{a_x}{1-\frac{P}{P_x}},\quad c_y=\frac{a_y}{1-\frac{P}{P_y}}. \tag{4.4}$$

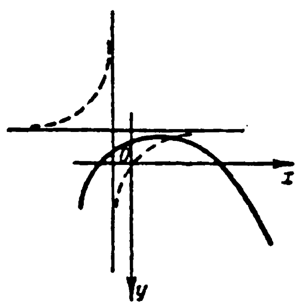


Figure 158

To each critical force there will obviously correspond its center of rotation determining an axis of rotation parallel to the beam axis. For the three critical forces P_1, P_2 and P_3, obtained from equation (3.2) for a given number n of half-waves of the sinusoidal function, we thus get three centers of rotation.

Eliminating the force P from expressions (4.4), we obtain the locus of the centers of rotation which will only depend on the dimensions of the section. This equation has the form

$$c_xa_yP_y-c_ya_xP_x=c_xc_y\,(P_y-P_x),$$

or

$$c_xa_yJ_y-c_ya_xJ_x=c_xc_y\,(J_y-J_x).$$

where J_x and J_y are the moments of inertia with respect to the principal axes.

Equation (4.5) represents an equilateral hyperbola having asymptotes parallel to the principal axes and passing through the coordinate origin (Figure 158).

§ 5. Design of an axially compressed beam with asymmetrical cross section

We shall examine an angle with unequal sides (Figure 159). We shall assume that the end sections of the angle are restrained from displacements in their planes (translations ξ, η and rotation θ) and are free from normal stresses σ.

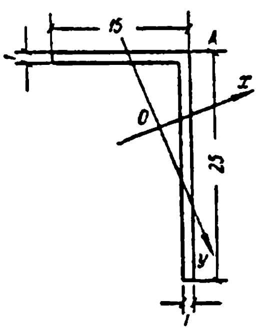


Figure 159

Equation (3.5) is the equation for the case of axial compression. We refer our section to the principal axes Ox and Oy and calculate the geometrical characteristics of the section which enter equation (3.5).

In our case, the coordinates of the shear center in the principal axes will be

$$a_x = 5.2\text{cm}, \quad a_y = -6.3\text{cm}. \qquad (5.1)$$

The remaining geometrical characteristics will have the following values

$$\left.\begin{aligned} &J_x = 2898\text{cm}^4, \quad J_y = 434\text{cm}^4, \\ &J_d = 13\text{cm}^4, \qquad F = 39\text{cm}^2, \\ &r^2 = a_x^2 + a_y^2 + \frac{J_x + J_y}{F} = 152\text{cm}^2. \end{aligned}\right\} \qquad (5.2)$$

The sectorial moment of inertia J_ω vanishes, since if the shear center is at the vertex of the angle the sectorial areas which determine the torsional stresses are equal to zero.

Suppose the elastic moduli to be

$$E = 2.1 \cdot 10^6 \text{kg/cm}^2, \quad G = 0.4E = 0.84 \cdot 10^6 \text{kg/cm}^2. \qquad (5.3)$$

The cubic equation (3.5) will have in our case the following form

$$-\frac{J_x + J_y}{F} P^3 + \left\{ GJ_d + E\left[(J_x + J_y) r^2 - (J_x a_y^2 + J_y a_x^2)\right] \lambda^2 \right\} P^2 -$$
$$- E\lambda^2 \left[EJ_x J_y r^2 \lambda^2 + GJ_d (J_x + J_y) \right] P + E^2 J_x J_y G J_d \lambda^4 = 0. \qquad (5.4)$$

The coefficients of equation (5.4) are quantities which depend on the geometrical characteristics of the section, the elastic moduli and, through the parameter λ, on the beam span and the number of half-waves of the sinusoidal function. Introducing the data from (5.1), (5.2) and (5.3) in equation (5.4) leaving the quantity λ still undetermined and dividing throughout by $\frac{J_x + J_y}{F}$, we obtain

$$P^3 - (1.28 \cdot 10^5 + 0.932 \cdot 10^{10} \lambda^2) P^2 +$$
$$+ (8.92 \cdot 10^{14} + 0.991 \cdot 10^{19} \lambda^2) \lambda^2 P - 7.10 \cdot 10^{22} \lambda^4 = 0. \qquad (5.5)$$

According to this equation the critical force P is determined as a function of $\lambda = \frac{n\pi}{l}$. We determine the critical forces for five values of the beam span:

$$l = 100,\ 200,\ 300,\ 400,\ 500 \text{cm}.$$

Calculating for these spans $\lambda^2 = \left(\frac{n\pi}{l}\right)^2$ for $n = 1$ and then inserting the results in equation (5.5), we obtain five cubic equations for the critical force P. We note in passing that the force P is found in kilograms. In order to avoid large numbers we shall calculate P in tons, reducing the coefficients by factors of 10^3, 10^6 and 10^9, correspondingly.

The equations for the given five cases will be the following:

$$\left.\begin{array}{l}
1)\ \text{for } l = 100\,\text{cm} \\
\quad P^3 - 93.3\cdot 10^2 P^2 + 1050\cdot 10^4 P - 692\cdot 10^6 = 0; \\
2)\ \text{for } l = 200\,\text{cm} \\
\quad P^3 - 24.3\cdot 10^2 P^2 + 82.5\cdot 10^4 P - 43.2\cdot 10^6 = 0; \\
3)\ \text{for } l = 300\,\text{cm} \\
\quad P^3 - 11.5\cdot 10^2 P^2 + 21.8\cdot 10^4 P - 8.59\cdot 10^6 = 0; \\
4)\ \text{for } l = 400\,\text{cm} \\
\quad P^3 - 7.03\cdot 10^2 P^2 + 9.26\cdot 10^4 P - 2.71\cdot 10^6 = 0; \\
5)\ \text{for } l = 500\,\text{cm} \\
\quad P^3 - 4.96\cdot 10^2 P^2 + 5.06\cdot 10^4 P - 1.11\cdot 10^6 = 0.
\end{array}\right\} \qquad (5.6)$$

To each of the equations (5.6) there correspond three positive real roots P_1, P_2, P_3. Solving these equations, we obtain for the critical forces the following values in tons:

$$\left.\begin{array}{llll}
1)\ \text{for } l = 100\,\text{cm} & P_1 = 70; & P_2 = 1226, & P_3 = 8034; \\
2)\ \text{for } l = 200\,\text{cm} & P_1 = 65; & P_2 = 330, & P_3 = 2035; \\
3)\ \text{for } l = 300\,\text{cm} & P_1 = 54; & P_2 = 172, & P_3 = 924; \\
4)\ \text{for } l = 400\,\text{cm} & P_1 = 42; & P_2 = 120, & P_3 = 541; \\
5)\ \text{for } l = 500\,\text{cm} & P_1 = 34; & P_2 = 95, & P_3 = 367.
\end{array}\right\} \qquad (5.7)$$

The critical forces found correspond to a sinusoidal form of instability with a single half-wave, characterized by the number $n = 1$. In general we have an infinite number of values n and, consequently, we may obtain an infinite number of critical forces. However, in each case we are interested only in the smallest critical force. It is easy to show that the smallest critical forces correspond to the value $n = 1$. Indeed, from an examination of the solution (5.7) we see that by increasing l the critical forces P_1, P_2, P_3 are reduced and a decrease in l increases the critical forces. However, in the expression $\lambda = \frac{n\pi}{l}$ an increase of l for constant n may be regarded as a decrease of n for constant l and, conversely, an increase of n for constant l can be regarded as a decrease of l for constant n. Consequently, the smallest critical force will correspond to the smallest value of n, i. e. $n = 1$.

The critical Euler forces are calculated from equations (3.3),

$$P_x = EJ_x\lambda^2, \quad P_y = EJ_y\lambda^2.$$

In our five cases these critical Euler forces will have the following values in tons:

1) for $l = 100\text{cm}$ $P_x = 6007$, $P_y = 900$;
2) for $l = 200\text{cm}$ $P_x = 1513$, $P_y = 225$;
3) for $l = 300\text{cm}$ $P_x = 669$, $P_y = 100$;
4) for $l = 400\text{cm}$ $P_x = 376$, $P_y = 56$;
5) for $l = 500\text{cm}$ $P_x = 240$, $P_y = 36$.

For comparison, we compute the critical force according to Wagner. This force is calculated by equation (3.4):

$$P_{\text{Wag}} = \frac{1}{r^2}(EJ_{\omega}\lambda^2 + GJ_d).$$

In our case the sectorial moment of inertia $J_{\omega} = 0$, and we obtain the following critical force according to Wagner's equation:

$$P_{\text{Wag}} = \frac{GJ_d}{r^2} = 72\,\text{ton}.$$

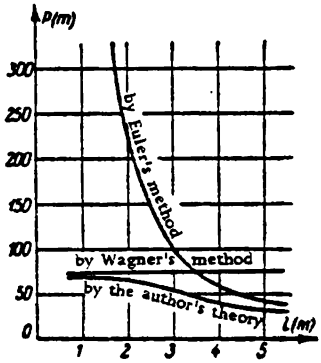


Figure 160

We see from this equation that the force P does not depend on the beam span. This erroneous result stems from Wagner's incorrect assumption concerning the position of the torsion center in buckling.

Figure 160 shows graphs of the smallest critical forces calculated as a function of the beam span by the different methods given here. It is seen from these graphs that the critical Euler forces as determined from the condition of buckling by bending in the plane of the smallest flexural stiffness are larger than the smallest critical forces determined by the theory of flexural-torsional modes of buckling. On increasing the span, the difference between the forces determined from the Euler equation and the cubic equation (5.4) is reduced. This means that by increasing the beam span the center of rotation, determined by equations (4.4), recedes and the flexural-torsional mode of buckling approaches the flexural mode.

§ 6. Stability of plane bending under eccentric compression

We shall examine the case of eccentric force P (see Figure 152). If e_x and e_y are the coordinates of the point of application of this force, then

$$M_x = -Pe_y, \quad M_y = Pe_x. \tag{6.1}$$

The differential equations (1.10) assume the following form

$$\left.\begin{aligned} &EJ_y\xi^{\text{IV}} + P\xi'' + (a_y - e_y)P\theta'' = 0,\\ &EJ_x\eta^{\text{IV}} + P\eta'' - (a_x - e_x)P\theta'' = 0,\\ &(a_y - e_y)P\xi'' - (a_x - e_x)P\eta'' + EJ_{\omega}\theta^{\text{IV}} +\\ &\quad + [P(r^2 + 2\beta_x e_x + 2\beta_y e_y) - GJ_d]\theta'' = 0. \end{aligned}\right\} \tag{6.2}$$

These equations and boundary conditions determine the additional bending and torsional deformations which appear under eccentric compression upon departure from plane bending.

If the force P is applied at the shear center, then $e_x = a_x$ and $e_y = a_y$. In this case the system of simultaneous differential equations (6.2) reduces to three independent equations,

$$\left.\begin{aligned} &EJ_y\xi^{IV} + P\xi'' = 0, \\ &EJ_x\eta^{IV} + P\eta'' = 0, \\ &EJ_\omega\theta^{IV} + [P(r^2 + 2\beta_x a_x + 2\beta_y a_y) - GJ_d]\theta'' = 0. \end{aligned}\right\} \qquad (6.3)$$

The first two equations (6.3) coincide in form with the Euler equation for the case of axial compression. These equations give solutions for the stability problem of the first kind when the bending deformations determined by them do not lie in the initial bending plane as caused by the eccentric (with respect to the centroid) thrust. If the bending deformations determined by these equations are in the initial plane, the solutions of these equations correspond to buckling of the second kind. The third equation determines a new equilibrium shape, for which the cross sections of the beam rotate about the shear center. This corresponds to stability of the first kind.

This leads us to the conclusion, that *for an open section of arbitrary form buckling by bending, characterized by translational displacements of the cross sections, is possible only under the condition that the longitudinal compressive force is applied at the shear center. If this force does not pass through the shear center, buckling involves not only the deflections ξ and η, but also rotation through an angle θ.*

It was noted in § 9 of Chapter I that the line of shear centers has a fundamental importance in the theory of thin-walled beams. This line, and not the line of centroids, should be taken as the axis of the beam. We see that this axis is of special importance also in the theory of stability of thin-walled beams. Indeed, only under the condition that the longitudinal compressive load is applied in the line of the shear centers of the beam is Euler buckling possible. In the case of a central compression of an asymmetrical section, pure bending, as we saw above, is impossible. Euler's equations are not applicable in this case*.

If the ends of the beam are restrained from displacements and free from normal stresses then, as we saw above from equations (2.2), the buckled beam has the form of a sinusoid with a whole number of half-waves. The magnitude of the eccentric longitudinal force is found from equations (2.4). Introducing in this equation the values of the moments M_x and M_y determined by equations (6.1), we obtain

$$\begin{vmatrix} P_y - P & 0 & -P(a_y - e_y) \\ 0 & P_x - P & P(a_x - e_x) \\ -P(a_y - e_y) & P(a_x - e_x) & P_\omega r^2 - P(r^2 + 2\beta_x e_x + 2\beta_y e_y) \end{vmatrix} = 0. \quad (6.4)$$

* An exception to this is the case of a section having a symmetry axis with respect to which the moment of inertia is least.

Expanding the determinant (6.4), we find

$$(P_x - P)(P_y - P)[r^2(P_\omega - P) - 2\beta_x e_x P - 2\beta_y e_y P] - \\ - (a_y - e_y)^2 (P_x - P) P^2 - (a_x - e_x)^2 (P_y - P) P^2 = 0. \quad (6.5)$$

Here the notations (3.3) and (3.4) are adopted.

For given eccentricities e_x and e_y and for a given sinusoidal buckling, equation (6.5) furnishes three critical forces P_1, P_2 and P_3. To each of these forces will correspond a buckling shape, characterized by the cross sections of the beam rotating about the line of centers of rotation. The torsion angle varies as a sine function along the beam. The coordinates of the center of rotation in the case of an eccentric longitudinal force can be computed, for small displacements ξ, η and θ, from the equations

$$c_x = \frac{a_x - \frac{P}{P_x} e_x}{1 - \frac{P}{P_x}}, \quad c_y = \frac{a_y - \frac{P}{P_y} e_y}{1 - \frac{P}{P_y}}. \quad (6.6)$$

To the three critical forces P_1, P_2 and P_3, obtained from equation (6.5) for a given point of application E and for a given number n of half-waves, there correspond three rotation centers C_1, C_2 and C_3 determined by equations (6.6). If the point of application E is displaced in the cross section of the beam along the curve

$$F(e_x, e_y) = 0,$$

then the centers of rotation C_1, C_2 and C_3, which correspond to the three values of the critical force P_1, P_2 and P_3, will shift along the three curves

$$\varphi_1(c_x, c_y) = 0, \quad \varphi_2(c_x, c_y) = 0, \quad \varphi_3(c_x, c_y) = 0.$$

Since one of the three forces P_1, P_2 and P_3 will have the smallest value, upon loss of stability the center of rotation will be situated on the curve $\varphi(c_x, c_y) = 0$ which corresponds to this smallest force.

§ 7. "Isostabs" of eccentric critical forces

For given elastic and geometrical characteristics of the beam and a given sinusoidal variation of the displacements ξ, η and θ along the beam, equation (6.5) establishes the relation between the value of the critical force P and the coordinates of the point of application e_x and e_y. Assuming $P = P_0 =$ const in equation (6.5) and dividing by $P_0^2 (P_x - P_0)(P_y - P_0)(P_\omega - P_0)$, we obtain

$$\frac{(a_x - e_x)^2}{(P_x - P_0)(P_\omega - P_0)} + \frac{(a_y - e_y)^2}{(P_y - P_0)(P_\omega - P_0)} + \frac{2\beta_x e_x}{P_0 (P_\omega - P_0)} + \\ + \frac{2\beta_y e_y}{P_0 (P_\omega - P_0)} - \frac{r^2}{P_0^2} = 0. \quad (7.1)$$

Equation (7.1) represents a curve in the plane of the cross section which is the locus of the points of application for which the critical force has a constant value $P = P_0$. We can, therefore, call the curve expressed by this equation for $P_0 =$ const the i s o s t a b [i. e. line of iso-stability] of

the critical forces in the case of eccentric compression. Considering in equation (7.1) $P = P_0$ as a parameter and assigning to it various values, we obtain a family of such curves. All these curves, as evident from equation (7.1), will be ellipses or hyperbolas, depending on the value of the parameter P_0.

If P_0 is the smallest of the three critical forces, then

$$(P_x - P_0)(P_\omega - P_0) > 0,$$
$$(P_y - P_0)(P_\omega - P_0) > 0,$$

and the curve expressed by equation (7.1) will be an ellipse.

If P_0 is the largest or second largest critical force, the isostabs will be either ellipses or hyperbolas depending on the magnitude of P_ω as compared with P_x and P_y. The axes of the ellipse and hyperbolas, obtainedfrom equation (7.1) for various values of the parameter P_0, will be parallel to the principal axes of the section, since in this equation the products of the eccentricities e_x and e_y do not appear. Denoting the half-axes of the isostab by a and b, we obtain from equation (7.1) after the necessary calculations

$$\left.\begin{aligned} a^2 &= \frac{r^2(P_x - P_0)(P_\omega - P_0)}{P_0^2} + \frac{\beta_x^2(P_x - P_0)^2}{P_0^2} + \\ &+ \frac{\beta_y^2(P_x - P_0)(P_y - P_0)}{P_0^2} - \frac{2a_x\beta_x(P_x - P_0)}{P_0} - \frac{2a_y\beta_y(P_x - P_0)}{P_0}, \\ b^2 &= \frac{r^2(P_y - P_0)(P_\omega - P_0)}{P_0^2} + \frac{\beta_y^2(P_y - P_0)^2}{P_0^2} + \\ &+ \frac{\beta_x^2(P_x - P_0)(P_y - P_0)}{P_0^2} - \frac{2a_x\beta_x(P_y - P_0)}{P_0} - \frac{2a_y\beta_y(P_y - P_0)}{P_0}. \end{aligned}\right\} \quad (7.2)$$

The coordinates of the isostab center will be

$$i_x = \frac{a_x P_0 - \beta_x(P_x - P_0)}{P_0}, \quad i_y = \frac{a_y P_0 - \beta_y(P_y - P_0)}{P_0}.$$

For a beam with equal moments of inertia the critical Euler forces will be

$$P_x = P_y = \frac{EJ\pi^2 n^2}{l^2}.$$

In this case the elliptical isostabs reduce to circles. The radius of the circle is calculated from the equation

$$R^2 = \frac{r^2(P_x - P_0)(P_\omega - P_0)}{P_0^2} + \frac{(\beta_x^2 + \beta_y^2)(P_x - P_0)^2}{P_0^2} - \frac{(2a_x\beta_x + 2a_y\beta_y)(P_x - P_0)}{P_0}.$$

In this equation P_0 must be such that R^2 is positive. In the opposite case we obtain an imaginary circle.

If the beam has two symmetry axes in the cross section, $a_x = a_y = 0$; $\beta_x = \beta_y = 0$. In this case equation (7.1) has a simpler form

$$\frac{e_x^2}{(P_x - P_0)(P_\omega - P_0)} + \frac{e_y^2}{(P_y - P_0)(P_\omega - P_0)} - \frac{r^2}{P_0^2} = 0.$$

The center of the curves expressed by these equations (for different values of the critical force P_0) coincides with the centroid of the section.

§ 8. Stability of plane bending in beams under eccentric extension. Stability circle

The flexural-torsional forms of buckling which we discussed are possible not only in the case of a compressive force acting on the beam, but also in the case of eccentric extension. The phenomenon of stability loss under eccentric extension is expressed by stating that the curve described by (7.1) retains its meaning (does not become imaginary) also for $P_0 < 0$. It follows that points can be found such that a tensile force applied at them will cause stability loss. This phenomenon is explained physically by that under eccentric application of longitudinal tension not only tensile but also compressive stresses may appear in the cross sections of the beam before it loses stability. These compressive stresses give rise to unstable equilibrium shapes of a beam in eccentric tension.

It is perfectly clear that no application of this force in the plane of the cross section can cause buckling of a beam under eccentric extension. If the external tension passes inside the section core, the whole section is subjected to tensile stresses only. In this case the beam cannot lost its stability for any magnitude of the force. The section core is thus the *stability domain* of the beam under extension.

The question naturally arises: is this domain limited only to the section core or does it extend beyond it? Is buckling possible for a beam when the longitudinal tensile force is applied close to but outside the core, i. e. when compressive stresses appear on the opposite side of the section? Which zone in the cross section should contain these compressive stresses for the beam to buckle by plane bending under a finite tensile force (allowing, of course, infinite elasticity of the beam)? We obtain the answer to these questions from equation (7.1) which determines the curve ("isostab") of the constant critical forces in the variables e_x and e_y (for a given parameter P_0).

The domain of stable equilibrium of a beam under eccentric extension is determined by the fact that the critical force becomes infinite on its boundary.

Equation (7.1) can be written in the following form

$$\frac{(a_x - e_x)^2}{\left(\frac{P_x}{P_0} - 1\right)\left(\frac{P_\omega}{P_0} - 1\right)} + \frac{(a_y - e_y)^2}{\left(\frac{P_y}{P_0} - 1\right)\left(\frac{P_\omega}{P_0} - 1\right)} + \frac{2\beta_x e_x}{\left(\frac{P_\omega}{P_0} - 1\right)} + \frac{2\beta_y e_y}{\left(\frac{P_\omega}{P_0} - 1\right)} - r^2 = 0.$$

Assuming in this equation $P_0 = \infty$, we obtain the equation $F(e_x, e_y) = 0$ representing an analytical curve which delimits the domain of stability under extension. This equation has the form

$$(a_x - e_x)^2 + (a_y - e_y)^2 - 2\beta_x e_x - 2\beta_y e_y - r^2 = 0$$

or, using the notation adopted before (1.6)

$$\beta_x = \frac{U_y}{2J_y} - a_x, \quad \beta_y = \frac{U_x}{2J_x} - a_y, \quad r^2 = a_x^2 + a_y^2 + \frac{J_x + J_y}{F},$$

we obtain

$$e_x^2 + e_y^2 - \frac{U_y}{J_y} e_x - \frac{U_x}{J_x} e_y - \frac{J_x + J_y}{F} = 0. \tag{8.1}$$

The curve determined by (8.1) is a circle. The radius of this circle and the coordinates of its center are determined from

$$R^2 = \frac{U_x^2}{4J_x^2} + \frac{U_y^2}{4J_y^2} + \frac{J_x + J_y}{F}, \tag{8.2}$$

$$k_x = \frac{U_y}{2J_y} = a_x + \beta_x, \quad k_y = \frac{U_x}{2J_x} = a_y + \beta_y, \tag{8.3}$$

where U_x and U_y are geometrical characteristics representing the higher order moments of inertia calculated from equations (1.7).

We may therefore conclude that the phenomenon of buckling under eccentric applications of a longitudinal force can appear in both compression and extension. Under compression the beam can buckle with any position of the force in the cross section. In the case of extension buckling can set in only if the longitudinal force is applied outside the stability domain. This domain is a circle independent of the form of the beam cross section.

In future we shall call the stability domain bounded by the circle (8.1) the stability circle. In the most general case the radius of the stability circle and the position of its center are determined by equations (8.2) and (8.3).

If the cross section of the beam has one axis of symmetry, the center of the stability circle lies on it. For a section with two axes, the geometrical characteristics U_x and U_y vanish, as seen from equation (1.7)*. Equation (8.1) will have the form

$$e_x^2 + e_y^2 - \frac{J_x + J_y}{F} = 0.$$

In this case, the radius of the stability circle is equal to the radius of gyration.

§ 9. Stability of a rectangular strip

As a first example we examine a narrow rectangular plate having hinged ends and loaded in the middle plane by a longitudinal compressive (or tensile) force. Let d and δ be, respectively, the width and thickness of the plate cross section. The principal axes coincide with the symmetry axes of the section (Figure 161) so that

$$a_x = a_y = \beta_x = \beta_y = 0.$$

Since in the present case the section consists of a single narrow rectangle, we obtain for the sectorial moment of inertia the value 0,

$$J_\omega = 0.$$

* This follows from the fact that the integrands for symmetrical sections in equations (1.7) are odd functions (being the product of an even function $\rho^2 = x^2 + y^2$ and an odd function $x = x(s)$ or $y = y(s)$. The integrals of odd functions vanish over a symmetrical domain, such as the area bounded by a skew-symmetrical curve.

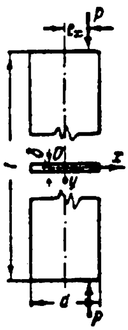


Figure 161

The fact that the characteristic J_ω vanishes shows that only tangential stresses, forming a closed circulation and corresponding to pure torsion, appear in the cross sections of a thin strip in torsion. A uniform distribution of tangential stresses over the section appears only upon bending of the plane.

The differential stability equations for the strip under an eccentric longitudinal force in the middle plane are obtaintained from the general equations (6.2), if we take there $a_x = a_y = e_y = J_\omega = \beta_x = \beta_y = 0$:

$$\left.\begin{aligned} EJ_y\xi^{\mathrm{IV}} + P\xi'' &= 0, \\ EJ_x\eta^{\mathrm{IV}} + P\eta'' + e_x P\theta'' &= 0, \\ e_x P\eta'' + (Pr^2 - GJ_d)\theta'' &= 0. \end{aligned}\right\} \tag{9.1}$$

The first equation determines the critical state of the strip due to bending in the plane Oxz. This equation is approximate since, upon applying a longitudinal force with eccentricity e_x, the strip is bent and loses stability. The additional deflections ξ lie in the plane of the basic bending equilibrium shape. The second and third equations (9.1), together with the boundary conditions, determine the flexural-torsional instability shape. The critical forces, corresponding to these shapes, are determined from equation (6.4) which assumes a simpler form for the given section and loading,

$$\begin{vmatrix} P_x - P & -Pe_x \\ -Pe_x & (P_\omega - P)r^2 \end{vmatrix} = 0, \tag{9.2}$$

where

$$P_x = EJ_x\lambda^2, \quad P_\omega = \frac{1}{r^2}GJ_d.$$

Solving equation (9.2) for P, we find

$$P = \frac{1}{2\left(1 - \frac{e_x^2}{r^2}\right)}(P_x + P_\omega)\left[1 \pm \sqrt{1 - 4\left(1 - \frac{e_x^2}{r^2}\right)\frac{P_xP_\omega}{(P_x + P_\omega)^2}}\right]. \tag{9.3}$$

Equation (9.3) expresses the critical force P as a function of the coordinate e_x which determines the point of application of this force on the axis Ox.

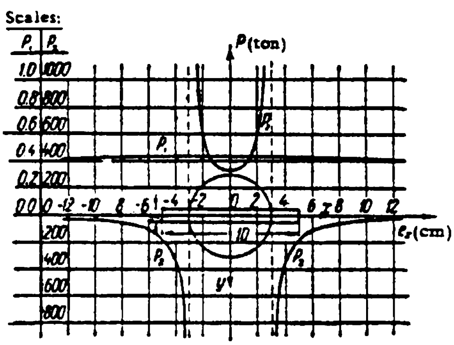


Figure 162

Keeping the eccentricity e_x fixed in this equation, we obtain two values for the critical force corresponding, for a given sinusoidal variation of the displacements η and θ, to two flexural-torsional modes of buckling.

Figure 162 shows graphs of the critical forces, calculated from equation (9.3) as functions of the eccentricity. These graphs were constructed for a rectangular steel strip having the following dimensions: $d = 10$ cm, $\delta = 1$ cm, $l = 200$ cm. The elastic moduli

were taken as

$$E = 2.1 \cdot 10^6 \text{kg/cm}^2,$$
$$G = 0.84 \cdot 10^6 \text{kg/cm}^2.$$

The ends of the strip are hinged so that there can be no deflection nor torsion. The critical forces are given on these graphs in tons; the eccentricities e_x are in centimeters. The smallest critical force P_1 will be compressive (positive). This force is shown in Figure 162 by a continuous symmetrical curve P_1. The curve marked by the letter P_2 refers to the second critical point. This force is negative (tension) outside the stability circle. In Figure 162 the scales of P_1 and P_2 are different.

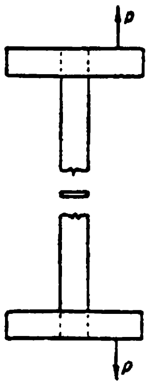


Figure 163

When the eccentricity e_x is decreased, i. e. when the point of application of the force approaches the stability circle, the critical tensile force P_2 increases. On the boundray of the stability circle this force becomes infinite, has a discontinuity and changes sign. It must be noted that the critical tensile forces exceed the compressive many times over.

This phenomenon of stability loss under eccentric compression or tension can be demonstrated on an ordinary wooden T-square having cross-pieces at the ends (Figure 163).

§ 10. Stability of a T-beam

We examine a T-beam having a single axis of symmetry in its cross section (Figure 164). The shear center is situated at the intersection of the axes of the web and the flange of the T-beam. Taking the direction of the axis Ox along the axis of the web we obtain

$$a_y = 0.$$

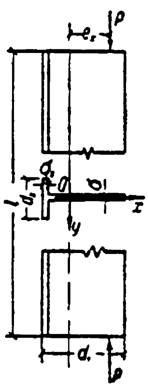


Figure 164

The coordinate a_x is the distance between the centroid of the section and the point where the web joins the flange. For the direction of the axis Ox shown in Figure 164, this coordinate is negative.

In the present case the sectorial moment of inertia also vanishes:

$$J_\omega = 0.$$

The stability equation for plane bending under the action of the force on Ox (i. e. for $e_y = 0$) has the following form

$$EJ_y\xi^{IV} + P\xi'' = 0,$$
$$EJ_x\eta^{IV} + P\eta'' - (a_x - e_x)P\theta'' = 0,$$
$$-(a_x - e_x)P\eta'' + [P(r^2 + 2\beta_x e_x) - GJ_d]\theta'' = 0.$$

The equation for determining the critical forces when the beam is hinged at both ends will be

$$\begin{vmatrix} P_x - P & P(a_x - e_x) \\ P(a_x - e_x) & P_\omega r^2 - P(r^2 + 2\beta_x e_x) \end{vmatrix} = 0. \qquad (10.1)$$

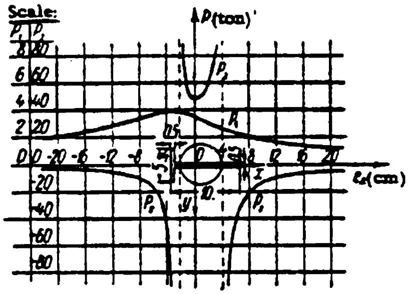


Figure 165

Figure 165 shows the graphs of the critical tension and compression forces, plotted for a T-beam having the following dimensions: $d_1 = 10$ cm, $d_2 = 5$ cm, $\delta_1 = \delta_2 = 0.5$ cm, $l = 167$ cm. The elastic moduli were taken as

$$E = 2.1 \cdot 10^6 \text{kg/cm}^2, \quad G = 0.84 \cdot 10^6 \text{kg/cm}^2.$$

§ 11. Stability of a compressed chord (of box-like section) of a railway bridge

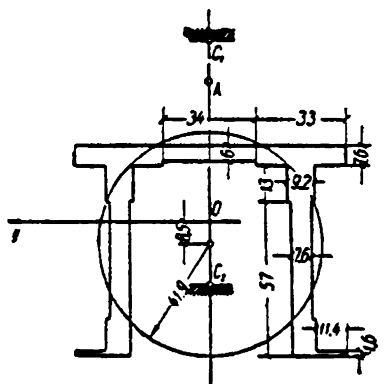


Figure 166

We examine now a box-like beam forming the compressed chord of a bridge. The dimensions of the cross section of the beam are shown in Figure 166. From these dimensions the following values are obtained for the geometrical characteristics:

$$J_x = 1.943 \cdot 10^6 \text{ cm}^4,$$
$$J_y = 1.173 \cdot 10^6 \text{ cm}^4,$$
$$J_d = 0.03557 \cdot 10^6 \text{ cm}^4,$$
$$J_\omega = 1747 \cdot 10^6 \text{ cm}^6.$$

Introducing these data in equation (10.1) and taking, for structural steel, $E = 2.1 \cdot 10^6$ kg/cm^2, $G = 0.84 \cdot 10^6$ kg/cm^2, we obtain an equation for the following three quantities: the critical force P, the length l and the eccentricity e_x (the force is assumed to be applied on the symmetry axis). Assigning in this equation various values to the length l of the beam we obtain a series of graphs representing the critical forces P as a function of the eccentricity. The graphs of the compressive design forces P_1 for beam spans of 8, 12, 16, 20 m are drawn in Figure 167. The graph of the tensile force P_2, calculated for $l = 8$ m, is drawn in this figure below the abcissa axis (the axis of the eccentricities). The critical forces are given in the graphs as fractions of the linear stiffness EF.

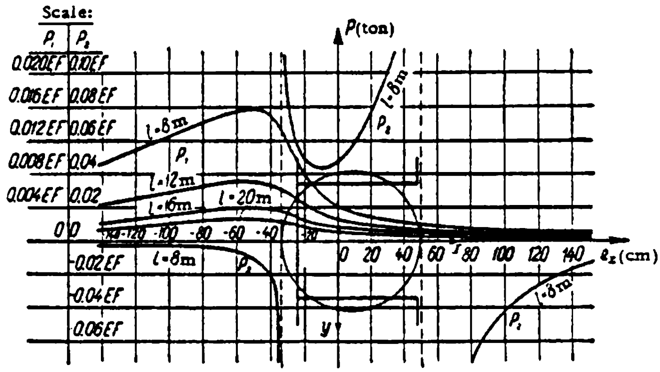


Figure 167

For the stability circle of the given box section we obtain

$$R=\sqrt{\frac{J_x+J_y}{F}+\frac{U_y^2}{4J_y^2}}=41.93\text{cm},\quad k_x=\frac{U_y}{2J_y}=8.475\text{cm}.$$

It should be noted that as in the previous case the critical force P_1 reaches at the shear center its largest value (equal to the Euler force). When the force is applied at the centroid of the section, the critical value of the compressive force is considerably smaller than the Euler force.

The elastic stability in the plane of symmetry of the beam is thus considerably increased if this force is applied at the shear center. If the force is transmitted to the shear center, the flexural-torsional shape passes into that of pure bending, since in this case the center of rotation c_x moves to infinity. The cross sections are progressively displaced in a direction perpendicular to the axis of symmetry.

The Euler buckling in the plane Oyz is thus a particular case of the flexural-torsional shape when the force is applied not at the centroid of the section but at the shear center.

§ 12. Stability of plane bending under pure bending

Assuming, in equations (1.10),

$$P=0,\quad M_x=M\sin\alpha,\quad M_y=-M\cos\alpha,$$

where M is the bending moment acting in a plane at an angle α with the axis Ox (Figure 168), we obtain

$$\left.\begin{aligned} EJ_y\xi^{\text{IV}}+M\sin\alpha\cdot\theta''=0,\\ EJ_x\eta^{\text{IV}}-M\cos\alpha\cdot\theta''=0,\\ M\sin\alpha\cdot\xi''-M\cos\alpha\cdot\eta''+EJ_\omega\theta^{\text{IV}}-\\ -[2(\beta_x\cos\alpha+\beta_y\sin\alpha)M+GJ_d]\theta''=0.\end{aligned}\right\}\qquad(12.1)$$

The differential equations (12.1) refer to the case of stability loss under pure bending. The magnitude of the bending moment M represents the parameter of the external load. The critical values of this moment are

determined by the system of differential equations (12.1) and the conditions of constraint at the ends of the beam. If the ends of the beam are fixed so that displacements ξ, η and the angles of rotation θ are eliminated, and if the additional stresses σ, corresponding to the required deformed state, vanish at these ends the critical value of the moment is determined from the determinant of the equation (2.4) in which it is necessary to set $P=0$, $M_x = M\sin\alpha$, $M_y = -M\cos\alpha$:

$$\begin{vmatrix} P_y & 0 & -M\sin\alpha \\ 0 & P_x & M\cos\alpha \\ -M\sin\alpha & M\cos\alpha & r^2P_\omega + 2M(\beta_x\cos\alpha + \beta_y\sin\alpha) \end{vmatrix} = 0, \quad (12.2)$$

here, as before (§ 3),

$$P_x = EJ_x\lambda^2, \quad P_y = EJ_y\lambda^2, \quad P_\omega = \frac{1}{r^2}(EJ_\omega\lambda^2 + GJ_d) \text{ and } \lambda = \frac{n\pi}{l}.$$

Equation (12.2) was derived for an arbitrary section. If the section has one axis of symmetry (say, the axis Oy) and the moment M acts in the symmetry plane, the following equation results from equation (12.2) for $\beta_x = 0$ and $\alpha = 90°$, in case the bending is no more plane

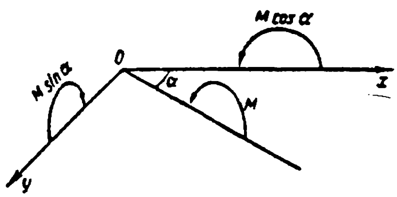


Figure 168

$$\begin{vmatrix} P_y & -M \\ -M & r^2P_\omega + 2\beta_yM \end{vmatrix} = 0.$$

Solving, we obtain an expression for the moment

$$M = P_y\left(\beta_y \pm \sqrt{\beta_y^2 + r^2\frac{P_\omega}{P_y}}\right). \quad (12.3)$$

Equation (12.3) yields for the moment M and for a given value of $\lambda = \frac{n\pi}{l}$ $(n = 1, 2, 3, \ldots)$ two values which differ in magnitude and in sign. As a rule, the smallest moment is obtained for $\lambda = \frac{\pi}{l}$, i.e. in the case of a sinusoidal function with a single half-wave.

If the moment M acts in the plane Oxz, perpendicular to the axis of symmetry of the section, equation (12.2) yields for $\beta_x = 0$ and $\alpha = 0$

$$M = \pm\sqrt{r^2P_xP_\omega}. \quad (12.4)$$

Equations (12.3) and (12.4) can be applied to any open symmetrical section of rigid contour. In particular, for a wide-flanged beam with equal flanges, as follows from sub-sec. 1, § 1, Chapter II,

$$J_\omega = \int_F \omega^2 dF = \frac{J_1d_2^2}{2}, \quad (12.5)$$

where J_1 is the moment of inertia of one flange with respect to the vertical axis y, and d_2 is the depth of the web.

Introducing expression (12.5) in (12.3) and assuming $\lambda = \frac{\pi}{l}$ and $\beta_y = 0$, we obtain

$$M = \frac{\pi}{l}\sqrt{EJ_yGJ_d}\sqrt{1 + \frac{EJ_1d_2^2\pi^2}{2GJ_dl^2}}. \quad (12.6)$$

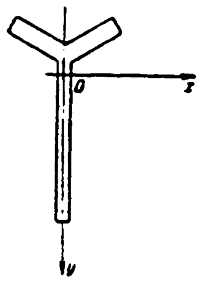


Figure 169

Equation (12.6) is the same as the well known equation first derived by Timoshenko in his dissertation /178/.

Equations (12.3) and (12.4) can be also applied to sections consisting of a radial bundle of plates intersecting in one rib and having Oy for a symmetry axis (Figure 169). Assuming in this case $J_{\omega}=0$, we obtain from equation (12.3)

$$M=\frac{\pi^2}{l^2}EJ_y\left[\beta_y\pm\sqrt{\beta_y^2+\frac{GJ_d l^2}{EJ_y\pi^2}}\right]. \qquad (12.7)$$

Here, as before, the torsional rigidity of the section for pure torsion is denoted by GJ_d. For the particular case of a rectangular strip, equation (12.7) is identical with the well known Prandtl equation /178/.

§ 13. Determination of the critical forces from the end conditions of the beam /68/

1. Equation (2.4), the general equation for the critical forces, refers to a beam having hinged ends that are fixed with respect to all three displacements. These terminal restraints are represented by the boundary conditions (2.1).

Depending on the method of restraint of the supported sections, the boundary conditions for the required functions ξ, η, θ of equations (1.10) may take various forms, different, in general, from the conditions (2.1). Thus, for example, in the case of a beam completely built-in at the ends the boundary conditions will have the form

$$\begin{array}{lll} \text{for } z=0 & \xi=\eta=\theta=0, & \xi'=\eta'=\theta'=0; \\ \text{for } z=l & \xi=\eta=\theta=0, & \xi'=\eta'=\theta'=0. \end{array}$$

If one end of the beam, for example $z=0$, is fully hinged and the other ($z=l$) is built-in, the boundary conditions take the following form:

$$\begin{array}{lll} \text{for } z=0 & \xi=\eta=\theta=0, & \xi''=\eta''=\theta''=0; \\ \text{for } z=l & \xi=\eta=\theta=0, & \xi'=\eta'=\theta'=0. \end{array}$$

Other forms of boundary conditions are also possible. Thus, for example, if the supported section $z=0$ is hinged with respect to the deflections ξ, η, and fixed to restrain torsion, it remains plane (having a rigid stiffening plate). For the other section $z=l$, hinged only with respect to the deflection ξ and rigidly clamped with respect to the other two displacements η, θ, the boundary conditions will be the following

$$\begin{array}{llll} \text{for } z=0 & \xi=\eta=\theta=0, & \xi''=\eta''=0, & \theta'=0; \\ \text{for } z=l & \xi=\eta=\theta=0, & \xi''=0, & \eta'=\theta'=0. \end{array}$$

In all the cases described here of end constraints of the beam (and in similar cases) the critical load for a longitudinal force P and end moments M_x, M_y, is determined by solving the homogeneous differential equations (1.10). These are given up to some load parameter for the corresponding homogeneous conditions (in the given boundary-value problem). The exact

solution of the problem (excluding the case, examined above, of fully hinged ends) involves laborious mathematical calculations connected with the solution of rather cmoplicated transcendental equations.

Therefore, in order to solve the problem of spatial stability of a thin-walled beam for various boundary conditions we shall start with an approximate method based on the application of the principle of virtual displacements to a discrete-continuous system.

Let

$$\xi(z)=A\chi(z),\quad \eta(z)=B\varphi(z),\quad \theta(z)=C\psi(z), \tag{13.1}$$

where $\chi(z)$, $\varphi(z)$, $\psi(z)$ are given functions satisfying all the necessary boundary conditions. There will be four conditions for each of the three functions (two conditions for each end of the beam). As the functions χ, φ, ψ we shall choose the so-called eigenfunctions of the transverse vibrations of the beam, i. e. functions satisfying the homogeneous differential equations*:

$$\left.\begin{aligned}\chi^{IV}-\lambda^4\chi=0,\\ \varphi^{IV}-\mu^4\varphi=0,\\ \psi^{IV}-\nu^4\psi=0,\end{aligned}\right\} \tag{13.2}$$

and the homogeneous boundary conditions which have the same form as the conditions of the examined boundary problem for the stability of a beam. A transcendental characteristic equation is obtained for the parameter λ in the first equation of (13.2). This equation gives an infinite number of solutions for the eigenvalue λ The infinite series of eigenvalues λ correspond to an infinite series of the eigenfunctions χ, which are orthogonal and constitute a complete set. Table 33 gives the eigenfunctions χ for various cases of boundary conditions, together with the corresponding eignevalues. From the complete set of eigenfunctions χ we choose one function corresponding to the first harmonic. The eigenvalue λ for this function is the smallest and corresponds to the ground state of vibration.

* The differential equation for free vibration of a beam has the following form

$$EJ\frac{\partial^4 w}{\partial z^4}+\rho S\frac{\partial^2 w}{\partial t^2}=0.$$

In the present case the deflection function $w(z, t)$ depends on the longitudinal coordinate of the beam z and the time t; EJ is the rigidity of the beam; ρS is the mass of the beam per unit length. In order to solve the vibration equation we use Fourier's method, i. e. we write the deflection function $w(z, t)$ in the following way

$$w(z,t)=Z(z)\,T(t).$$

Introducing the expression for w in the vibration equation, we obtain

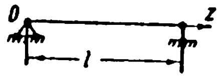
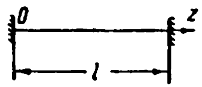
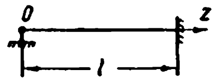
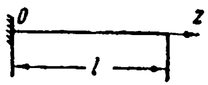
$$EJZ^{IV}T+\rho SZT''=0 \quad \text{or} \quad \frac{Z^{IV}}{Z}=-\frac{\rho S}{EJ}\frac{T''}{T}=\lambda^4,$$

where λ is a constant quantity since the left part of the last equation depends only on z, the right part only on t. For the functions $Z(z)$ and $T(t)$ we obtain the following equations

$$Z^{IV}-\lambda^4 Z=0,\quad T''+\lambda^4\frac{\rho S}{EJ}T=0.$$

The first equation determines the form of the transverse vibrations of the beam. For a more detailed examination of these functions see the author's work /51/.

Table 33

Number	Boundary conditions			Eigenfunctions		Characteristic equation	Roots of the characteristic equation (eigenvalues)		
	Scheme	$z=0$	$z=l$	χ	a		λ_1	λ_2	$\lambda_n (n>2)$
1		$\chi=0$ $\chi''=0$	$\chi=0$ $\chi''=0$	$\sin\frac{\lambda_n z}{l}$		$\sin\lambda_n=0$	π	2π	$n\pi$
2		$\chi=0$ $\chi'=0$	$\chi=0$ $\chi'=0$	$\sin\frac{\lambda_n z}{l}-\mathrm{sh}\frac{\lambda_n z}{l}-a\left(\cos\frac{\lambda_n z}{l}-\mathrm{ch}\frac{\lambda_n z}{l}\right)$	$\frac{\mathrm{sh}\lambda_n-\sin\lambda_n}{\mathrm{ch}\lambda_n-\cos\lambda_n}$	$\mathrm{ch}\lambda_n\cos\lambda_n=1$	4.730	7.8532	$\frac{2n+1}{2}\pi$
3		$\chi=0$ $\chi''=0$	$\chi=0$ $\chi'=0$	$\sin\frac{\lambda_n z}{l}+a\,\mathrm{sh}\frac{\lambda_n z}{l}$	$-\frac{\sin\lambda_n}{\mathrm{sh}\lambda_n}$	$\mathrm{tg}\,\lambda_n=\mathrm{th}\,\lambda_n$	3.9266	7.0685	$\frac{4n+1}{2}\pi$
4		$\chi=0$ $\chi'=0$	$\chi''=0$ $\chi'''=0$	$\sin\frac{\lambda_n z}{l}-\mathrm{sh}\frac{\lambda_n z}{l}-a\left(\cos\frac{\lambda_n z}{l}-\mathrm{ch}\frac{\lambda_n z}{l}\right)$	$\frac{\mathrm{sh}\lambda_n+\sin\lambda_n}{\mathrm{ch}\lambda_n+\cos\lambda_n}$	$\mathrm{ch}\lambda_n\cos\lambda_n=-1$	1.8751	4.6941	$\frac{2n-1}{2}\pi$

The same considerations can be repeated for the second and third equation of (13.2). Thus, the parameters λ, μ, ν in equations (13.2) serve as quantities characterizing the eigenfrequencies in the general case of three different beams, i.e. beams of different constraint conditions at the ends of the span l. Since we examine here boundary conditions which represent various combinations of rigid and hinged supports, of the eigenfunctions of the transverse vibrations of the beam we only need those corresponding to the three different cases of boundary conditions, viz.

1) Both ends hinged;
2) Both ends rigidly clamped;
3) One end hinged and the other rigidly clamped*.

We turn now to the basic stability equations (1.10). These equations express the equilibrium conditions, viz. all forces as well as all moments acting on an elementary transverse strip of unit width make up zero. Multiplying, for example, the first equation (1.10) by the function $\chi(z)\,dz$, and considering (13.1), we obtain the work done by all forces acting on our transverse strip of width dz in the direction of the axis Oz in their corresponding virtual displacements $\chi(z)$. Since we cannot satisfy the condition of this work vanishing in each cross section of the beam (the functions $\chi(z)$, $\varphi(z)$ and $\psi(z)$ are not integrals of equations (1.10)), we have to require this condition to be satisfied only in the integral form, using the Bubnov-Galerkin variational method with the exact fulfillment of the geometrical and statical conditions at the ends of the beam.

Thus, introducing (13.1) in equation (1.10), multiplying, in accordance with the physical meaning, the first of these equations by $\chi\,dz$, the second by $\varphi\,dz$, and the third by $\psi\,dz$, then integrating each of the three expressions thus obtained over the whole length of the beam l and, applying now the principle of virtual displacements, equating to zero the result in each of the three cases, we obtain, after using equations (13.2) and transforming the two relations**

$$\int_0^l \chi\chi''\,dz = -\int_0^l (\chi')^2\,dz,$$

$$\int_0^l \chi\varphi''\,dz = -\int_0^l \chi'\varphi'\,dz$$

and the four analogous relations for φ and ψ, for A, B, C a system of three linear homogeneous equations which can be written

$$\left.\begin{aligned}
&\left(k_{11}EJ_y\frac{\pi^2}{l^2}-P\right)A-k_{13}(a_yP+M_x)C=0,\\
&\left(k_{22}EJ_x\frac{\pi^2}{l^2}-P\right)B+k_{23}(a_xP-M_y)\,C=0,\\
&-k_{31}(a_yP+M_x)A+k_{32}(a_xP-M_y)B+\\
&+\left\{k_{33}EJ_\omega\frac{\pi^2}{l^2}-[r^2P+2(\beta_xM_y-\beta_yM_x)-GJ_d]\right\}C=0.
\end{aligned}\right\}\qquad(13.3)$$

* All our considerations are also valid for beams with free ends.

** These equalities are obtained by integrating by parts, taking into account the fact that the functions χ, φ, ψ satisfy the geometrical conditions at the ends of the beam. Thus, for example

$$\int_0^l \chi\chi''\,dz = \chi\chi'\Big|_0^l - \int_0^l (\chi')^2\,dz = -\int_0^l (\chi')^2\,dz.$$

Coefficents of reduced moments of inertia and eccentricities for

Boundary conditions	for $z = +\frac{l}{2}$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta'' = 0$ $\theta = 0;\ \theta'' = 0$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta'' = 0$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta'' = 0$	3 $\xi = 0;\ \xi' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta'' = 0$
	for $z = -\frac{l}{2}$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta'' = 0$ $\theta = 0;\ \theta'' = 0$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta'' = 0$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta'' = 0$ $\theta = 0;\ \theta'' = 0$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta'' = 0$ $\theta = 0;\ \theta'' = 0$
Coefficients	k_{11}	1	1	1	2.092
	k_{22}	1	4.1223	2.092	2.092
	k_{33}	1	1	1	1
	$k_{31} = k_{13}$	1	1	1	0.9041
	$k_{32} = k_{23}$	1	0.8834	0.9041	0.9041

Boundary conditions	for $z = +\frac{l}{2}$	4 $\xi = 0;\ \xi' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta' = 0$	4 $\xi = 0;\ \xi' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta' = 0$	4 $\xi = 0;\ \xi' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta' = 0$	2 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta'' = 0$ $\theta = 0;\ \theta' = 0$
	for $z = -\frac{l}{2}$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta'' = 0$ $\theta = 0;\ \theta'' = 0$	1 $\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta'' = 0$	3 $\xi = 0;\ \xi' = 0$ $\eta = 0;\ \eta' = 0$ $\theta = 0;\ \theta'' = 0$	$\xi = 0;\ \xi'' = 0$ $\eta = 0;\ \eta'' = 0$ $\theta = 0;\ \theta' = 0$ 2
Coefficients	k_{11}	2.092	2.092	4.1223	1
	k_{22}	2.092	4.1223	4.1223	1
	k_{33}	2.092	2.092	2.092	4.1223
	$k_{31} = k_{13}$	1	1	0.8746	0.8834
	$k_{32} = k_{23}$	1	0.8746	0.8746	0.8834

Table 34

the spatial stability of a beam under various boundary conditions

3 $\xi=0;\ \xi'=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta''=0$	3 $\xi=0;\ \xi'=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta''=0$	2 $\xi=0;\ \xi''=0$ $\eta=0;\ \eta''=0$ $\theta=0;\ \theta'=0$	$\xi=0;\ \xi''=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$ 2	$\xi=0;\ \xi''=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$ 2
1 $\xi=0;\ \xi''=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta''=0$	3 $\xi=0;\ \xi'=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta''=0$	3 $\xi=0;\ \xi'=0$ $\eta=0;\ \eta''=0$ $\theta=0;\ \theta''=0$	$\xi=0;\ \xi''=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta''=0$ 1	$\xi=0;\ \xi''=0$ $\eta=0;\ \eta''=0$ $\theta=0;\ \theta''=0$ 1
2.092	4.1223	1	1	1
4.1223	4.1223	1	4.1223	2.092
1	1	2.092	2.092	2.092
0.9041	0.8834	0.9041	0.9041	0.9041
0.8834	0.8834	0.9041	0.8746	1

Continuation

2 $\xi=0;\ \xi''=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$	2 $\xi=0;\ \xi''=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$	4 $\xi=0;\ \xi'=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$	4 $\xi=0;\ \xi'=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$	4 $\xi=0;\ \xi'=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$
2 $\xi=0;\ \xi''=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$	2 $\xi=0;\ \xi''=0$ $\eta=0;\ \eta''=0$ $\theta=0;\ \theta'=0$	2 $\xi=0;\ \xi''=0$ $\eta=0;\ \eta''=0$ $\theta=0;\ \theta'=0$	2 $\xi=0;\ \xi''=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$	4 $\xi=0;\ \xi'=0$ $\eta=0;\ \eta'=0$ $\theta=0;\ \theta'=0$
1	1	2.092	2.092	4.1223
4.1223	2.092	2.092	4.1223	4.1223
4.1223	4.1223	4.1223	4.1223	4.1223
0.8834	0.8834	0.8746	0.8746	1
1	0.8746	0.8746	1	1

here $k_{11}, k_{12}, \ldots, k_{33}$ are dimensionless quantities depending only on the boundary conditions and determined in the general case from the equations

$$k_{11}=\frac{l^2}{\pi^2}\lambda^4\frac{\int\limits_0^l \chi^2 dz}{\int\limits_0^l (\chi')^2 dz}, \quad k_{22}=\frac{l^2}{\pi^2}\mu^4\frac{\int\limits_0^l \varphi^2 dz}{\int\limits_0^l (\varphi')^2 dz}, \quad k_{33}=\frac{l^2}{\pi^2}\nu^4\frac{\int\limits_0^l \psi^2 dz}{\int\limits_0^l (\psi')^2 dz},$$

$$k_{13}=k_{31}=\frac{\int\limits_0^l \chi'\psi' dz}{\sqrt{\int\limits_0^l (\chi')^2 dz \int\limits_0^l (\psi')^2 dz}}, \quad k_{23}=k_{32}=\frac{\int\limits_0^l \varphi'\psi' dz}{\sqrt{\int\limits_0^l (\varphi')^2 dz \int\limits_0^l (\psi')^2 dz}}.$$

Table 34 gives the values of these coefficients for various cases of boundary conditions (the upper rows of the table).

Equating to zero the determinant of equations (13.3) and introducing the notations

$$\left.\begin{aligned} P_x^* &= k_{22}\,EJ_x\frac{\pi^2}{l^2}, \\ P_y^* &= k_{11}\,EJ_y\frac{\pi^2}{l^2}, \\ P_\omega^* &= \frac{1}{r^2}\left(k_{33}\,EJ_\omega\frac{\pi^2}{l^2} + GJ_d\right), \end{aligned}\right\} \tag{13.4}$$

we obtain

$$\begin{gathered}(P_x^*-P)(P_y^*-P)[r^2(P_\omega^*-P)+2(\beta_y M_x-\beta_x M_y)]- \\ -k_{13}^2(P_x^*-P)(a_yP+M_x)^2-k_{23}^2(P_y^*-P)(a_xP-M_y)^2=0.\end{gathered} \tag{13.5}$$

Equations (13.4) determine the critical forces: Euler's P_x^*, P_y^* and the torsional forces P_ω^*. The quantities k_{11}, k_{22}, k_{33} (which enter these equations) having the values given in Table 34, allow the determination of the forces P_x^*, P_y^*, P_ω^* for the most varied end constraints of the beam. For the case of a fully hinged end of the beam the coefficients k_{11}, k_{22}, k_{33} become equal to one, and equations (13.4) pass into equations (3.3) and (3.4).

With equations (13.4) and the coefficients $k_{11}, k_{22}, \ldots, k_{33}$ given in Table 34 equation (13.5) is the general equation for the critical forces and moments and allows the design for stability of a thin-walled beam for most diverse conditions as given by the elastic characteristics of the material of the beam E, G, and by the geometrical characteristics J_x, J_y, J_ω, J_d, a_x, a_y, β_x, β_y, r^2. These in turn depend on the form and dimensions of the cross section of the beam, on the length of the beam l, on the statical quantities P, M_x, M_y, which determine up to one parameter the load that gives rise to purely normal stresses only up to the moment of buckling. Finally the conditions are given by the quantities $k_{11}, k_{22}, \ldots, k_{33}$ depending only on the boundary conditions. By using the values of the coefficients $k_{11}, k_{22}, \ldots, k_{33}$ given in Table 34 and by varying the geometrical and statical parameters enumerated above, we can, with the help of the single equation (13.5), attack a very great number of practically important problems concerning the stability of a thin-walled beam as a shell with a rigid contour loaded at the ends either by the force P only or by the bending moment M only, etc.

For $k_{11} = k_{22} = k_{33} = k_{13} = k_{23} = 1$ equation (13.5) passes into equation (2.4) for the case of hinges at both ends of the beam. It follows that the design of a beam for other boundary conditions, for example in the case of an eccentric application of a longitudinal force P, can be carried out on the basis of equations given in previous sections and referring to the fundamental case of the indicated conditions (2.1). It is only necessary to replace in all these equations the critical forces P_x, P_y, P_ω by the forces P_x^*, P_y^*, P_ω^*, determined by equations (13.4) for these boundary conditions, and to replace the relative eccentricities $e_x - a_x$, $e_y - a_y$ by the reduced eccentricities $k_{23}(e_x - a_x)$ and $k_{13}(e_y - a_y)$.

It should be mentioned that instead of the vibration functions the beam functions (flexures) can be chosen which are due to a uniformly distributed statical load corresponding to the conditions for each of the three displacements examined in our problem. In other words, each of the three functions can be given (up to an arbitrary factor) in the form of a fourth degree polynomial. The coefficients of the polynomial must be chosen for each of the functions χ, φ, ψ separately, in accordance with the boundary conditions.

2. The above variational method is easily extended to thin-walled struts reinforced by transverse bimoment connections. If these connections are in the form of transverse bars with their axes situated in the sections $L = L_i$ $(i = 1, 2, 3, \ldots, n)$ the quantity GJ_d in the last of equations (13.3) is to be taken as the reduced torsional rigidity determined by the equation

$$\overline{GJ}_d = GJ_d + \sum_{i=1,2}^{n} b_i \int_0^l \chi_i''dx,$$

where b_i is the reduced rigidity of the bimoment connection referring to the i-th bar and calculated by the methods described before; χ' is the value of the derivative of the function $\chi_i(z)$ at $z = z_i$.

§ 14. Experimental verification of the theory on structural and aircraft metal beams

Experimental studies for the verification of the general theory of spatial stability in thin-walled struts under axial and eccentric compression were conducted at the Laboratory of Construction Engineering of the Central Scientific Research Institute of Industrial Constructions under the supervision of the author.

1. Axial compression*. Welded metallic test pieces of 5-millimeter St 3 sheet steel of two types (box- and T-sections) were designed and built at the workshops of the TsNIPS. Three specimens of each section were prepared (Figure 170).

In order to avoid deformation of the section contour and the loss of stability, 3-millimeter steel ribs were fastened along the piece at 250 mm intervals, having dimensions such that as far as possible they did not prevent longitudinal deformations in torsion.

* The results of experimental studies on the spatial stability of thin-walled metal beams under axial compression were obtained by N. G. Dobudoglo, Candidate of Technical Sciences.

The dimensions of the specimens were chosen with the aim that flexural-torsional buckling should occur in the elastic range.

The mechanical characteristics of the metal were determined on corresponding test pieces cut from the same sheet of steel: the strength limit $\sigma_{sl} = 4{,}050$ kg/cm^2, the yield point $\sigma_y = 2{,}870$ kg/cm^2, percentage elongation $\varepsilon = 23\%$, Young's modulus $E = 2.14 \cdot 10^6$ kg/cm^2; the shear modulus G was taken equal to $0.4\,E$.

The boundary conditions in the experiments correspond to hinged ends. The fulfillment of these requirements under the experimental conditions did not present any difficulties since the bearing supports on which the test pieces were placed had spherical or cylindrical hinges. A great difficulty arose in the practical fulfillment of another requirement resulting from the theoretical premises underlying the design of the sections, viz. of no normal stresses at the supported sections. In other words, to preserve the condition of nondeformability of the test piece contour in the plane of the cross section it was necessary to permit free warping (through torsion) for separate elements of the supported sections.

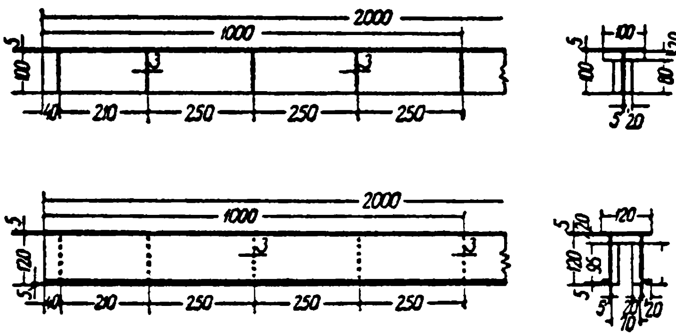


Figure 170

For the T-profile, consisting of just two elements (web and flange), the requirement of free warping of its supported section was automatically fulfilled due to the direct support of the beam by the bearing cushion since for sections consisting of straight elements intersecting at one point, the sectorial moment of inertia, J_ω (which governs warping) equals zero.

In order to fulfill the requirement for a box section special support arrangements consisting of interchangeable clamps with steel balls were designed and built separately for each element of the section (Figure 171). The upper and lower ends of the test piece rested on three steel balls, each at the centroid of its element. Consequently, the resultant of the pressure on the ball passed through the centroid of the supported section of the specimen. Such mounting permitted individual elements of the supported section free warping by rotating about the ball as a center. Since the balls were placed symmetrically with respect to the plane of symmetry of the beam there was no initial bimoment load and the beam was not in torsion until loss of stability.

The testing was carried out with 500-ton press of the a "MAN" type [Latin characters in Russian text]. The theoretically calculated centroid of the section was made to coincide with the center of the support cushions

of the press. The piece rested, as shown above, either directly on the cushion (the T-section), or on the steel balls (the box section).

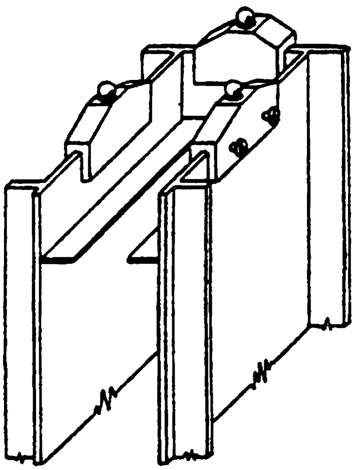


Figure 171

During testing measurements were taken at three sections (at heights of 1/4, 1/2 and 3/4 l): 1) of the horizontal deflections (by Maksimov deflection meters) in two directions, 2) of fiber deformations (by Hugenberger strain gauges), 3) of the rotation of the section in its plane. In order to measure the deflections and the rotations of the section, three deflection meters were installed: one of them determined the deflection in the direction of the symmetry axis, the second and third determined the deflections in directions perpendicular to this axis, indicating the rotation of the section. The position of the section in space was determined from the readings of the deflection meters.

The behavior of the specimens under load depended on their straightness and on the form of the section. Box-sections, having a stronger section, buckled gradually, slowly bending and twisting with the approach of the critical state. Specimens having a T-section seemed to lose stability abruptly, sometimes with a noticeably sharp snap.

Sometimes the initial deflection reversed its sign near the moment of buckling. This phenomenon was caused by the twisting of the beam observed in all the specimens. Only in two cases did twisting appear right from the beginning of compression. In all the other specimens it became noticeably only with the approach of the critical state.

The results of the experiments confirm that all the tested specimens buckled in the flexural-torsional mode.

The basic factor for comparing theoretical with experimental results is the magnitude of the critical load: any relation between the experimental and the theoretical magnitude of this load enables to assess the accuracy of the theoretical method of determining the critical force corresponding to flexural-torsional buckling.

In addition to this fundamental verification, yet another theoretical conclusion was experimentally verified, which bears directly on the phenomenon of beam torsion, and in particular on the properties of the center of rotation and the axis of rotation.

It was shown above that under torsion separate sections of a beam rotate about the center of rotation determined by equation (4.4), and that the coordinates of this center do not depend on the position of the section along the height of the beam on the z-axis, i.e. the beam as a whole rotates about some axis parallel to the beam axis. Both these statements were verified experimentally.

The theoretical critical load was determined by equation (3.9). The elastic and geometrical characteristics in this equation, and also the values of the coefficients of the equation for the box and T-sections are given in Table 35.

Table 35

Section	Characteristics							
	E kg/cm^2	G kg/cm^2	l cm	$\lambda = \frac{\pi}{l}$ cm^{-1}	F cm^2	J_y cm^4	J_x cm^4	J_d cm^4
Box-like	$2.14 \cdot 10^6$	$0.856 \cdot 10^6$	198.4	0.01583	19.6	365.5	285.2	1.83
T	$2.14 \cdot 10^6$	$0.856 \cdot 10^6$	199.2	0.01516	9.7	101.2	38.13	1.12

Section	Characteristics							
	Slenderness ratio $\frac{l}{r_x}$	I_ω cm^6	Centroid x_0 cm	a_x cm	$r^2 = \frac{J_x + J_y}{F} + a_x^2$ cm^2	$\frac{J_x + J_y}{F}$ cm^2	$r^2(P_x + P_\omega)$ kg·cm^2	$r^2 P_x P_\omega$ kg·cm^2
Box-like	52.0	3931	4.83	9.375	121.18	33.29	$22.168 \cdot 10^6$	$0.5568 \cdot 10^{12}$
T	100.0	0	2.80	2.80	20.86	14.36	$1.1934 \cdot 10^6$	$0.01457 \cdot 10^{12}$

The following values for the critical loads for a box and T-section were obtained from the data of this Table:

1) for a box-like section $P_1 = 26{,}140$ kg, $P_2 = 639{,}800$ kg;

2) for T-section $P_1 = 14{,}850$ kg, $P_2 = 68{,}250$ kg.

The smaller values of these forces are the critical design forces.

The critical stresses for the box and T-section corresponding to these forces are equal to 1,335 and 1,530 kg/cm^2 respectively.

The experimental critical stessres corresponding to the experimental critical loads, given in Table 36, vary within the limits of 1,33C-1,440 kg/cm^2.

It follows that, according to the theory described and to the experiments, the phenomenon of flexural-torsional buckling takes place in the elastic range.

We obtain a different picture if we examine our beams from the point of view of the usual theory and if we consider that they buckle by bending and without torsion.

It is, for example, already clear from Table 35 that a box-like piece, whose slenderness-ratio* is equal to 52, should buckle beyond the elastic limit.

Indeed, its smallest critical Euler force, calculated under the assumption of a constant modulus E, equals 152,88 t, and the corresponding critical stress 7,810 kg/cm^2. As these values are fictitious, we change them into realsitic ones by multiplying the ratio of the variable modulus T** to the constant modulus E. According to the design norms of metal structures this ratio can be taken equal to 0.36 for St 3 steel and a slenderness-ratio of 52.

The critical force and stress then equal, for the case of buckling by bending, respectively

$$P_{cr} \approx 55\text{ton}, \sigma_{cr} \approx 2800\,\text{kg/cm}^2,$$

i.e. the beam buckles at the yield point.

For a T-section Euler buckling takes place in the elastic range but for a larger critical force, equal to 18.75 tons.

Table 36

Section	Critical loads P, ton						Ratio of average experimental critical load to that calculated by the author's theory
	theoretical		experimental				
	Euler value	by the author's theory	specimen No 1	specimen No 2	specimen No 3	average of the three	
Channel	55.0	26.14	24.7	26.0	26.0	25.57	0.98
T	18.75	14.85	-	-	14.0	14.0	0.94

* In the theory of strength of materials slenderness-ratio designates the ratio of the length of the beam or column to its radius of gyration. In bending in the plane Oxz the radius of gyration is determined by the relation $r_x^2 = \frac{J_y}{F}$.

** Here T is the so-called reduced modulus of elasticity depending on the modulus of elasticity E, on the variable modulus (beyond the elastic limit) $E_\sigma = \frac{d\sigma}{d\varepsilon}$, and on the dimensions of the section.

Table 36 shows the theoretical and experimental results. It is seen from this table that, first, the critical Euler forces differ considerably from the experimental and the theoretical ones found by our method. The disparity is especially large in the case of a box section: the critical Euler force is more than twice the real one. However, for a real beam forming part of the structure and having stiffeners in the form of longitudinal or transverse connections reinforcing the open part of the section, the difference in the critical forces, as found by the usual theory of longitudinal buckling and by the theory described here, should considerably decrease since such connections increase the torsional stiffness of the beam.

Secondly, it is seen from the table that the experimental critical loads are very close to the theoretical values calculated by our method. The ratio of the average values of the first to the second for a box section is equal to 0.98, and for a T-section to 0.94. Such agreement can be considered as very good and the experiments thus confirm the theory completely.

The theoretical values of the coordinates c_x and c_y of the center of rotation of separate sections of the beam were determined by equations (4.5). The readings of the deflection meters, which determined the position in space of the walls of the box-like section and of the flange of the T-section at all loading stages including the critical load, were used to determine experimentally these coordinates.

The results of the experimental and theoretical determinations $c_x (c_y = 0)$ and their comparison are given in Table 37.

Table 37

Section	Distance between center of rotation and centroid, c_x (in mm)						
	box-section					T-section	
	experimental results				theoretical values	experimental values	theoretical values
	specimen no 1	specimen no 2	specimen no 3	average of the three			
Top	131	120	94	115	113	75	134
Middle	122	90	115	109	113	120	134
Bottom	111	104	85	100	113	83	134

It is seen from Table 37 that the experimental values of the coordinates of the center of rotation for the box section are very close to the theoretical ones in all three specimens.

The experimental axis of rotation is very near the theoretical axis. This is yet another confirmation of the theory described, not less important than the agreement in the critical load.

We obtained a worse picture for the T-section. Here apparently the production was to blame: the influence of the welding and alignment of the test piece on the clamp, to which such a simple section of only two elements is especially sensitive.

2. Eccentric compression. Welded pieces of steel wide-flange columns with different flanges were tested in the Amsler press*. The pieces were loaded axially and eccentrically. In eccentric loading the longitudinal thrust was transmitted to the column at a point of the web

* These studies were conducted by D. V. Bychkov, Doctor of Technical Sciences.

lying away from the sectional centroid towards the narrow flange. Figure 172 shows a specimen placed in the press, having buckled by flexural torsion under eccentric compression. The experimental studies, conducted on columns with high slenderness-ratio, confirm the theoretical results.

3. Stability of columns reinforced by braces. In order to verify our statement that in the spatial behavior of thin-walled columns and beams of open section an essential role is played by the braces and not by the diaphragms, some of the test specimens were reinforced by braces welded symmetrically to the flanges (with four braces on each side)*. The influence of the braces on the stability of a column as a thin-walled spatial structure was studied experimentally in the following way.

Figure 172

The column test piece initially underwent compression with all the four braces attached. The longitudinal compressive force of the press was raised to the critical value under which the tested specimen reached a new flexural-torsional equilibrium shape.

The flexural-torsion deformation in the critical state was determined by strain gauges and deflection meters placed on the webs and flanges of three sections: the middle and two terminal ones, at a distance of one fourth of the length from the ends of the beam. The value of the eccentricity of the compression force was chosen so as to cause the specimen, of given dimensions of the cross section and elastic characteristics of the material, to buckle in the elastic range. This specimen recovered (even with the braces) its initial unstrained state upon unloading. The measuring instruments indicated that the flexural-torsion deformations observed upon buckling disappeared completely after unloading.

This specimen was then tested again for compression but with only the two extreme braces, preventing the warping at the ends. The corresponding bimoment connections (the two middle braces) were severed. As could be expected, the critical compression force was smaller in this weakened test piece than the one observed in the first case with all four braces in place. After removing the load, the specimen once more recovered its initial unstrained state. The last two extreme braces were then severed. The model of the wide-flange column with the transverse bimoment connections (braces) eliminated was again tested in compression. In this case the buckling occurred also in a spatial flexural-torsional form, but at a smaller critical compressive force than in the first two cases described above.

Figure 173a shows a photograph of the specimen with the braces severed, taken at the moment of buckling, when the initial precritical equilibrium shape, characterized in our case by compression and bending in the symmetry plane of the wide-flange section, passed into the new flexural-torsional shape, a combination of bending away from the symmetry plane of the section and of restrained torsion. At buckling this spatial flexural-torsional equilibrium shape of the column is also associated with warping.

* The studies were conducted by S.I. Stel'makh, Candidate of Technical Sciences.

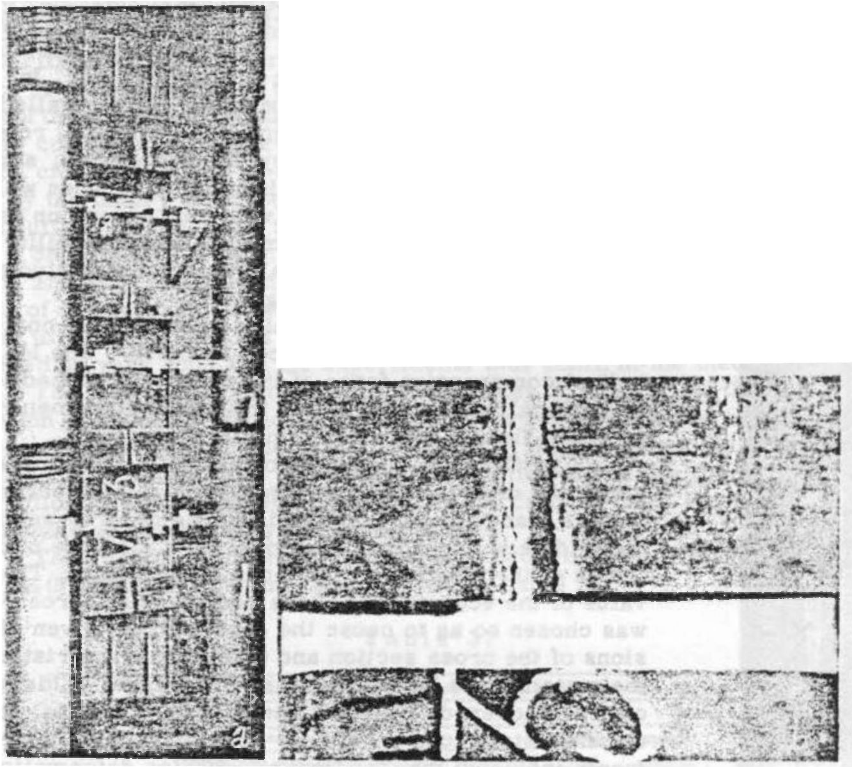


Figure 173

The transverse braces at the cuts undergo, as a result of warping, relative displacement in the longitudinal direction.

Figure 173b shows an enlarged photograph of one of the cut braces (3) of the buckled specimen.

Values of critical forces, calculated theoretically and obtained experimentally by the method described above, are given below.

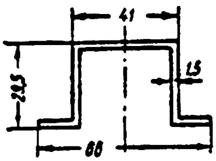
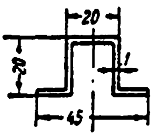
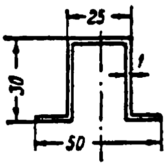
	Column without braces	Column with two braces	Column with four braces
P_{cr} theoretical (in tons)	24.4	30.2	39.3
P_{cr} experimental (in tons)	25.8	29.8	41.0
P_{cr} calculated by the law of plane sections (in tons)	57.7	57.7	57.7

Values of critical forces are also given for comparison, calculated for columns of small slenderness-ratio, taking account of the plastic deformations by the well-known equation of the law of plane sections given in textbooks and standard manuals.

4. Stability of aircraft beams. Besides the above results of experimental work conducted in the Laboratory of Construction Engineering of TsNIPS, we have also used experimental data on the stability of duraluminum aircraft beams obtained in 1935 at the Institute of Dirigible Building and at TsAGI /18/.

The dimensions of aircraft beams in cross section and the results of the corresponding theoretical and experimental studies are shown in Table 38 in which the critical forces obtained from the usual theory of lateral buckling are also given for comparison.

Table 38

Dimensions of the section, mm	No of specimen	Length l, mm	Design critical thrust P, kg		
			Euler value	By the author's method	TsAGI empirical data
	1	1500	3007	900	920
	2	1300	4398	1174	1150
	3	1000	6766	1637	1400
	1	1000	1423	421	400
	2	750	2531	654	630
	3	500	5694	1318	1200
	1	1000	4026	461	480
	2	750	7158	744	780
	3	500	16105	1550	1420

It is seen from this table that the author's theory of spatial stability of thin-walled beams, which allows in addition to bending also the consideration of torsion of these beams, agrees well with the experimental data on aircraft beams.

The theory of lateral buckling, when applied to nonsymmetrical beams or to beams having a single symmetry axis and least moment of inertia with respect to the symmetry plane, leads to incorrect results and gives for the critical force exaggerated values which disagree with experimental data.

The experimental studies can be taken as proof that the critical thrust for thin-walled beams of open section has the Euler critical value only when the thrust is applied in the shear center of the section. For other values of the eccentricity of the application point of the force and also for central compression (except when the section centroid coincides with the shear center), the critical force is smaller than the Euler force. The beam buckles not only by bending but also by torsion. These conclusions, following from the theory of stability of thin-walled sections, were

confirmed not only qualitatively but also quantitatively since the experimental and theoretical values for the critical forces agreed well for comparable slenderness-ratios.

One of the consequences of the theory is also confirmed by the experiments: the existence of a rotation axis for every section, about which the torsion of the beam takes place. This axis is parallel to the beam axis and is the locus of the rotation centers of the separate cross sections.

The results obtained should be taken into account when designing and constructing metal structures of rolled sections; however, one must remember that here the influence of the lattices and braces was not taken into account, for these, bringing the thin-walled open section near to a closed one, eliminate to a considerable extent the dangerous influence of torsion due to longitudinal compression. At the same time this reduced sharply the difference between the values of the critical forces calculated according to Euler and according to our stability theory, independent of the eccentricity of the thrust.

§ 15. Stability of beams, loaded at the ends by bimoments

In the previous sections we have examined th detail the problem of stability of thin-walled beams subjected to the action of compressive forces applied at the ends of the beam. We imposed on these longitudinal forces one very essential limitation, namely, we considered that up to the moment of buckling the beam was in a state of central transverse bending with axial extension, and that there was no flexural torsion. With the problem posed in this way, the bent equilibrium shape of the beam passes, at the moment of buckling, into another flexural-torsional equilibrium shape, essentially different from the first.

We shall now examine a more general case of transmission of an external compressive load to the beam. We shall consider that, up to buckling, the transverse sections of the beam do not remain plane, and that, consequently, the bending deformations of the beam are accompanied by torsion.

Let the longitudinal compressive force, applied at an arbitrary point (in the principal generalized coordinates of the section), give rise at the beam end to an axial compressive force P, to bending moments M_x, M_y and to a bimoment B. The longitudinal normal stresses, which appear in the beam before buckling, will be determined in this case by the four-term equation

$$n = -\frac{P}{F} + \frac{M_x}{J_x} y - \frac{M_y}{J_y} x + \frac{B}{J_\omega} \omega. \tag{15.1}$$

Here, the bending moments M_x, M_y remain constant along the beam (they do not depend on the coordinate z). The bimoment B is a variable quantity (being a function of the coordinate z) and is determined by the equation

$$B = -P\omega_0 \frac{\operatorname{ch} \frac{kz}{l}}{\operatorname{ch} \frac{k}{2}}. \tag{15.2}$$

where ω_0 is the value of the principal generalized coordinate of warping ω for the point of application of the compressive force P; k is the generalized flexural-torsional characteristic, determined by equation (2.2) of Chapter II; $2l$ is the length of the beam-shell.

The origin of the coordinate z is chosen in the mid-length of the beam. It is easy to obtain equation (15.2), using the arguments of sub-sec. 5, § 7, Chapter II.

Equation (15.1) differs from equation (1.1) in that the fourth term refers to the normal stresses due to a bimoment. Determining the additional reduced load due to these stresses, as obtained from an infinitely small variation of the bending and torsional deformation of an initially stressed beam, and keeping in mind that the bimoment stresses depend also on the coordinate z, we obtain

$$p_{xB} = \delta\,(n\xi_s')',$$
$$p_{y3} = \delta\,(n\eta_s'),$$

where n are the additional normal stresses caused by the bimoment load.

Introducing in these equations the values of ξ_s and η_s from equations (1.2), determining the linear transverse loads q_{xB}, q_{yB} and the linear torsion moment m_B by the equations

$$q_{xB} = \int_L p_{xB} ds,$$
$$q_{yB} = \int_L p_{yB} ds,$$
$$m_B = \int_L [p_{yB}(x - a_x) - p_{xB}(y - a_y)]\,ds,$$

and recalling the orthogonality condition of the functions 1, x, y, ω over the whole area F of the cross section, we obtain

$$q_{xB} = 0, \quad q_{yB} = 0,$$
$$m_B = \frac{U_\omega}{J_\omega}(B\theta')'.$$

Here U_ω is a new geometrical characteristic of the section, determined by the equation

$$U_\omega = \int_F \omega\,(x^2 + y^2)\,dF,$$

where the definite integral is evaluated over the whole area of the cross section.

Having determined the additional torsional moment m_B due to a bimoment laod, the stability equation (1.10) assumes the more general form

$$\left.\begin{aligned} EJ_y\xi^{IV} + P\xi'' + (M_x + a_yP)\,\theta'' = 0,\\ EJ_x\eta^{IV} + P\eta'' + (M_y - a_xP)\,\theta'' = 0,\\ (M_x + a_yP)\,\xi'' + (M_y - a_xP)\,\eta'' + EJ_\omega\theta^{IV} +\\ + (r^2P + 2\beta_xM_y - 2\beta_yM_x - GJ_d)\,\theta'' - \frac{U_\omega}{J_\omega}(B\theta')' = 0. \end{aligned}\right\} \quad (15.3)$$

These equations will refer, for $B \neq 0$, to a case of spatial stability, for which the critical state is characterized by a variation of the beam deformation only in the quantitative but not in the qualitative sense. In accordance with the normal stresses n, determined by the general four-term equation (15.1), the beam, in addition to the bending deformations given by the equations

$$\xi_0'' = \frac{M_y}{EJ_y}, \quad \eta_0'' = -\frac{M_x}{EJ_x},$$

which it undergoes up to buckling, will also undergo an initial torsional deformation, caused by the bimoments and determined by the equation

$$\theta_0'' = -\frac{B}{EJ_\omega}.$$

The functions $\xi = \xi(z)$, $\eta = \eta(z)$, $\theta = \theta(z)$ in equations (15.3) (referring these equations to the problem of stability, i. e. the determination of the critical value of the parameter of the external load) represent the variations of the deflections and of the torsional angle, which do not vanish in the pre-critical state.

Equations (1.10) refer to another case of spatial stability of a beam, namely to the case where the beam undergoes only compression and bending deformations up to buckling. There is no torsional deformation.

As already agreed in § 1, equations (1.10) will refer to spatial stability of the first kind, since a new equilibrium shape, essentially different from the initial one, appears at the moment of buckling. For $B \neq 0$ equations (15.3) refer to spatial stability of the second kind since the stability loss of the beam is not associated with the setting-in of a new equilibrium shape.

Chapter VI

GENERAL THEORY OF STABILITY OF PLANE BENDING IN THIN-WALLED BEAMS AND GIRDERS

§ 1. General differential equations of stability for plane bending

We have examined in the previous chapter the problem of stability of thin-walled beams subjected to an axial and an eccentric longitudinal force. Only normal stresses $n(s)$, determined by equation (V.1.1), were considered to appear in the beam before buckling. They remained constant along the beam. The tangential stresses were equal to zero before buckling. This fact allowed a considerable simplification of the general problem of stability of thin-walled beams with a nondeformable contour and its mathematical reduction to the integration of the system of differential equations with constant coefficients (V.1.10).

We shall now examine the stability problem of beams, proceeding from more general assumptions with respect to the external load.

Let a load, producing not only normal but also tangential stresses, act on an open thin-walled beam of arbitrary cross section.

We shall assume that the transverse load and the balancing support reactions pass through the line of shear centers. Under these assumptions the beam will be in a condition of central transverse bending before buckling. In other words, the cross sections of the beam undergo only translational displacements in their plane, without warping.

The normal and axial-tangential stresses in the cross section are determined by the equations

$$\left.\begin{aligned} n(z,\ s) &= \frac{N(z)}{F} + \frac{M_x(z)}{J_x}y(s) - \frac{M_y(z)}{J_y}x(s), \\ t(z,\ s) &= -\frac{M'_x(z)}{J_x}\frac{S_x(s)}{\delta(s)} + \frac{M'_y(z)}{J_y}\frac{S_y(s)}{\delta(s)}, \end{aligned}\right\} \tag{1.1}$$

where $N(z)$, $M_x(z)$ and $M_y(z)$ are, respectively, the normal force and the bending moments taken in the general case as functions of z; $M'_x(z)$ and $M'_y(z)$ are the derivatives with respect to z of the corresponding moments; $x(s)$ and $y(s)$ are the coordinates of the contour point for which the stresses $n(z,\ s)$ and $t(z,\ s)$ are calculated; these coordinates are measured from the principal axis of the section and are functions of the arc-length s; $\delta(s)$ is the wall thickness, which in the general case is also a function of s; $S_x(s)$ and $S_y(s)$ are the statical moments of the area of the cut-off part of the section, calculated with respect to the axes Ox and Oy.

In the critical state the beam may pass from one form of equilibrium to another, differing from the first by infinitely small additional deformations and stresses.

Let the additional deformations be determined by the functions $\xi(z)$, $\eta(z)$ and $\theta(z)$, of which $\xi(z)$ and $\eta(z)$ are the displacements of the shear center in the direction of the principal axes x and y and $\theta(z)$ the torsion angle.

Additional normal and tangential stresses $\sigma(z, s)$ and $\tau(z, s)$ and torsional moments $H(z)$ appear in the cross sections of the beam with the displacements $\xi(z)$, $\eta(z)$ and $\theta(z)$. These stresses and moments must be in equilibrium with the given basic stresses $n(z, s)$ and $t(z, s)$ referring to the deformation state of the beam after buckling (i.e. to the state determined by the basic plus the additional displacements).

For small deformations the equilibrium conditions of the beam, as shown before (see (I.7.3)), can be described by linear equations:

$$EJ_y\xi^{IV} = q_x, \quad EJ_x\eta^{IV} = q_y, \quad EJ_\omega\theta^{IV} - GJ_d\theta'' = m, \qquad (1.2)$$

where q_x and q_y are the magnitudes of the reduced additional transverse loads per unit length obtained from the given stresses $n(z, s)$ and $t(z, s)$ by varying the basic deformation state; m is the magnitude of the additional external torsional moment per unit length, obtained from the given state of stress and from the external load in the required state of deformation.

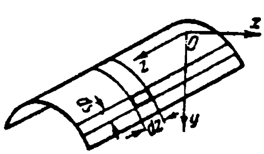


Figure 174

In order to determine the quantities q_x, q_y and m in equation (1.2) we shall examine the deformed state of an elementary longitudinal strip enclosed between two generators of the middle surface: $s =$ const and $s + ds =$ const. The width of this strip is equal to the differential ds of the arc of the the section contour (Figure 174). With the transition of the beam from one state of elastic equilibrium to a neighboring state, the elementary strip undergoes, besides elongation, also bending and torsional deformations. Let ξ_s and η_s be the displacements of the point $M(z, s)$ in the directions of the principal axes of the section Ox and Oy. As shown before (see (I.3.4)) these displacements are expressed for a beam with a nondeformable contour by the displacements of the shear center ξ, η and the torsion angle θ:

$$\left.\begin{aligned}\xi_s &= \xi - (y - a_y)\theta,\\ \eta_s &= \eta + (x - a_x)\theta.\end{aligned}\right\} \qquad (1.3)$$

Here the coordinates of the point $M(z, s)$ in the plane of the section $z =$ const were denoted by $x = x(s)$ and $y = y(s)$. The displacements ξ_s and η_s, for a given value of the independent variable s, are functions of z and determine the space curve into which the generator of the middle surface $s =$ const is deformed on buckling. The angle between the tangent to this curve and the z-axis is determined by their projections on the principal coordinate planes of the beam. Retaining only first-order quantities, we may write for these projections the following expressions:

$$\left.\begin{aligned}\alpha &= \xi_s' = \xi' - (y - a_y)\theta',\\ \beta &= \eta_s' = \eta' + (x - a_x)\theta',\end{aligned}\right\} \qquad (1.4)$$

where ξ', η' and θ' are the derivatives with respect to z of the required functions ξ, η and θ.

We isolate an infinitely small strip element $ABCD$ (Figure 175). The forces, which in the deformed state of the beam consist of the fundamental forces referring to the state of the beam before buckling and of the additional forces, which arise when the basic equilibrium chape changes, will act on the lateral sides of this element AB and CD. The fundamental forces are expressed by the stresses n and t, determined in the statical calculation from equations (1.1). The additional forces, corresponding to the displacements ξ, η and θ, are determined by the stresses σ and τ.

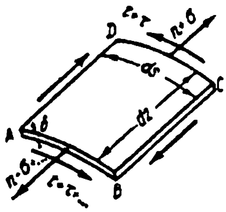


Figure 175

Figure 175 shows the normal and shear forces acting on the sides AB and CD of the isolated element of the middle surface and which refer to the final deformation state of the beam. The magnitude of these forces is obtained by multiplying the corresponding stresses by the thickness of the section wall $\delta(s)$ so that they are in kg/cm. In passing from the section $z = \text{const}$ to the section $z + dz = \text{const}$ the normal and shear forces, which are functions of the position of the point on the longitudinal elementary strip, change by certain increments. These increments are proportional to the differential dz.

The normal forces, acting on the sides AB and CD of the infinitely small element, will be inclined at certain angles to the generator of the middle surface of the undeformed beam as a result of the bending of this element in planes parallel to the principal planes Oxz and Oyz.

These angles will be equal to α and β for the point M situated on the section $z = \text{const}$, and to $\alpha + d\alpha$, $\beta + d\beta$ for the point M_1 for the other (adjacent) section $z + dz = \text{const}$ (Figure 176).

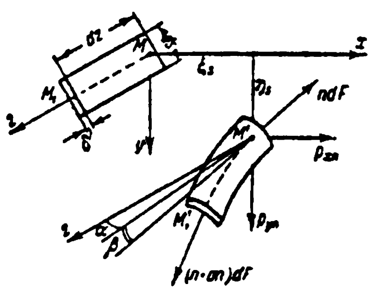


Figure 176

The normal forces acting on the side CD have projections on the axes Ox and Oy. Owing to the smallness of the deformation we can replace $\sin\alpha$ and $\sin\beta$ by α and β and obtain expressions for the projections of the forces on the axes Ox and Oy:

$$\left.\begin{aligned} &-(n+\sigma)\,\delta\sin\alpha\cdot ds \approx \\ &\qquad\approx -(n\alpha+\sigma\alpha)\,\delta\,ds, \\ &-(n+\sigma)\,\delta\sin\beta\cdot ds \approx \\ &\qquad\approx -(n\beta+\sigma\beta)\,\delta\,ds. \end{aligned}\right\} \tag{1.5}$$

Passing to the side AB, replacing now $\sin(\alpha + d\alpha)$ by $\alpha + d\alpha$ and $\sin\beta + d\beta$ by $(\beta + d\beta)$, we shall consider α, β, σ and dn to be quantities of the same order and $d\alpha$, $d\beta$, $d\sigma$ quantities of a higher order. Retaining only the principal terms we obtain the following expressions for the projections of the normal forces acting on the side AB on the axes Ox and Oy,

$$\left.\begin{aligned} &[n+\sigma+d(n+\sigma)]\,\delta\sin(\alpha+d\alpha)\,ds \approx \\ &\qquad\approx (n\alpha+\sigma\alpha+\alpha dn+n d\alpha)\,\delta ds, \\ &[n+\sigma+d(n+\sigma)]\,\delta\sin(\beta+d\beta)\,ds \approx \\ &\qquad\approx (n\beta+\sigma\beta+\beta dn+n d\beta)\,\delta ds. \end{aligned}\right\} \tag{1.6}$$

With the help of equations (1.5) and (1.6) it is now possible to determine the projections on the axes Ox and Oy of the reduced surface load due to the given normal stresses n, since the difference of the normal components of the forces for the sides AB and CD refers to the deformed state of the beam. Denoting the values of these projections by p_{xn} and p_{yn}, we may may write

$$\left.\begin{aligned} p_{xn}\,dzds &= (\alpha dn + nd\alpha)\,\delta ds, \\ p_{yn}\,dzds &= (\beta dn + nd\beta)\,\delta ds. \end{aligned}\right\} \qquad (1.7)$$

Dividing both sides fo (1.7) by $dzds$ and noting that the expressions $nd\alpha + \alpha dn$ and $nd\beta + \beta dn$ on the right-hand sides of the equalities are partial differentials of the products $n\alpha$ and $n\beta$ with respect to the variable z, we obtain:

$$p_{xn} = \frac{\partial}{\partial z}(n\alpha)\,\delta,$$

$$p_{yn} = \frac{\partial}{\partial z}(n\beta)\,\delta.$$

We shall now determine the projections on the axes Ox and Oy of the reduced surface load due to the shear forces acting on the sides AB and CD of the deformed element.

In each point of the beam cross section the shear forces act in the direction of the tangent to the arc of the section contour at this point. As the beam passes into the deformed state this tangent rotates by an angle equal to the twist angle of the longitudinal elementary strip. As a result of the torsional deformation, the shear forces have a non-zero projection on the direction of the normal to the middle surface of the undeformed beam. This projection on the inward normal to the side CD is

$$-(t+\tau)\,\delta\sin\theta\cdot ds \approx -(t\theta + \tau\theta)\,\delta ds. \qquad (1.8)$$

Passing to the side AB, we note that the values of the shear forces and the torsion angle differ for this side from the corresponding values for the side CD by the partial differentials proportional to the differential dz. Projecting the shear forces acting on the side AB on the direction of the inward normal at the point A, discarding (as before) terms of higher order and replacing $\sin(\theta + d\theta)$ by its argument $\theta + d\theta$, we obtain

$$[t + dt + (\tau + d\tau)]\,\delta\sin(\theta + d\theta)\,ds \approx (t\theta + \tau\theta + \theta dt + td\theta)\,\delta ds. \quad (1.9)$$

Equations (1.8) and (1.9) determine the normal components of the shear forces acting on the sides AB and CD of the deformed element. These components have different signs: for CD the normal component for positive t and θ is in the direction of the outward normal. For AB this component acts in the direction of the inward normal. We can consider the difference between the normal components for the sides AB and CD as an additional surface load on the isolated element, which appears in the deformed beam due to the tangential stresses t.

Denoting the magnitude of this load by p_t and considering it to be positive when directed along the inward normal, we may write the difference in the normal components on the right-hand side of equalities (1.8) and (1.9) in the following form:

$$p_t dzds = (\theta dt + td\theta)\,\delta ds. \qquad (1.10)$$

Dividing now both sides of the obtained equality by $dz\,ds$ and noting that the expression $\theta dt + t d\theta$ on the right-hand side of equation (1.10) is a partial differential of the product $t\theta$ with respect to the variable z, we obtain for p_t the equation

$$p_t = \frac{\partial}{\partial z}(t\theta)\,\delta.$$

We can represent the surface load p_t acting in the direction of the inward normal by its components along the axes Ox and Oy of a stationary coordinate system.

Denoting these components by p_{xt} and p_{yt}, respectively, we find

$$p_{xt} = -p_t \sin\psi = -\frac{\partial}{\partial z}(t\theta)\,\delta \sin\psi,$$

$$p_{yt} = p_t \cos\psi = \frac{\partial}{\partial z}(t\theta)\,\delta \cos\psi.$$

Here we denoted by $\psi = \psi(s)$ the angle formed by the tangent to the arc of the beam contour and the axis Ox. This angle, like the thickness $\delta = \delta(s)$, depends only on the one variable s.

For the components of the total reduced surface load due to the stresses n and t we now obtain the equations

$$\left.\begin{aligned} p_x &= p_{xn} + p_{xt} = \left[\frac{\partial}{\partial z}(n\alpha) - \frac{\partial}{\partial z}(t\theta)\sin\psi\right]\delta, \\ p_y &= p_{yn} + p_{yt} = \left[\frac{\partial}{\partial z}(n\beta) + \frac{\partial}{\partial z}(t\theta)\cos\psi\right]\delta. \end{aligned}\right\} \qquad (1.11)$$

These equations determine the components of the additional load per unit area of the middle surface which is caused by the stresses n and t due to a variation in the basic deformation of the beam. This reduced additional load must be in equilibrium with the additional normal and tangential stresses σ and τ and the torsional moments H which build up in the cross sections on departure from plane bending.

From the components p_x and p_y we can determine the components of the transverse load acting on an elementary transverse curved strip enclosed between two adjacent transverse sections $z = \text{const}$ and $z + dz = \text{const}$. Denoting the magnitude of these components by q_x and q_y and keeping in mind that they are found by integrating the surface loads p_x and p_y around the contour of the cross section, we may write

$$\left.\begin{aligned} q_x &= \int_L p_x\,ds = \int_F \frac{\partial}{\partial z}(n\alpha - t\theta\sin\psi)\,dF, \\ q_y &= \int_L p_y\,ds = \int_F \frac{\partial}{\partial z}(n\beta + t\theta\cos\psi)\,dF, \end{aligned}\right\} \qquad (1.12)$$

where L and F denote the circumference of the contour and the area of the cross section respectively. The stresses $n(z, s)$ and $t(z, s)$, referring to the state of the beam before buckling, and the angular displacements $\alpha(z, s)$, $\beta(z, s)$ and $\theta(z, s)$, referring to the sought deformed state after buckling, enter the integrands.

We insert in the right-hand side of (1.12) the values of n and t from equations (1.1), of α and β from equations (1.4), and note that N, M_x, M_y, ξ, η and θ are functions of z only, and x, y, δ and ψ of s only. Differentiating with respect to z, integrating over the area of the cross section F

and using the relations*

$$\int_F \frac{S_x}{\delta}\sin\psi\, dF = \int_F S_x dy = -J_x, \quad \int_F \frac{S_y}{\delta}\cos\psi dF = \int_F S_y dx = -J_y,$$

$$\int_F \frac{S_x}{\delta}\cos\psi dF = \int_F S_x dx = 0, \quad \int_F \frac{S_y}{\delta}\sin\psi dF = \int_F S_y dy = 0$$

we obtain

$$\left.\begin{aligned} q_x &= [N(\xi' + a_y\theta')]' - (M_x\theta)'', \\ q_y &= [N(\eta' - a_x\theta')]' - (M_y\theta)''. \end{aligned}\right\} \tag{1.13}$$

We shall now determine for the buckled beam the additional "external" torsional moment per unit length from the third equation of the system (1.2).

For this purpose we examine the forces and loads which act on an elementary transverse strip $s\,dz$. Normal and shear forces act on the curved edges of this strip. As a result of the bending and torsional deformation of the longitudinal elementary strip ds these forces give rise, as we saw above, to an additional transverse load. Choosing as the moment axis a line parallel to the generator of the cylinder and passing through the shear center, we obtain for the torsional moment for a load acting on a shell element $dsdz$, and for a width $dz=1$ of the transverse strip, the following expression

$$dm_1 = [p_y(x-a_x) - p_x(y-a_y)]\,ds.$$

Substituting here the components p_x and p_y of the load as determined from equation (1.11), and expressing then α and β by the displacements ξ, η and θ according to equations (1.4), we find

$$dm_1 = \frac{\partial}{\partial z}\Big\{-\xi' n(y-a_y) + \eta' n(x-a_x) + \theta'[n(x-a_x)^2 + n(y-a_y)^2] + \theta[t_x(x-a_x) + t_y(y-a_y)]\Big\}\,dF, \tag{1.14}$$

where t_x and t_y are the components of the axial tangential stress at an arbitrary point of the contour:

$$t_x = t\cos\psi, \quad t_y = t\sin\psi. \tag{1.15}$$

Integrating both sides of (1.14) over the area of the section F and keeping in mind that the displacements ξ, η and θ do not depend on the coordinate s, we can write

$$m_1 = \frac{\partial}{\partial z}\Big\{-\xi'\int_F n(y-a_y)\,dF + \eta'\int_F n(x-a_x)\,dF + \theta'\int_F [n(x-a_x)^2 + n(y-a_y)^2]\,dF + \theta\int_F [t_x(x-a_x) + t_y(y-a_y)]\,dF\Big\}.$$

This equation determines the additional torsional moment due to the surface load obtained as a result of projecting the normal and shear forces on the plane of the cross section of the beam, if we refer these forces to the buckled state of the beam and if we take into account their change in direction.

The additional torsional moment is also due to the fact that the shear forces take new positions with respect to the stationary system of coordinates Oxy as the beam passes into another state of deformation. The points

* It is easy to obtain these relations if we use integration by parts (see footnote to § 8, Chapter I).

of application of these forces shift. As a result, the given torsional moment, expressed by the internal shear forces, is changed by a certain amount. In order to determine this change we resolve the shear forces acting on the sections $z = \text{const}$ and $z + dz = \text{const}$ into components along the axes Ox and Oy of the stationary system of coordinates. Referring these components to a unit arc-length of the contour and expressing them in terms of tangential stresses, we can write

	Projection on Ox	Projection on Oy
Section $z = \text{const}$	$(t_x + \tau_x)\,\delta$	$(t_y + \tau_y)\,\delta$
Section $z + dz = \text{const}$	$[t_x + \tau_x + d(t_x + \tau_x)]\,\delta$	$[t_y + \tau_y + d(t_y + \tau_y)]\,\delta$

We shall calculate the increment of the torsional moment due to the shear forces that act on an infinitely small element $dzds$ of the isolated transverse strip. Noting that, in passing from the section $z = \text{const}$ to the section $z + dz = \text{const}$, the displacements ξ_s and η_s of any point of the contour have also increments $d\xi_s$ and $d\eta_s$, we can express the additional torsional moment due to the shear forces acting along the curved sides of an element $dzds$ as follows

$$dm_z = \left(\xi_s + \frac{\partial \xi_s}{\partial z}\,dz\right)\left[t_y + \tau_y + \frac{\partial}{\partial z}(t_y + \tau_y)\,dz\right]\delta ds - \xi_s (t_y + \tau_y)\,\delta ds -$$
$$- \left(\eta_s + \frac{\partial \eta_s}{\partial z}\,dz\right)\left[t_x + \tau_x + \frac{\partial}{\partial z}(t_x + \tau_x)\,dz\right]\delta ds + \eta_s (t_x + \tau_x)\,\delta ds. \qquad (1.16)$$

The first and second terms in this equation express the torsional moment due to the shear forces parallel to the axis Oy. The third and fourth terms give the torsional moment due to the shear forces parallel to the axis Ox.

The increments of the forces and displacements are expressed by the partial differentials of the corresponding functions with respect to z.

Performing in (1.16) the obvious simplifications, dropping second-order quantities (products of the additional tangential stresses τ_x and τ_y and the displacements ξ_s and η_s) and setting $dz = 1$, we obtain

$$dm_z = \left[\frac{\partial (t_y \xi_s)}{\partial z} - \frac{\partial (t_x \eta_s)}{\partial z}\right]\delta ds. \qquad (1.17)$$

Substituting in (1.17) for ξ_s and η_s the right-hand members (1.3) and integrating both sides of (1.17) around the cross section contour, we obtain the following expression for the torsional moment acting on an isolated transverse strip of unit width ($dz = 1$):

$$m_z = \frac{\partial}{\partial z}\left(\xi \int_F t_y dF\right) - \frac{\partial}{\partial z}\left(\eta \int_F t_x dF\right) -$$
$$- \frac{\partial}{\partial z}\left\{\theta\left[\int_F t_y (y - a_y)\,dF + \int_F t_x (x - a_x)\,dF\right]\right\}.$$

The torsional moment m_z, like the moment m_1, is expressed as a function of the given stresses and of the required displacements. The moment m_z is the sought quantity, into which the initial torsional moment is

transformed as a result of the change of the given shear forces under a parallel displacement of these forces to another position, following stability loss.

The third additional moment remains to be determined; this is the increment of the torsional moment m due to the external surface forces p_x° and p_y°, which appears as a result of these forces being shifted to another position with respect to the shear center. The increment of the torsional moment m due to the given load p_x° and p_y° and acting on an element of the middle surface $dzds$ has the following form, due to the displacements ξ_s and η_s of this element in the plane of the cross section:

$$dm_3\,dz = (p_y^\circ \xi_s - p_x^\circ \eta_s)\,dzds.$$

Inserting the values of ξ_s and η_s from (1.3), reducing by dz and integrating around the contour of the cross section, we obtain for the increment of the external moment per unit length the following equation:

$$m_3 = \xi \int_L p_y^\circ ds - \eta \int_L p_x^\circ ds - \theta \left[\int_L p_x^\circ (x - a_x)\,ds + \int_L p_y^\circ (y - a_y)\,ds \right].$$

We have determined the additional torsional moments due to all the forces acting on the strip $dz = 1$. The total additional torsional moment in the third equation of the system (1.2) is equal to the sum of the three moments

$$\begin{aligned} m = m_1 + m_2 + m_3 = \frac{\partial}{\partial z}\Big[-\xi' \int_F n(y - a_y)\,dF + \eta' \int_F n(x - a_x)\,dF\Big] + \\ + \frac{\partial}{\partial z}\left\{ \theta' \int_F [n(x - a_x)^2 + n(y - a_y)^2]\,dF \right\} + \frac{\partial}{\partial z}\Big[\xi \int_F t_y dF - \eta \int_F t_x dF\Big] + \\ + \xi \int_L p_y^\circ ds - \eta \int_L p_x^\circ ds - \theta \left[\int_L p_x^\circ (x - a_x)\,ds + \int_L p_y^\circ (y - a_y)\,ds \right]. \qquad (1.18) \end{aligned}$$

We insert in (1.18) the normal and tangential stresses $n(z, s)$ and $t(z, s)$ determined from (1.1) for the case of central transverse bending, and calculate the integrals on the right-hand side of (1.18). For the integrals of the normal stresses we obtain

$$\left.\begin{aligned} \int_F n(y - a_y)\,dF = \frac{N(z)}{F} \int_F (y - a_y)\,dF + \frac{M_x(z)}{J_x} \int_F y(y - a_y)\,dF - \\ - \frac{M_y(z)}{J_y} \int_F x(y - a_y)\,dF, \\ \int_F n(x - a_x)\,dF = \frac{N(z)}{F} \int_F (x - a_x)\,dF + \frac{M_x(z)}{J_x} \int_F y(x - a_x)\,dF - \\ - \frac{M_y(z)}{J_y} \int_F x(x - a_x)\,dF, \\ \int_F [n(x - a_x)^2 + n(y - a_y)^2]\,dF = \frac{N(z)}{F} \int_F [(x - a_x)^2 + \\ + (y - a_y)^2]\,dF + \frac{M_x(z)}{J_x} \int_F y[(x - a_x)^2 + (y - a_y)^2]\,dF - \\ - \frac{M_y(z)}{J_y} \int_F x[(x - a_x)^2 + (y - a_y)^2]\,dF. \end{aligned}\right\} \qquad (1.19)$$

Using the abbreviated notations (V.1.6), (V.1.7), (V.1.8), we can write (1.19) more simply:

$$\left.\begin{aligned}\int_F n(y-a_y)\,dF&=-a_yN(z)+M_x(z),\\ \int_F n(x-a_x)\,dF&=-a_xN(z)-M_y(z),\\ \int_F n[(x-a_x)^2+(y-a_y)^2]\,dF&=r^2N(z)+2\beta_yM_x(z)-2\beta_xM_y(z).\end{aligned}\right\}\tag{1.20}$$

In the same way we can simplify in (1.18) the integrals containing the tangential stresses t_x and t_y. In view of (1.15) we can write

$$\left.\begin{aligned}\int_F t_x dF&=\int_L t\delta\cos\psi ds,\\ \int_F t_y dF&=\int_L t\delta\sin\psi ds.\end{aligned}\right\}\tag{1.21}$$

In these equations the expressions $ds\cos\psi$ and $ds\sin\psi$ represent the projections of the differential ds of the contour arc on the axes Ox and Oy:

$$ds\cos\psi=dx,\quad ds\sin\psi=dy.\tag{1.22}$$

Multiplication of the axial tangential stress t by the wall thickness δ gives the shear force, i.e. the force in the cross section of the beam per unit length of the arc of the contour, directed along the tangent to this arc. By virtue of (1.1)

$$t\delta=-\frac{M_x'(z)}{J_x}S_x(s)+\frac{M_y'(z)}{J_y}S_y(s).\tag{1.23}$$

Introducing expressions (1.22) and (1.23) in (1.21) and noting that for the whole cross section

$$\int_F S_x\,dx=0,\qquad \int_F S_y\,dx=-J_y,$$

$$\int_F S_x\,dy=-J_x,\quad \int_F S_y\,dy=0,$$

we finally write the integrals of the tangential stresses in the form

$$\int_F t_x\,dF=-M_y'(z),\qquad \int_F t_y\,dF=M_x'(z).\tag{1.24}$$

Equations (1.24) express the well-known equilibrium conditions of an elementary transverse strip. The integrals on the left-hand side of these equations, taken over the whole area of the cross section F, give the projections on the coordinate axes of the transverse force acting in the plane $z=\text{const}$.

In the case of transverse bending these projections are expressed by the derivatives with respect to z of the corresponding transverse bending moments. Taking into consideration that the projections of the transverse force Q_x and Q_y for the section $z=\text{const}$ having a positive outward normal directed along the z-axis coincide, for positive values of these projections, with the directions of the coordinate axes Ox and Oy, and using

for the moments $M_x(z)$ and $M_y(z)$ the sign rule adopted before, we can write the conditions of moment equilibrium of the transverse elementary strip of the middle surface of the beam $dz=1$ with respect to the axes Ox and Oy in the following form

$$Q_x=-M'_y, \quad Q_y=M'_x. \tag{1.25}$$

Equations (1.25) express the transverse forces Q_x and Q_y by the derivatives of the moment $M_y(z)$ and $M_x(z)$. Conversely, these transverse forces can be obtained as the integrals of the shear forces acting in the direction of the axes Ox and Oy and evaluated over the cross section area F.

We can also simplify the last integral on the right-hand side of (1.18) which contains terms with the components of the transverse surface load p_x^0 and p_y^0.

Denoting by q_x^0 and q_y^0 the components of the transverse load acting not on the element $dzds$ of the middle surface of the beam but on a transverse strip of unit width $(dz=1)$, we can write

$$\left.\begin{aligned}\int_L p_x^0 ds=q_x^0,\\ \int_L p_y^0 ds=q_y^0,\\ \int_L p_x^0(x-a_x)\,ds+\int_L p_y^0(y-a_y)\,ds=q_x^0(e_x-a_x)+q_y^0(e_y-a_y),\end{aligned}\right\} \tag{1.26}$$

where e_x and e_y are the coordinates of the point of application of the given transverse load per unit length in the plane of the cross section*.

Applying (1.20), (1.24) and (1.26), expression (1.18) for the additional external torsional moment becomes

$$\begin{aligned}m=\frac{\partial}{\partial z}[\xi'(a_yN-M_x)-\eta'(a_xN+M_y)]+\frac{\partial}{\partial z}[\theta'(r^2N+2\beta_yM_x-\\ -2\beta_xM_y)]+\frac{\partial}{\partial z}(\xi M'_x+\eta M'_y)+\xi q_y^0-\eta q_x^0-\theta[q_x^0(e_x-a_x)+\\ +q_y^0(e_y-a_y)].\end{aligned}$$

Performing the indicated differentiation with respect to z and keeping in mind that on the basis of the equilibrium condition of the elementary transverse strip dz

$$M''_x=Q'_y=-q_y^0, \quad M''_y=-Q'_x=q_x^0,$$

we finally write the expression for the torsional moment in the following form

$$\begin{aligned}m=a_y(N\xi')'-a_x(N\eta')'-M_x\xi''-M_y\eta''+[(r^2N+2\beta_yM_x-\\ -2\beta_xM_y)\theta']'-[q_x^0(e_x-a_x)+q_y^0(e_y-a_y)]\theta.\end{aligned} \tag{1.27}$$

Equations (1.13) and (1.27) determine the additional "external" forces q_x and q_y and the moment m due to the given stressed condition of the beam when its state of deformation changes. These forces and this moment

* This is so since $\int_L p_x^0 x\,ds=q_x^0e_x$ and $\int_L p_y^0 y\,ds=q_y^0e_y$,

where

$$q_x^0=\int_L p_x^0 ds \quad \text{and} \quad q^0=\int_L p_y^0\,ds.$$

should be in equilibrium with the additional internal forces which also arise as a result of the change in deformation.

Introducing (1.13) and (1.27) in equation (1.2), we write the equilibrium condition of the beam after buckling in the following form

$$\left.\begin{aligned} EJ_y\xi^{IV}-[N(\xi'+a_y\theta')]'+(M_x\theta)''=0,\\ EJ_x\eta^{IV}-[N(\eta'-a_x\theta')]'+(M_y\theta)''=0,\\ EJ_\omega\theta^{IV}-GJ_d\theta''-[(r^2N+2\beta_yM_x-2\beta_xM_y)\theta']'+\\ +[q_x^\circ(e_x-a_x)+q_y^\circ(e_y-a_y)]\theta-a_y(N\xi')'+\\ +a_x(N\eta')'+M_x\xi''+M_y\eta''=0. \end{aligned}\right\} \qquad (1.28)$$

We have obtained the **general differential stability equations for an arbitrary open thin-walled section under conditions of combined resistance to central transverse bending together with compression or extension.**

In these equations, the required functions are the displacements $\xi=\xi(z)$, $\eta=\eta(z)$ and $\theta=\theta(z)$ appearing as a result of the change in the basic equilibrium shape of the beam on buckling. The coefficients of the differential equations (1.28) are determined from the dimensions of the cross sections of the beam, the elastic constants E and G and the functions $N=N(z)$, $M_x=M_x(z)$, $M_y=M_y(z)$, $q_x^\circ=q_x^\circ(z)$ and $q_y^\circ=q_y^\circ(z)$ which depend on the given external load and on the end conditions.

In calculating the stability of elastic systems it is usually assumed that the external load is given to within a general proportionality factor, characterizing the magnitude of the load. All the forces constituting the external load are in a given constant ratio to one another. The design for stability of an elastic system leads, in this case, to a search for the smallest critical value of the generalized load, i.e., of the proportionality factor.

The general differential equations (1.28) that we obtained allow the investigation of the stability problem in a more general manner. We can consider the external load which acts on the elastic system as being linearly dependent on one or several parameters. The case in which the external load depends linearly on a single parameter is of practical interest. We meet this kind of load when calculating the stability of an elastic system initially loaded by a given constant load (for example, its own weight) and then subjected to the action of a temporary load. The constant load has a completely determined value. The transient load can be considered as given up to a single parameter, which is the proportionality factor for this load.

The force factors N, M_x, M_y, q_x° and q_y° which depend on the external load and on the boundary conditions and which, together with the dimensions of the cross section of the beam and the elastic characteristics of its material, determine the coefficients of the differential equations (1.28) are in this case linear in the load parameter. As this parameter varies, so does the external load acting on the elastic system. Consequently, the coefficients of the differential equations (1.28) vary too. For some definite value of the parameter a state of unstable equilibrium of the system ensues, in which the basic equilibrium shape may pass into a neighboring one, characterized by the variation in the displacements ξ, η and θ and described by the differential equations (1.28).

Such a state of unstable equilibrium is mathematically expressed by the fact that for some (critical) value of the parameter λ and for homogeneous boundary conditions the system of linear homogeneous differential equations (1.28) may have nonvanishing solutions.

There exists an infinite number of solutions for the functions $\xi(z)$, $\eta(z)$ and $\theta(z)$ satisfying, for certain values of the parameter λ, the differential equations (1.28) and the boundary conditions. All these solutions form a so-called set of *eigenfunctions*. Each particular solution of these differential equations, for given homogeneous boundary conditions, is determined by its eigenvalue, proportional to the parameter λ of the critical load.

The infinite number of eigenvalues will correspond to an infinite number of critical loads and an infinite number of equilibrium shapes of the elastic system after buckling. Each of these possible equilibrium shapes is determined, up to an arbitrary constant, by the eigenfunctions of (1.28). The indeterminacy of the solution of the stability problem examined here is due to the fact that in deriving these equations we expressed the equilibrium state of the elastic system in the coordinates of the undeformed state because of the smallness of the displacements ξ, η and θ.

The eigenvalues determined from (1.28) and the boundary conditions can have positive and negative values. In order to determine the critical design load we should choose from the infinite number of positive and negative values of the parameter λ those which are smallest in absolute value*.

§ 2. Stability under longitudinal forces arbitrarily distributed along the beam

The differential equations (1.28) are general and include, in particular, a wide class of stability problems of open thin-walled sections for an arbitrary action of a load (longitudinal and transverse), causing in the section (before buckling) normal stresses $n(z, s)$, linearly distributed over a section $z = \text{const}$.

If the external load consists of longitudinal forces only, arbitrarily distributed along the beam and causing in the section (before buckling) normal stresses of linear distribution, we have

$$q_x^0 = q_y^0 = 0, \quad M_x = Ne_y, \quad M_y = -Ne_x, \tag{2.1}$$

where e_x and e_y are the eccentricities of the applied longitudinal force $N = N(z)$.

Equations (1.28) acquire the following form under conditions (2.1):

$$\left.\begin{aligned} EJ_y\xi^{\mathrm{IV}} - [N(\xi' + a_y\theta')]' + (Ne_y\theta)'' = 0,\\ EJ_x\eta^{\mathrm{IV}} - [N(\eta' - a_x\theta')]' - (Ne_x\theta)'' = 0,\\ EJ_\omega\theta^{\mathrm{IV}} - GJ_d\theta'' - [N(r^2 + 2\beta_y e_y + 2\beta_x e_x)\theta']' +\\ + a_x(N\eta')' - a_y(N\xi')' + Ne_y\xi'' - Ne_x\eta'' = 0. \end{aligned}\right\} \tag{2.2}$$

* The homogeneous boundary-value problem described here, involving eigenfunctions and eigenvalues, can be solved with the help of integral equations.

These equations represent the generalization of equations (V.6.2) obtained before in the case where the longitudinal force is applied only on the ends of the beam and, consequently, remains constant along the beam.

If $e_x = e_y = 0$, then equations (2.2) passing to the following stability equations for axial compression:

$$\left.\begin{aligned} EJ_y\xi^{IV} - [N(\xi' + a_y\theta')]' = 0, \\ EJ_x\eta^{IV} - [N(\eta' - a_x\theta')]' = 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' - r^2(N\theta')' - a_y(N\xi')' + a_x(N\eta')' = 0. \end{aligned}\right\} \qquad (2.3)$$

Assuming that the longitudinal force $N = N(z)$ is linearly given up to a single parameter, we can write

$$N(z) = N^0(z) + \lambda N_1(z),$$

where $N^0(z)$ is a well-defined constant load causing in the beam given initial stresses; $\lambda N_1(z)$ is the temporary load varying along the beam according to some well-defined function and given up to a proportionality factor λ, characterizing the magnitude of this load.

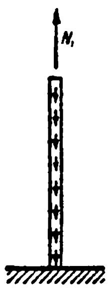


Figure 177

As an illustration, consider a column having weight and loaded at the end by an axial longitudinal force whose critical value is sought. The longitudinal force $N(z)$ for an arbitrary section $z =$ const is determined here by the equation

$$N(z) = -\gamma Fz + N_1, \qquad (2.4)$$

where γFz is the weight of the length z of the column; N_1 is the longitudinal force applied at the end of the column (Figure 177).

Introducing expression (2.4) in equations (2.3) we obtain:

$$\left.\begin{aligned} EJ_y\xi^{IV} - [(N_1 - \gamma Fz)(\xi' + a_y\theta')]' = 0, \\ EJ_x\eta^{IV} - [(N_1 - \gamma Fz)(\eta' - a_x\theta')]' = 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' - r^2[(N_1 - \theta Fz)\theta']' - a_y[(N_1 - \\ - \gamma Fz)\xi']' + a_x[(N_1 - \gamma Fz)\eta']' = 0. \end{aligned}\right\} \qquad (2.5)$$

For $\gamma = 0$, i.e. in the case of a weightless column, equations (2.5) reduce to (V.3.1) for a beam loaded at the ends by a compressive force $P \equiv -N_1$.

Assuming in equations (2.5) N_1 equal to zero and taking $\gamma F = g$ as the load parameter, we obtain the stability equations for the case when the column is subjected to an axial longitudinal load of constant magnitude distributed along it. These equations will have the following form:

$$\left.\begin{aligned} EJ_y\xi^{IV} + [gz(\xi' + a_y\theta')]' = 0, \\ EJ_x\eta^{IV} + [gz(\eta' - a_x\theta')]' = 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' + r^2(gz\theta')' - a_x(gz\eta')' + a_y(gz\xi')' = 0. \end{aligned}\right\} \qquad (2.6)$$

The critical value of the magnitude of the load g is determined from the condition of the solutions of the system of homogeneous linear equations (2.6) vanishing for homogeneous boundary conditions.

Equations (2.3), as the more general initial equations (1.28), apply to a beam of arbitrary cross section.

If the cross section is such that the shear center coincides with the centroid, as is the case, for example, in sections with two axes of symmetry, then $a_x = a_y = 0$. In this case the system of simultaneous equations (2.3) is separated into three independent differential equations:

$$\left.\begin{aligned} EJ_y\xi^{IV} - (N\xi')' &= 0, \\ EJ_x\eta^{IV} - (N\eta')' &= 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' - r^2(N\theta')' &= 0. \end{aligned}\right\} \tag{2.7}$$

The first two equations refer to buckling by bending with respect to the principal axes of the section. The third equation describes buckling by twisting about the shear center.

In the case of a uniform load distribution equations (2.7) become

$$\begin{aligned} EJ_y\xi^{IV} + (gz\xi')' &= 0, \\ EJ_x\eta^{IV} + (gz\eta')' &= 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' + r^2(gz\theta')' &= 0. \end{aligned}$$

After integrating once with respect to z, these equations can be written

$$\left.\begin{aligned} EJ_y\xi''' + gz\xi' + C_1 &= 0, \\ EJ_x\eta''' + gz\eta' + C_2 &= 0, \\ EJ_\omega\theta''' - GJ_d\theta' + r^2gz\theta' + C_3 &= 0, \end{aligned}\right\} \tag{2.8}$$

where C_1, C_2 and C_3 are integration constants. From a physical viewpoint the first two of these constants stand for the transverse forces Q_x^0 and Q_y^0 and the third stands for the torsional moment H_0 acting in the origin section $z = 0$. If the origin section is free from transverse forces and torsional moments, the integration constants vanish and (2.8) become

$$\left.\begin{aligned} EJ_y\xi''' + gz\xi' &= 0, \\ EJ_x\eta''' + gz\eta' &= 0, \\ EJ_\omega\theta''' - GJ_d\theta' + r^2gz\theta' &= 0. \end{aligned}\right\} \tag{2.9}$$

The first two of these equations are identical with the well-known differential equations for a beam subjected to a uniformly distributed longitudinal load that buckles by bending*. The third equation refers to the torsion of the beam on buckling and is of the same form as the first two equations. Introducing into these first two equations the expression

$$u = \frac{2}{3}\sqrt{\frac{g}{B}z^3},$$

and into the third the expression

$$u = \frac{2}{3}\sqrt{\frac{gr^2}{B}\left(z - \frac{GJ_d}{gr^2}\right)^3},$$

where B is the corresponding elastic characteristic (EJ_x, EJ_y or EJ_ω) we obtain each of equations (2.9) in the form

$$v''' + \frac{1}{u}v'' + \left(1 - \frac{1}{9u^2}\right)v' = 0, \tag{2.10}$$

where v stands for ξ, η and θ, respectively.

* See /84/ Chapter IV, § 1.

This is a Bessel equation for v. The general integral of equation (2.10) for the derivative of the sought function is

$$\frac{dv}{du} = C_1 J_{1/3}(z) + C_2 J_{-1/3}(z), \tag{2.11}$$

where $J_{1/3}(z)$ and $J_{-1/3}(z)$ are Bessel functions depending not only on z, but also on the parameter g of the external load which enters in the arguments of these functions. C_1 and C_2 are integration constants.

In the examined boundary-value problem the Bessel functions form, for corresponding homogeneous boundary conditions, a set of eigenfunctions, determining (up to a constant factor) the possible forms of buckling for our elastic beam. The eigenvalues of these functions determine the magnitudes of the critical loads. The smallest of these must be chosen for the design load.

§ 3. Stability of plane bending of thin-walled girders subjected to a transverse load. General case

Assuming in equations (1.28) that the longitudinal force N vanishes, we obtain the following general differential equations of stability for a beam subjected to an arbitrary transverse load,

$$\left.\begin{aligned} EJ_y\xi^{IV} + (M_x\theta)'' = 0, \\ EJ_x\eta^{IV} + (M_y\theta)'' = 0, \\ EJ_\omega\theta^{IV} + [(2\beta_x M_y - 2\beta_y M_x - GJ_d)\,\theta']' + \\ + [q_x^\circ(e_x - a_x) + q_y^\circ(e_y - a_y)]\,\theta + M_x\xi'' + M_y\eta'' = 0, \end{aligned}\right\} \tag{3.1}$$

where $q_x^\circ(z)$ and $q_y^\circ(z)$ are the components of the given transverse load per unit length, applied along the lines $x = e_x$ and $y = e_y$ and passing through the shear center; $M_x(z)$ and $M_y(z)$ are the bending moments due to the given load, determined by statical calculations and taking into account the end conditions of the squared beam.

The external transverse load in (3.1) can be given as an arbitrary function. The magnitude and direction of this load can be a function of the abscissa z, since the components $q_x^\circ(z)$ and $q_y^\circ(z)$ entering equations (3.1) can be prescribed independently of one another.

If the direction of the transverse load along the square beam remains constant, the load components $q_x^\circ(z)$ and $q_y^\circ(z)$ and, consequently, the moment components $M_x(z)$ and $M_y(z)$ will be in a given constant ratio.

Equations (3.1) apply to a beam having an arbitrary cross section. In the case of a section with two symmetry axes the quantities a_x, a_y, β_x and β_y vanish. Equations (3.1) then assume the form

$$\left.\begin{aligned} EJ_y\xi^{IV} + (M_x\theta)'' = 0, \\ EJ_x\eta^{IV} + (M_y\theta)'' = 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' + (q_x^\circ e_x + q_y^\circ e_y)\,\theta + M_x\xi'' + M_y\eta'' = 0. \end{aligned}\right\}$$

If the transverse load acts in a plane parallel to Oyz, then $q_x^\circ = 0$, $M_y = 0$. The second equation of the system (3.2) is eliminated and the

remaining two form a system determining (together with the boundary conditions) the buckling shape for bending in the plane Oyz. This system has the form

$$\left.\begin{aligned} EJ_y\xi^{IV} + (M_x\theta)'' = 0, \\ EJ_\omega\theta^{IV} - GJ_d + q_y^0 e_y\theta + M_x\xi'' = 0. \end{aligned}\right\} \tag{3.3}$$

For $e_y = 0$, i. e. in the case of a transverse load $q_y^0(z)$ applied at the centroid of the section, equations (3.3) become

$$\left.\begin{aligned} EJ_y\xi^{IV} + (M_x\theta)'' = 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' + M_x\xi'' = 0. \end{aligned}\right\} \tag{3.4}$$

The differential equations (3.4) apply to a beam having two axes of symmetry in their cross section and loaded by a transverse load $q_y^0(z)$, distributed according to an arbitrary function along the line of centroids and acting in one of the symmetry axes Oyz. In the general case the bending moment M_x in equations (3.4) is considered to be a given function. This function is determined by the load $q_y^0(z)$ and by the end conditions of the beam.

§ 4. Stability of a wide-flanged beam in plane bending. Generalization of Timoshenko's problem

The differential equations (3.4), obtained as a particular case of the more general equations (3.1), are, in turn, a generalization of the well known stability equations for wide-flanged beams, obtained by S. P. Timoshenko /181/. The sectorial moment of inertia J_ω for such a beam is determined from the equation

$$J_\omega = \int_F \omega^2\, dF = \frac{b^3h^2\delta}{24}, \tag{4.1}$$

where b and δ are respectively the width and the thickness of the beam flanges and h is the depth of the web.

Introducing the expression $J_y = \frac{\delta b^3}{12}$ in the right-hand side of (4.1), we can express J_ω as

$$J_\omega = \frac{h^2}{2} J_y. \tag{4.2}$$

Introducing (5.2) in (3.4) we obtain the stability equations for a wide-flanged beam,

$$\left.\begin{aligned} EJ_y\xi^{IV} + (M_x\theta)'' = 0, \\ \frac{h^2}{2} EJ_y\theta^{IV} - GJ_d\theta'' + M_x\xi'' = 0. \end{aligned}\right\} \tag{4.3}$$

The equations derived by Timoshenko for a cantilever have the following form:

$$\left.\begin{aligned} EJ_y\xi'' + M_x\theta = 0, \\ \frac{h^2}{2}EJ_y\theta''' - GJ_d\theta' + M_x\xi' - M_x'\xi + \int_0^z M_x''\,\xi dz = 0. \end{aligned}\right\} \tag{4.4}$$

Comparing the equations (4.3) with Timoshenko's equations (4.4), we see that they differ in the order of the derivatives of the required functions. This is due to the fact that Timoshenko, in deriving equation (4.4), started from the equilibrium conditions for a finite part of the beam and first satisfied the statical conditions on the free end of the cantilever. The first of equations (4.4) expresses the vanishing of the moments with respect to the axis Oy, for the free end-span of the beam. The second equation is obtained from the condition that the sum of the torsional moments referring to the same part of the beam before buckling must vanish.

The differential equations (4.3) express the equilibrium of an infinitely small element $dz=1$. The first equation represents the vanishing of the variation of the transverse loads in the direction of the axis Ox due to the change of the deformed state of the beam. The second equation refers to the torsion moments also acting on an infinitely small element dz. Integrating the first equation (4.3) twice, the second equation once and applying in the second equation integration by parts, we can write:

$$\left.\begin{aligned} EJ_y\xi'' + M_x\theta &= Az + B, \\ \frac{h^2}{2}EJ_y\theta''' - GJ_d\theta' + M_x\xi' - M_x'\xi + \int_0^z M_x''\xi dz &= C, \end{aligned}\right\} \quad (4.5)$$

where A, B and C are arbitrary constants. These constants, as evident from the statical meaning of the transformation of equations (4.3) into equations (4.5), are respectively equal to the variations of the transverse force Q_x acting in the initial section $z=0$ in the direction of the axis Ox, of the bending moment M_y with respect to the axis Oy and of the torsional moment H acting in the same section. If the variations of the statical factors Q_x, M_y and H vanish in the initial section $z=0$, which is the case, for example, in a cantilever, then the integration constants A, B and C are equal to zero, and equations (4.5) reduce to (4.4) for a beam whose statical factors at the free end do not change on buckling, i. e. with the appearance of the displacements ξ and θ.

If the beam has at the ends a rigid or elastic fixing to restrain translation and rotation, the integration constants A, B and C will not vanish. Instead of equations (4.4) it will be necessary to use the more general equations (4.3) which are not based on any assumptions concerning the initial conditions, since these general equations express the equilibrium of an infinitely small element of the beam.

The differential equations (3.4) obtained from the more general equations (3.1) for a beam with two symmetry axes do not differ in form from equations (4.3) for a wide-flanged beam. It follows that all the studies of Timoshenko on the stability of plane bending of a wide-flanged beam can be freely used for the stability calculation of thin-walled beams with two axes of symmetry. It is only necessary to substitute for the quantity $\frac{h^2}{2}EJ_y$ in Timoshenko's equations the corresponding value of the sectorial rigidity EJ_ω.

The differential equations (4.3) are of general character and do not restrict the external transverse load causing the bending moment $M_x(z)$ in the cross section.

These equations are applicable for an external transverse load distributed along the beam axis Oz according to an arbitrary function. In

particular, if the external load consists of two equal moments applied at the ends of the beam and acting in opposite directions, the moment remains constant in any section of the beam and (4.3) are transformed into equations with constant coefficients:

$$\left.\begin{aligned} EJ_y\xi^{IV}+M_x\theta''&=0,\\ EJ_\omega\theta^{IV}-GJ_d\theta''+M_x\xi''&=0.\end{aligned}\right\}\tag{4.6}$$

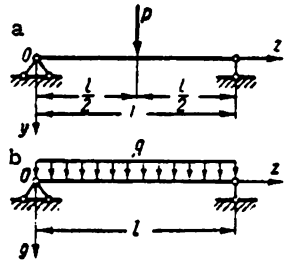


Figure 178

Equations (4.6) express the equilibrium of a beam after buckling by pure bending. We examined these equations in detail earlier, in § 12 of Chapter V.

If the beam is loaded by a concentrated force P in its span, the moment $M_x(z)$ between the support and the force will vary linearly. In the case of a beam with both ends fixed we obtain the following equation for the part on the left of the force P, if it is applied in the middle and the origin of the coordinates is at the left support (Figure 178a)

$$M_x(z)=\frac{Pz}{2}.$$

Introducing this expression in (4.3), we obtain

$$\left.\begin{aligned} EJ_y\xi^{IV}+\frac{1}{2}P(z\theta)''&=0,\\ EJ_\omega\theta^{IV}-GJ_d\theta''+\frac{1}{2}Pz\xi''&=0.\end{aligned}\right\}\tag{4.7}$$

Equations (4.7) were integrated by Timoshenko in terms of infinite series.

The critical force P_{cr}, determined by the homogeneous differential equations (4.7) and by the homogeneous boundary conditions is expressed, in the general case, by the equation

$$P_{cr}=\frac{K\sqrt{EJ_yGJ_d}}{l^2}.\tag{4.8}$$

The stability coefficient K depends on the quantity

$$m^2=\frac{GJ_dl^2}{EJ_\omega}.\tag{4.9}$$

Table 39 presents a series of values of the coefficient K for various values of the elastic characteristic m^2. These data, calculated by Timoshenko for a wide-flanged beam, can be used on the basis of the generalization indicated above in the stability calculation of a beam of arbitrary section with two symmetry axes.

Table 39

m^2	0.4	4	8	16	24	32	48	64	80	96	160	240	320	400
K	86.4	31.9	25.6	21.8	20.3	19.6	18.8	18.3	18.1	17.9	17.5	17.4	17.2	17.2

In the case of a uniformly distributed load the moment M_x in an arbitrary section of a beam with hinged supports is determined by the equation

$$M_x = \frac{qz(l-z)}{2}, \tag{4.10}$$

where q is the load. The coordinate origin is chosen at the left support (Figure 178b). Introducing equation (4.10) in equations (4.3) we obtain the equations:

$$EJ_y\xi^{IV} + \frac{1}{2}q[z(l-z)\theta]'' = 0,$$
$$EJ_\omega\theta^{IV} - GJ_d\theta'' + \frac{1}{2}qz(l-z)\xi'' = 0.$$

The critical value of the load q can be determined here from the equation

$$ql = \frac{K\sqrt{EJ_yGJ_d}}{l^2}.$$

The stability coefficient K depends on the same auxiliary quantity m^2 as in the case of a concentrated load examined above, i. e. the quantity determined by equation (4.9). The values of the coefficient K for the case of a uniformly distributed load are given in Table 40.

Table 40

m^2	0.4	4	8	16	24	32	48	64	80	96	160	240	320	400
K	143	53.0	42.6	36.3	33.8	32.6	31.5	30.5	30.1	29.4	29.0	28.8	28.6	28.6

If the external forces acting on the beam consist of a load of magnitude q uniformly distribtued along the whole span and of a concentrated load P applied at midspan, then we obtain for the moment $M_x(z)$ the expression

$$M_x = \frac{1}{2}[Pz + qz(l-z)].$$

In this case equations (4.3) acquire the following form:

$$\left.\begin{aligned} EJ_y\xi^{IV} + \frac{1}{2}\{[Pz + qz(l-z)]\theta\}'' = 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' + \frac{1}{2}[Pz + qz(l-z)]\xi'' = 0. \end{aligned}\right\} \tag{4.11}$$

In these equations the external load is represented by the two parameters q and P. Taking the parameter q as a constant having a determinate numerical value, and P variable, we obtain the stability equations for a beam subjected to initial stresses due to a given load q (for example the dead weight) and loaded by a concentrated force P. The critical values of this force are determined from the differential equations (4.11) and the boundary conditions.

Depending on the value of the load q, the critical force P can be positive or negative. In particular it can vanish, if by chance the given load happens to be just the critical one.

§ 5. Stability of plane bending of beams with zero sectorial rigidity. Generalization of Prandtl's problem

Setting in equations (3.1) the sectorial rigidity EJ_ω equal to zero, we obtain the stability equations in transverse bending for beams whose torsional resistance is expressed only by the torsional rigidity GJ_d. These equations have the form

$$\left.\begin{aligned} EJ_y\xi^{IV} + (M_x\theta)'' &= 0,\\ EJ_x\eta^{IV} + (M_y\theta)'' &= 0,\\ M_x\xi'' + M_y\eta'' + [(2\beta_x M_y - 2\beta_y M_x - GJ_d)\,\theta']' &+ \\ + [q_x^0(e_x - a_x) + q_y^0(e_y - a_y)]\,\theta &= 0. \end{aligned}\right\} \qquad (5.1)$$

If the cross sections of a square beam have two symmetry axes then $a_x = a_y = 0$, $\beta_x = \beta_y = 0$. In this case equations (5.1) take the form

$$\left.\begin{aligned} EJ_y\xi^{IV} + (M_x\theta)'' &= 0,\\ EJ_x\eta^{IV} + (M_y\theta)'' &= 0,\\ M_x\xi'' + M_y\eta'' - GJ_d\theta'' + (q_x^0 e_x + q_y^0 e_y)\,\theta &= 0. \end{aligned}\right\} \qquad (5.2)$$

If the transverse load acts in the symmetry plane Oyz, then $q_x^0 = M_y = 0$.

The second equation of the system (5.2) is eliminated in this case, and the first and third equations form a system which, for $e_y = 0$, i.e. for the case when the load q_y is applied along the line of centroids, will have the form

$$\left.\begin{aligned} EJ_y\xi^{IV} + (M_x\theta)'' &= 0,\\ M_x\xi'' - GJ_d\theta'' &= 0. \end{aligned}\right\} \qquad (5.3)$$

The equations derived here cover a very wide class of practically important problems. These equations are applicable to all cases where the sectorial rigidity EJ_ω vanishes, i.e. where only nonuniformly distributed tangential stresses appear under torsion in a thin-walled beam with a nondeformable section contour. The normal and tangential stresses, determined by the law of sectorial areas, are equal to zero.

It was shown in § 1 of Chapter II that the sectorial normal stresses approximately vanish in beams consisting of a single or several very thin plates with a common intersection line (Figure 179). In this case the sectorial rigidity EJ_ω must be taken equal to zero. The torsional rigidity for pure torsion is determined as a product of the elastic shear modulus G and the torsional moment of inertia J_d. For the sections shown in Figure 179 the value of J_d can be calculated, with practically sufficient accuracy, from the equation

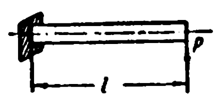
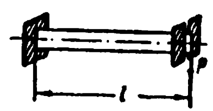
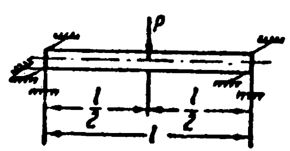
$$J_d = \frac{1}{3}\sum d_i\delta_i^3, \qquad (5.4)$$

where d_i and δ_i are respectively the width and thickness of the plate with the index i.

Equations (5.3) refer to a section having two axes of symmetry. Prandtl's equations for a beam of rectangular section, one of whose principal moments of inertia is small in comparison with the other, are obtained from these equations as a particular case.

After twice integrating the first differential equation of the system

Table 41

Load diagram and boundary conditions	1st critical force P	2nd critical force P	Higher critical forces P
	$4.01\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$	$10.24\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$	$\frac{\pi}{4}\,(8n-3)\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$
	$5.56\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$	$11.82\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$	$\frac{\pi}{4}\,(8n-1)\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$
	$16.94\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$	$68.6\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$	$2\pi\,(8n-5)\,\frac{\sqrt{EJ_yGJ_d}}{l^2}$

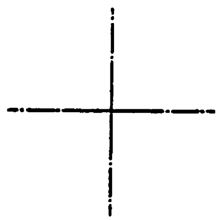


Figure 179

(5.3) we can write it in the form:

$$\left.\begin{aligned} EJ_y\xi'' + M_x\theta = Az + B, \\ GJ_d\theta'' - M_x\xi'' = 0, \end{aligned}\right\} \tag{5.5}$$

where A and B are arbitrary constants, respectively equal to the variations of the transverse force Q_x and the moment M_y in the origin section of the beam, $z = 0$. If these variations vanish, which is the case, for example, at the free end of a cantilever, then the constants A and B vanish and equations (5.5) take a simple form:

$$\left.\begin{aligned} EJ_y\xi'' + M_x\theta = 0, \\ GJ_d\theta'' - M_x\xi'' = 0. \end{aligned}\right\} \tag{5.6}$$

Expressing ξ'' in the first equation in terms of θ and substituting this expression in the second equation, we obtain the basic differential stability equation for sections of the type examined here. This equation can be written

$$EJ_yGJ_d\theta'' + M^2\theta = 0. \tag{5.7}$$

Equations (5.8) and (5.7) are identical with Prandtl's equations for a rectangular strip. Since, in our case, the cross section of the squared beam can be chosen arbitrarily, under the one condition that it have two symmetry axes and be characterized by a vanishing sectorial rigidity EJ_ω, Prandtl's investigations of the stability of a rectangular strip can be completely applied to the stability calculations of more complicated symmetrical sections. We shall examine several particular loading cases.

a) Pure bending. Taking the value of the moment M in equation (5.7) to be constant, we obtain an equation with constant coefficients which represents pure bending and is integrated in terms of trigonometric functions. If the ends of the square beam are completely free,

$$M_{cr} = \frac{\pi}{l}\sqrt{EJ_yGJ_d}, \tag{5.8}$$

if they are clamped,

$$M_{cr} = \frac{2\pi}{l}\sqrt{EJ_yGJ_d}.$$

We derived equation (V.5.8) in § 12 of Chapter V as a particular case of the more general problem of pure bending under a longitudinal force.

b) Bending of beams by transverse concentrated forces*. If a system of concentrated forces are acting in the symmetry plane Oyz of the beam, then the moment $M_x = M_x(z)$ varies linearly. For any unloaded portion of the beam, we can write this moment in the form

$$M(z) = M_0 + M_1 z, \tag{5.9}$$

where M_0 and M_1 are quantities linearly dependent on the forces P_1, P_2,..., acting on the beam. Introducing (5.9) in (5.7) we obtain

$$EJ_yGJ_d\theta'' + (M_0 + M_1 z)^2\theta = 0. \tag{5.10}$$

* The stability problems of plane bending of a beam with a narrow rectangular cross section are also examined in the work of Korobov /106/

Equation (5.10) is integrated in terms of Bessel functions of order $\pm\frac{1}{4}$. The generalized critical force P_{cr} is given by the formula

$$P_{cr} = \frac{K\sqrt{EJ_y GJ_d}}{l^2}, \tag{5.11}$$

where the stability coefficient K depends on the position of the forces, on the ratio between these forces, and on the end conditions of the squared beam. This coefficient is expressed in terms of the smallest root of the corresponding Bessel function.

Table 41 gives the stability coefficients K for various cases of boundary conditions, the beam being loaded by a single concentrated force.

We prepared this table on the basis of data obtained by Academician A.N. Dinnik for a strip consisting of a single narrow plate*. In the first column of Table 41 are shown the loading diagrams and the boundary conditions. The second column gives formulas for the determination of the critical design force corresponding to the smallest root of the Bessel function. The third column contains equations of the critical force corresponding to the second root of the Bessel function.

The last (fourth) column gives expressions for determining the higher critical forces. These expressions are obtained from asymptotic equations for the roots of the corresponding Bessel transcendental equation.

If the force P is applied at a distance a from one of the supports, then the critical force is expressed by the same equation (5.11). The coefficient K is determined from the transcendental equation obtained according to the boundary conditions from the Bessel function. Table 42 gives the values of this coefficient K for a beam with both ends free.

Table 42

$\frac{a}{l}$	0.5	0.4	0.3	0.2	0.1	0.05
K	16.94	17.82	21.01	29.11	56.01	112

c) The action of a uniformly distributed load. In this case the bending moment $M_x(z)$ varies quadratically. Denoting by q the uniform load and choosing the origin of the coordinates at the left end of the beam we can write (Figure 178b):

$$M = \frac{qz(l-z)}{2}.$$

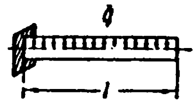
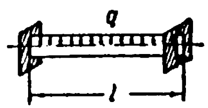
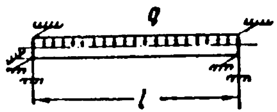
Introducing this expression in (5.7), we obtain

$$EJ_y GJ_d \theta'' + \frac{1}{4} q^2 z^2 (l-z)^2 \theta = 0. \tag{5.12}$$

Equation (5.12) is integrated in terms of Bessel functions of order $\pm\frac{1}{6}$. The critical value of the load q entering the argument of these functions is determined from the boundary conditions leading to the

* See Chapter IV of Dinnik's book /85/.

Table 43

Load diagram and boundary conditions	1st critical load q	2nd critical load q	Higher critical loads q
	$12.85\,\frac{\sqrt{EJ_yGJ_d}}{l^3}$	$38.56\,\frac{\sqrt{EJ_yGJ_d}}{l^3}$	$2\pi(3n-1)\,\frac{\sqrt{EJ_yGJ_d}}{l^3}$
	$15.95\,\frac{\sqrt{EJ_yGJ_d}}{l^3}$	$34.6\,\frac{\sqrt{EJ_yGJ_d}}{l^3}$	$\pi(6n-1)\,\frac{\sqrt{EJ_yGJ_d}}{l^3}$
	$28.3\,\frac{\sqrt{EJ_yGJ_d}}{l^3}$	—	—

transcendental Bessel equation. Table 43 gives the expressions for the critical loads for various cases of boundary conditions.

§ 6. Application of the method of virtual displacements to the spatial stability of beams

In general, equations (1.28) have variable coefficients. The inquiry after the eigenfunctions and eigenvalues with the help of these equations and boundary conditions is a complicated mathematical problem requiring much computation.

The determination of the critical numbers in this problem is considerably simplified if we use the principle of virtual displacements according to the Bubnov-Galerkin method. The essence of this principle as applied to the present problem consists in giving the deflections $\xi(z)$, $\eta(z)$ and the torsion angle $\theta(z)$, characterizing the flexural-torsional buckling of the squared beam, in the form of series:

$$\left.\begin{aligned} \xi(z) &= \sum A_n \varphi_n(z), \\ \eta(z) &= \sum B_n \psi_n(z), \\ \theta(z) &= \sum C_n \chi_n(z), \end{aligned}\right\} \tag{6.1}$$

where the functions $\varphi_n(z)$, $\psi_n(z)$ and $\chi_n(z)$ are so chosen that the deformed state of the squared beam, expressed by each term of the series (6.1), should satisfy the given boundary conditions.

Introducing expressions (6.1) in equations (1.2) we approximate by the series (6.1) the variations of the transverse loads and torsional moments expressed by the left-hand side of these equations.

We thus obtain a mechanical system subjected to external forces and moments determined with the accuracy of the coefficients of the series (6.1). Since this system is in equilibrium, the work done by all external and internal forces in the virtual (infinitesimal) displacements must vanish. Choosing as the virtual displacements

$$\xi_k = \varphi_k(z), \quad \eta_k = \psi_k(z) \text{ and } \theta_k = \chi_k(z) \qquad (k = 1, 2, 3, \ldots)$$

and equating to zero the work done in these displacements, we obtain for the coefficients A, B, C a system of linear homogeneous equations. The number of these equations will be equal to the number of the required coefficients. Equating to zero the determinant of the system of equations with the unknowns A, B, C we obtain an equation for the critical force. In general, the order of this equation will be equal to $3n$*. If we retain in expressions (6.1) the first term of the series, i.e. if we put ξ, η and θ in the form:

$$\xi = A\varphi(z), \quad \eta = B\psi(z), \quad \theta = C\chi(z), \tag{6.2}$$

we obtain for A, B and C a system of three homogeneous equations. Thus, for example, in the case of axial compression the introduction of the expressions (6.2) in the left-hand sides of equations (2.3) yields for the

* Here we proceed exactly as in § 13 of Chapter V.

variations of the transverse loads and the torsional moments the following expressions

$$\begin{aligned}\delta q_x &= A[EJ_y\varphi^{IV} - (N\varphi')'] - Ca_y(N\chi')',\\ \delta q_y &= B[EJ_x\psi^{IV} - (N\psi')'] - Ca_x(N\chi')',\\ \delta m &= C[EJ_\omega\chi^{IV} - GJ_d\chi'' - r^2(N\chi')'] - Aa_y(N\varphi')' + Ba_x(N\psi')'.\end{aligned}$$

After integrating by parts several terms and applying the natural boundary conditions at the ends of the beam, Lagrange's equations for these forces and the virtual displacements $\varphi(z)$, $\psi(z)$ and $\chi(z)$ become*:

$$\left.\begin{aligned} &A[EJ_y\int_0^l(\varphi'')^2dz + \int_0^l N(\varphi')^2dz] + Ca_y\int_0^l N\varphi'\chi'dz = 0,\\ &B[EJ_x\int_0^l(\psi'')^2dz + \int_0^l N(\psi')^2dz] - Ca_x\int_0^l N\psi'\chi'dz = 0,\\ &C[EJ_\omega\int_0^l(\chi'')^2dz + GJ_d\int_0^l(\chi')^2dz + r^2\int_0^l N(\chi')^2dz] +\\ &\qquad + Aa_y\int_0^l N\varphi'\chi'dz - Ba_x\int_0^l N\psi'\chi'dz = 0.\end{aligned}\right\} \quad (6.3)$$

Equating to zero the determinant of the coefficients of the unknowns A, B, C we obtain a cubic equation for the parameter of the external load in the expression of the function $N=N(z)$:

$$\begin{vmatrix} EJ_y\int_0^l(\varphi'')^2dz + \int_0^l N(\varphi')^2dz & 0 & a_y\int_0^l N\varphi'\chi'dz\\ 0 & EJ_x\int_0^l(\psi'')^2dz + \int_0^l N(\psi')^2dz & -a_x\int_0^l N\psi'\chi'dz\\ a_y\int_0^l N\varphi'\chi'dz & -a_x\int_0^l N\psi'\chi'dz & \begin{aligned}&EJ_\omega\int_0^l(\chi'')^2dz +\\ &+GJ_d\int_0^l(\chi')^2dz + r^2\int_0^l N(\chi')^2dz\end{aligned}\end{vmatrix} = 0. \quad (6.4)$$

* Thus, for example,

$$\int_0^l \varphi^{IV}\varphi\,dz = [\varphi'''\varphi]_{z=0}^{z=l} - [\varphi''\varphi']_{z=0}^{z=l} + \int_0^l(\varphi'')^2dz = \int_0^l(\varphi'')^2dz,$$

where the integrated terms vanish since they express the work done by the external forces applied at the ends of the beam, this work being equal to zero as either the statical factors or their corresponding geometrical factors vanish there.

The roots of equation (6.4) are three values for the critical load, corresponding to three flexural-torsional modes of buckling. All these roots are real since the detreminant (6.4) is symmetrical.

If the section has two axes of symmetry, then $a_x = a_y = 0$. The system of equations (6.4) falls into three independent equations. In this case the equation for determining the critical forces has the form

$$\left.\begin{aligned} EJ_y \int_0^l (\varphi'')^2\,dz + \int_0^l N(\varphi')^2\,dz = 0, \\ EJ_x \int_0^l (\psi'')^2\,dz + \int_0^l N(\psi')^2\,dz = 0, \\ EJ_\omega \int_0^l (\chi'')^2\,dz + GJ_d \int_0^l (\chi')^2\,dz + r^2 \int_0^l N(\chi')^2\,dz = 0. \end{aligned}\right\} \qquad (6.5)$$

The first two equations (6.5) refer to bending in the principal planes for certain modes of buckling. The third equation refers to torsion. As the critical design force we choose from the three critical forces the one that has the smallest absolute value.

The stability problem of the plane bending of beams subjected to the action of a transverse load can thus be solved. Representing the sought displacements $\xi(z)$, $\eta(z)$ and $\theta(z)$ in the form of (6.2) we obtain, from equations (3.1), the following expressions for the variations of the transverse loads and torsional moments

$$\begin{aligned} \delta q_x &= AEJ_y\varphi^{\mathrm{IV}} + C(M_x\chi)'', \\ \delta q_y &= BEJ_x\psi^{\mathrm{IV}} + C(M_y\chi)'', \\ \delta m &= C\{EJ_\omega\chi^{\mathrm{IV}} + [(2\beta_x M_y - 2\beta_y M_x - GJ_d)\chi']' + \\ &\quad + [q_x^*(e_x - a_x) + q_y^*(e_y - a_y)]\chi\} + AM_x\varphi'' + BM_y\psi''. \end{aligned}$$

Writing the equations for the work done by these loads and moments in the displacements $\varphi(z)$, $\psi(z)$ and $\chi(z)$, using, as before, integration by parts and satisfying the natural boundary conditions in the integrated expressions, we obtain:

$$\left.\begin{aligned} AEJ_y \int_0^l (\varphi'')^2\,dz + C\int_0^l M_x\varphi''\chi\,dz = 0, \\ BEJ_x \int_0^l (\psi'')^2\,dz + C\int_0^l M_y\psi''\chi\,dz = 0; \\ C\{EJ_\omega \int_0^l (\chi'')^2\,dz - \int_0^l (2\beta_x M_y - 2\beta_y M_x - GJ_d)(\chi')^2\,dz + \\ + \int_0^l [q_x^*(e_x - a_x) + q_y^*(e_y - a_y)]\chi^2\,dz\} + \\ + A\int_0^l M_x\varphi''\chi\,dz + B\int_0^l M_y\psi''\chi\,dz = 0. \end{aligned}\right\} \qquad (6.6)$$

The equation for determining the critical value of the parameter of the external transverse load, under which buckling occurs in the beam bent in a plane, is obtained by equating to zero the determinant of the coefficients of equations (6.6). This equation has the form:

$$\begin{vmatrix} EJ_y\int\limits_0^l(\varphi'')^2dz & 0 & \int\limits_0^l M_x\varphi''\chi\,dz \\ 0 & EJ_x\int\limits_0^l(\psi'')^2dz & \int\limits_0^l M_y\psi''\chi\,dz \\ \int\limits_0^l M_x\varphi''\chi\,dz & \int\limits_0^l M_y\psi''\chi\,dz & \begin{matrix} EJ_\omega\int\limits_0^l(\chi'')^2dz-\int\limits_0^l(2\beta_x M_y- \\ -2\beta_y M_x-GJ_d)(\chi')^2dz+ \\ +\int\limits_0^l[q_x^\bullet(e_x-a_x)+q_y^\bullet(e_y-a_y)]\chi^2dz. \end{matrix} \end{vmatrix}=0. \quad (6.7)$$

Equation (6.7) also furnishes three values for the parameter of the critical load.

If the cross section has a single axis of symmetry and the transverse load acts in the plane of symmetry, the stability of plane bending is described by the equations

$$AEJ_y\int\limits_0^l(\varphi'')^2dz+C\int\limits_0^l M_x\varphi''\chi dz=0,$$

$$A\int\limits_0^l M_x\varphi''\chi\,dz+C[EJ_\omega\int\limits_0^l(\chi'')^2dz+\int\limits_0^l(2\beta_y M_x+GJ_d)(\chi')^2dz+$$
$$+\int\limits_0^l q_y^\bullet(e_y-a_y)\chi^2dz]=0.$$

The determinant corresponding to these equations and which gives the critical values for the transverse load is

$$\begin{vmatrix} EJ_y\int\limits_0^l(\varphi'')^2dz & \int\limits_0^l M_x\varphi''\chi\,dz \\ \int\limits_0^l M_x\varphi''\chi\,dz & \begin{matrix} EJ_\omega\int\limits_0^l(\chi'')^2dz+\int\limits_0^l(2\beta_y M_x+GJ_d)(\chi')^2dz+ \\ +\int\limits_0^l q_y^\bullet(e_y-a_y)\chi^2dz \end{matrix} \end{vmatrix}=0.$$

The method presented here, as well as the Ritz-Timoshenko method, gives an exact solution of the problem if the functions $\varphi(z)$, $\psi(z)$ and $\chi(z)$ (which characterize the variation of the deformation of the beam with respect to the variable z) satisfy the corresponding differential stability equations. Thus, for example, prescribing a sinusoidal form for the

deflections and the torsion angles, i. e. assuming

$$\varphi = A \sin \lambda z, \quad \psi = B \sin \lambda z, \quad \chi = C \sin \lambda z, \tag{6.8}$$

where $\lambda = \frac{n\pi}{l}$ $(n = 1, 2, 3, \ldots)$, and writing for these displacements the work equations, we obtain an exact solution for a beam hinged at the ends and loaded by a longitudinal force $N = \text{const.}$ This follows from the fact that, as we saw before, the functions (6.8) satisfy the differential equations (V.3.1) and the boundary conditions. The determinant of the system (6.3) coincides in this case with the determinant (V.3.2).

If the functions $\varphi(z)$, $\psi(z)$ and $\chi(z)$ are not solutions of the differential stability equations, the method of virtual displacements gives approximate values for the critical forces. These values will deviate the more from the exact ones the better selected are the buckling shapes.

It is possible to show that the critical forces, found by the method of virtual displacements, will have exaggerated values. Indeed, by approximately describing the form of buckling we replace, from the mechanical point of view, the elastic system with an infinite number of degrees of freedom by a system having a finite number of degrees of freedom. Such change is equivalent to introducing connections into the elastic system, thus increasing its stability.

The method of virtual displacements gives the most effective solutions if we choose well the eignefunctions of the transverse vibrations of the beam for $\varphi(z)$, $\psi(z)$ and $\chi(z)$, trying to approximate, according to (6.1), the deformed state of the beam on buckling. As we saw in § 13 of Chapter V, these functions satisfy a homogeneous differential equation of the form

$$X^{\mathrm{IV}} - \mu^4 X = 0 \tag{6.9}$$

and homogeneous boundary conditions.

Representing flexural-torsional forms of buckling by the beam eigenfunctions, we have to select these functions according to the boundary conditions. With the help of the functions written in Table 33 (§ 13, Chapter V), it is possible to write the work equations (6.3) or (6.6) for various cases of boundary conditions. These conditions can be presented in mixed form: some for the deflections and others for the torsion angles.

The characteristic equation, determining the critical forces, is obtained by equating to zero the determinant of the corresponding system. As an example we shall examine a beam having a single axis of symmetry. Let the transverse load act in the plane of symmetry. In this case the stability equations (3.1) are

$$EJ_y \xi^{\mathrm{IV}} + (M_x \theta)'' = 0,$$

$$EJ_\omega \theta^{\mathrm{IV}} - GJ_d \theta'' - 2\beta_y (M_x \theta')' + q_y^* e_y \theta + M_x \xi'' = 0 \qquad (0 < z < l).$$

For hinged ends the boundary conditions will be:

$$\xi(0) = \xi(l) = \xi''(0) = \xi''(l) = 0,$$
$$\theta(0) = \theta(l) = \theta''(0) = \theta''(l) = 0.$$

If we introduce the formal quantities

$$\left.\begin{array}{c} \varkappa = \dfrac{lM_x}{EJ_y}, \quad a = \dfrac{EJ_\omega}{l^2 GJ_d}, \quad b_1 = \dfrac{2\beta_y EJ_y}{lGJ_d}, \\ b_2 = \dfrac{e_y EJ_y}{lGJ_d}, \quad c = \dfrac{EJ_y}{GJ_d}, \end{array}\right\} \tag{6.10}$$

and change the independent variable, writing $s = l\zeta$, the equations become

$$\left.\begin{aligned} &\xi^{IV} + l(\varkappa\theta)'' = 0, \\ &a\theta^{IV} - \theta'' - b_1(\varkappa\theta')' + b_2\varkappa''\theta + \frac{c}{l}\varkappa\xi'' = 0 \quad (0 < \zeta < 1), \end{aligned}\right\} \tag{6.11}$$

where ζ is the argument of all the functions and we indicate by primes the differentiation with respect to this variable. The boundary conditions assume the form

$$\left.\begin{aligned} &\xi(0) = \xi(1) = \xi''(0) = \xi''(1) = 0, \\ &\theta(0) = \theta(1) = \theta''(0) = \theta''(1) = 0. \end{aligned}\right\} \tag{6.12}$$

It could have been possible to apply the Bubnov-Galerkin method to the system of equations (6.11) but then it would have been necessary to give the expressions of both quantities ξ and θ. This, however, would have impaired the accuracy of the result. In order to increase the accuracy it is possible to apply the Bubnov-Galerkin method to one differential equation, giving only one of the unknown functions, for example the function θ.

Instead of formally eliminating ξ from the system (6.11) (which would lead to a rather cumbersome equation and would require the elucidation of the mechanical meaning of the latter), we use the boundary conditions in part. That is, from the first equation of the system (6.11) we have

$$\xi'' + l\varkappa\theta = C_1 + C_2\zeta, \tag{6.13}$$

where C_1 and C_2 are arbitrary constants. But it is seen from the boundary conditions (6.12) that the left-hand side of equation (6.13) vanishes for $\zeta = 0$ and for $\zeta = 1$, and therefore $C_1 = C_2 = 0$. Consequently,

$$\xi'' = -l\varkappa\theta.$$

The introduction of this expression in the second of equations (6.11) leads to the following differential equation for θ:

$$a\theta^{IV} - \theta'' - b_1(\varkappa\theta')' + (b_2\varkappa'' - c\varkappa^2)\theta = 0. \tag{6.14}$$

We apply now the Bubnov-Galerkin method to the equation (6.14), putting

$$\theta = A\sin\pi\zeta. \tag{6.15}$$

We introduce for this purpose the last expression in equation (6.14), multiply the result by $\sin\pi\zeta$ and integrate with respect to ζ from 0 to 1:

$$(a\pi^2 + 1)\frac{\pi^2}{2} - \pi b_1\int_0^1(\varkappa\cos\pi\zeta)'\sin\pi\zeta\,d\zeta + $$

$$+ b_2\int_0^1\varkappa''\sin^2\pi\zeta\,d\zeta - c\int_0^1\varkappa^2\sin^2\pi\zeta\,d\zeta = 0. \tag{6.16}$$

In order to get rid of the derivatives of the function $\varkappa$, we rearrange the integrals, using integration by parts,

$$\int_0^1(\varkappa\cos\pi\zeta)'\sin\pi\zeta\,d\zeta = -\pi\int_0^1\varkappa\cos^2\pi\zeta\,d\zeta,$$

$$\int_0^1\varkappa''\sin^2\pi\zeta\,d\zeta = -2\pi\int_0^1\varkappa'\sin\pi\zeta\cos\pi\zeta\,d\zeta = 2\pi^2\int_0^1\varkappa(\cos^2\pi\zeta - \sin^2\pi\zeta)\,d\zeta.$$

The introduction of these expressions in equation (6.16) gives, after rearrangement,

$$\int_0^1 \left[(b_1+4b_2)\varkappa+\frac{c}{\pi^2}\varkappa^2\right]\cos 2\pi\zeta\, d\zeta+$$

$$+\int_0^1 \left(b_1\varkappa-\frac{c}{\pi^2}\varkappa^2\right)d\zeta+(a\pi^2+1)=0. \qquad (6.17)$$

This equation gives approximately the critical value of the load parameter which interests us and is included in the expression for $\varkappa$.

We apply now equation (6.17) to some particular cases.

a) Pure bending. In this case M_x=const; consequently, $\varkappa$ = const, too, and equation (6.17) becomes

$$b_1\varkappa-\frac{c}{\pi^2}\varkappa^2+a\pi^2+1=0,$$

whence

$$M_{x\,cr}=\frac{EJ_y}{l}\varkappa_{cr}=\frac{EJ_y}{l}\left[\frac{\pi^2 b_1}{2c}\pm\sqrt{\frac{\pi^4 b_1^2}{4c^2}+\frac{\pi^2(1+\pi^2 a)}{c}}\right],$$

or, introducing the expressions for a, b_1 and c and reducing, we obtain

$$M_{x\,cr}=\frac{\pi^2 EJ_y}{l^2}\beta_y\pm\sqrt{\frac{\pi^2 EJ_y}{l^2}\left(\frac{\pi^2 EJ_y}{l^2}\beta_y^2+\frac{\pi^2 EJ_\omega}{l^2}+GJ_d\right)}.$$

Since, in this case, the given form of buckling (6.15) is exact, the values obtained for $M_{x\,cr}$ are the exact absolute values of the two smallest critical moments.

b) A uniformly distributed load q. In this case

$$M_x=\frac{q}{2}z(l-z)=\frac{ql^2}{2}\zeta(1-\zeta), \quad \varkappa=\frac{\lambda}{2}\zeta(1-\zeta),$$

where $\lambda=\frac{ql^3}{EJ_y}$. Introducing in (6.17) the values of the integrals

$$\int_0^1 \varkappa\cos 2\pi\zeta\, d\zeta=-\frac{\lambda}{4\pi^2}, \quad \int_0^1 \varkappa^2\cos 2\pi\zeta\, d\zeta=-\frac{3\lambda^2}{8\pi^4},$$

$$\int_0^1 \varkappa\, d\zeta=\frac{\lambda}{12}, \qquad \int_0^1 \varkappa^2\, d\zeta=\frac{\lambda^2}{120},$$

we obtain

$$-\frac{c}{\pi^2}\left(\frac{3}{8\pi^4}+\frac{1}{120}\right)\lambda^2+\left[\left(-\frac{1}{4\pi^2}+\frac{1}{12}\right)b_1-\frac{b_2}{\pi^2}\right]\lambda+(a\pi^2+1)=0,$$

or

$$c\lambda^2+(82.08b_2-46.99b_1)\lambda-(7995a+810.1)=0. \qquad (6.18)$$

This equation gives the two approximate critical values of the parameter λ_{cr}, to which correspond two approximate critical values of the load q_{cr} (of positive and negative critical magnitude), determined by the equation

$$q_{cr}=\frac{EJ_y}{l^3}\lambda_{cr}.$$

In order to evaluate the accuracy of these results we shall examine some concrete problems. First, we shall examine the buckling under

transverse bending of a strip (length l, height h, and thickness δ) subjected to the action of a uniformly distributed load q, applied along the line of centroids. In this case

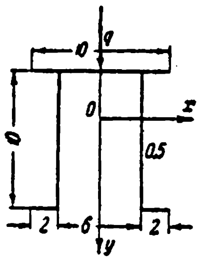


Figure 180

$$EJ_y = E\frac{h\delta^3}{12}, \quad GJ_d \approx G\frac{h\delta^3}{3}, \quad J_\omega = 0,$$
$$\beta_y = 0, \quad e_y = 0,$$

so that

$$a = 0, \quad b_1 = 0, \quad b_2 = 0, \quad c = \frac{E}{4G},$$

and equation (6.18) becomes

$$\frac{E}{4G}\lambda^2 - 810.1 = 0,$$

whence

$$\lambda_{cr} = \pm 56.92\sqrt{\frac{G}{E}},$$
$$q_{cr} = \pm\frac{h\delta^3 E}{12l^3}\lambda_{cr} = \pm 4.74\frac{h\delta^3}{l^3}\sqrt{GE}.$$

According to Timoshenko* this quantity equals $q_{cr} = \pm 4.72\frac{h\delta^3}{l^3}\sqrt{GE}$. The error, therefore, is smaller than half a percent.

We shall also examine the case of transverse bending of a beam of composite cross section freely hinged at the ends, of length $l = 100$ cm and loaded by a uniformly distributed load q, as shown in Figure 180. The thickness of the walls, forming the contour of the beam cross section, is equal to $\delta = 0.5$ cm. The other dimensions (in centimeters) are given in the figure.

Equation (6.10) here gives (taking $G = 0.4E$):

$$\frac{EJ_y}{GJ_d} = 290.0; \quad \frac{EJ_\omega}{GJ_d} = 1010; \quad \beta_y = 7.206; \quad e_y = -4.118;$$
$$a = 0.1010; \quad b_1 = 41.80; \quad b_2 = -11.94; \quad c = 290.0.$$

Equation (6.18) becomes

$$290\lambda^2 - 2944\lambda - 1618 = 0,$$

whence

$$\lambda_{cr}^{(1)} = -0.522; \quad \lambda_{cr}^{(2)} = +10.67,$$

and, since $J_y = 164.3$ cm^4; $l = 100$ cm, then $q_{cr}^{(1)} = -8.58 \cdot 10^{-5} E\frac{\text{kg}}{\text{cm}}$, $q_{cr}^{(2)} = +175 \cdot 10^{-5} E\frac{\text{kg}}{\text{cm}}$

A more accurate calculation (if we proceed from an expression for θ with three parameters: $\theta = A\sin\pi\zeta + B\sin 2\pi\zeta + C\sin 3\pi\zeta$) shows that the approximate absolute values obtained for the critical load are too large by 3% and 6%, respectively.

* S.P. Timoshenko. "On the Stability of Elastic Systems", Kiev, 1910, p 106.

Chapter VII

EQUILIBRIUM OF THIN-WALLED BEAMS UNDER COMBINED LOADING

§ 1. Bending and torsion of beams subjected to initial stresses

Let a beam having a rigid section contour be unstable under some critical load consisting, in general, of longitudinal forces and moments. The linear theory then shows that any transverse load applied to the beam will cause an infinitely large deformation. It follows that the state of stress due to any longitudinal load smaller than the critical one (but close to it) will have a marked influence on the deformation of a beam due to transverse loading.

Thus the problem arises of the spatial equilibrium of a beam subjected to stresses due to an external transverse load and also to a given initial state of stress, consisting solely of longitudinal normal stresses. Such a state will occur in the case of pure extension or compression or of pure bending by an external longitudinal eccentric or axial extension or compression and also in the case of pretension by a longitudinal reinforcement [in concrete] or thermal effects. We shall assume that not only the tensile force of the reinforcement but also the external longitudinal load are transmitted to the beam through rigid end diaphragms (Figure 181a).

We can, therefore, determine the normal initial stresses (constant along the beam) from the equations of the law of plane sections which will be in general of the form

$$n_p = -\frac{P}{F} + \frac{M_x}{J_x} y - \frac{M_y}{J_y} x, \tag{1.1}$$

$$n_R = -\frac{\sum R_k}{F} - \frac{\sum R_k y_k}{J_x} y - \frac{\sum R_k x_k}{J_y} x, \tag{1.2}$$

where x, y are position coordinates in the cross section of the beam-shell, referred to the principal central axes Ox, Oy; F is the cross sectional area of the beam; J_x, J_y are the principal moments of inertia of the section area with respect to the axes Ox, Oy; P is the external longitudinal compressive force applied at the centroid of the cross section; M_x, M_y are the external bending moments applied at the ends of the beam and acting relstive to the axes Ox, Oy respectively; R_k is the tension in the k-th reinforcing rod (Figure 181b); x_k, y_k are the coordinates of the point of application of the force R_k, and simultaneously also of the centroid of the cross section of the k-th rod.

Equation (1.1) determines the stresses n_p resulting from the external load. These stresses over the whole cross section are statically equivalent to the longitudinal force P and the bending moments M_x, M_y,

$$\int_F n_p dF = -P, \quad \int_F n_p y dF = M_x, \quad \int_F n_p x dF = -M_y.$$

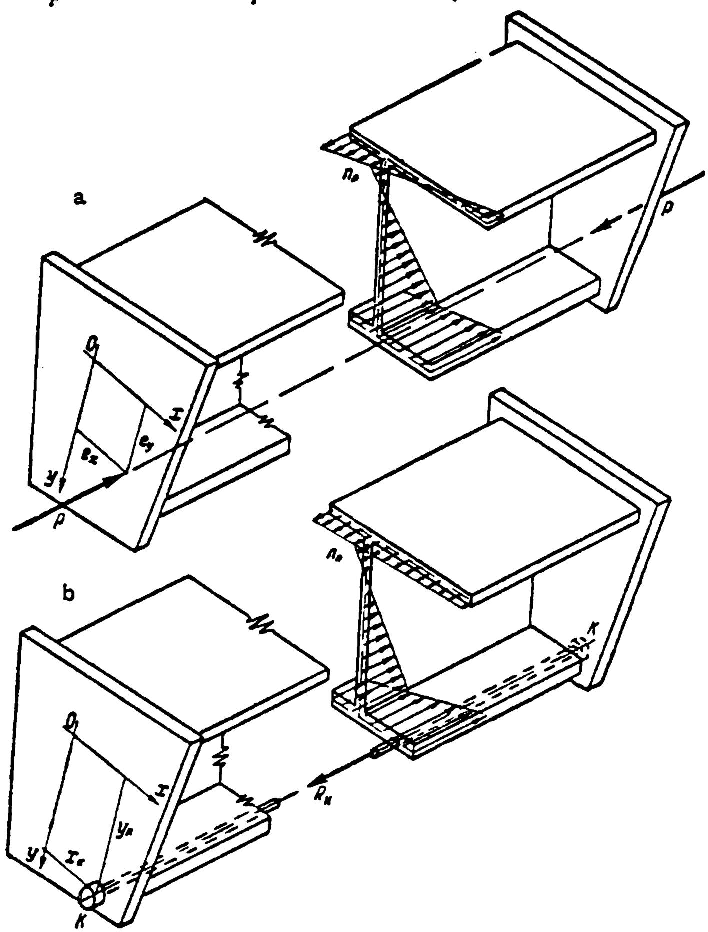


Figure 181

Equation (1.2) determines the stresses n_R due to pretension by k longitudinal reinforcing rods. These stresses, together with those in the cross sections of the reinforcement rods, lead to a system of forces statically equivalent to zero for each cross section, i.e. to bimoments. Consequently, in the case of initial tension due to longitudinal reinforcement a bimoment state of stress appears in each cross section of the beam. Since the area of the reinforcement is sufficiently small in comparison with the area of the whole cross section, the forces in the reinforcement can be taken as concentrated.

This constitutes the essential difference between the initial stresses due to an external longitudinal load, and those due to pretension by a

longitudinal reinforcement. It will be seen below that this also affects the equilibrium equations of a beam in a state of initial stress, depending on the nature of the causes of this state.

To derive the equilibrium equations of a beam in initial stress we shall start from the equations (I.7.3) for a beam subjected to an external transverse load

$$\left.\begin{aligned} EJ_y\xi^{IV} - q_x = 0, \\ EJ_x\eta^{IV} - q_y = 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' - m = 0. \end{aligned}\right\} \qquad (1.3)$$

To do this, we assume that the initial stresses in the beam, as determined by the three-term equation of the law of plane sections, cause during deformation the appearance of additional reduced distributed loads $\bar{q}_x$, $\bar{q}_y$ and $\bar{m}$. The equations for these can be obtained in complete analogy with the derivation of the stability equations in § 1, Chapter V.

These equations have the form (V.1.4),

$$\left.\begin{aligned} \bar{q}_x &= \xi'' \int_l n\delta ds - \theta'' \int_l n\delta(y - a_y)\, ds, \\ \bar{q}_y &= \eta'' \int_l n\delta ds + \theta'' \int_l n\delta(x - a_x)\, ds, \\ \bar{m} &= -\xi'' \int_l n\delta(y - a_y)\, ds + \eta'' \int_l n\delta(x - a_x)\, ds + \\ &\quad + \theta'' \int_l n\delta[(x - a_x)^2 + (y - a_y)^2]\, ds. \end{aligned}\right\} \qquad (1.4)$$

Depending on the causes of the initial stress, it is necessary to substitute in equation (1.4) for n either n_p from (1.1) or n_R from (1.2).

Substituting in the right-hand sides of (1.4) for n their values n_p from (1.1) and noting that the beam section is referred to the principal central axes, we obtain the expressions (V.1.5) for the components of the additional reduced distributed load due to an external compressive force P applied eccentrically at the ends of an initially stressed beam:

$$\left.\begin{aligned} \bar{q}_x &= -P\xi'' - (a_yP + M_x)\theta'', \\ \bar{q}_y &= -P\eta'' + (a_xP - M_y)\theta'', \\ \bar{m} &= -(a_yP + M_x)\xi'' + (a_xP - M_y)\eta'' + \\ &\quad + (-r^2P + 2\beta_yM_x - 2\beta_xM_y)\theta'', \end{aligned}\right\} \qquad (1.5)$$

where the coefficients a_x, a_y, r^2, β_x and β_y are calculated from (I.7.5) and (V.1.6).

Introducing the components (1.5) into (1.3) we obtain the differential equations for the analyzed case

$$\left.\begin{aligned} EJ_y\xi^{IV} + P\xi'' + (M_x + a_yP)\theta'' = q_x, \\ EJ_x\eta^{IV} + P\eta'' + (M_y - a_xP)\theta'' = q_y, \\ EJ_\omega\theta^{IV} + (r^2P + 2\beta_xM_y - 2\beta_yM_x - GJ_d)\theta'' + \\ + (M_x + a_yP)\xi'' + (M_y - a_xP)\eta'' = m. \end{aligned}\right\} \qquad (1.6)$$

If the initial stress of the beam is not caused by a compressive force P but by a longitudinal tensile force P applied eccentrically at the ends,

we obtain the equations for the beam from the system (1.6), changing the sign of P and putting $M_x = Pe_y$, $M_y = -Pe_x$.

If the external transverse load is only applied at the ends and not along the span, the system (1.6) will be homogeneous, i. e. we have to set in (1.6) $q_x = q_y = m = 0$, and the external transverse load applied at the ends will be included in the corresponding inhomogeneous boundary conditions.

When the compressive force P is close to the critical, the fundamental equilibrium shape of the beam, for which equations (1.6) were derived, becomes unstable. As shown above, another equilibrium shape is possible for such a critical state of the beam. Referring the system of differential equations (1.6) to the latter shape and setting in (1.6) $q_x = q_y = m = 0$, we come back to the previously derived stability equations (V. 1.10).

Finally, assuming in equations (1.6) the external longitudinal load equal to zero, i. e. $P = M_x = M_y = 0$, we are led to the initial equations (1.3).

Evidently the system of differential equations (1.6) has a rather general character and describes a fairly large class of practically important problems.

We shall now return to beams whose initial state of stress is caused by the prestressed reinforcement. We shall assume, for simplicity, that there is only one prestressed reinforcement rod in the beam. Let this be the k-th rod passing through the point with coordinates x_k and y_k (Figure 181b). We shall neglect the sign of the sums in equation (1.2).

Introducing the expression n_R (1.2) into the expressions (1.4) and remembering that together with the stresses n_R in the cross section of the beam, we have also to consider the stresses in the reinforcement itself, which we take as a concentrated force R_k applied at the point (x_k, y_k), and recalling that the cross section is referred to the principal central axes, we find

$$\bar{q}_x = 0, \quad \bar{q}_y = 0, \quad \bar{m} = -R_k \mathfrak{M}_k \theta'',$$

where

$$\mathfrak{M}_k = \frac{J_x + J_y}{F} + \frac{U_x}{J_x} y_k + \frac{U_y}{J_y} x_k - \rho_k^2, \tag{1.7}$$

and

$$U_x = \int_F y\rho^2 dF, \quad U_y = \int_F x\rho^2 dF,$$

$$\rho^2 = x^2 + y^2, \quad \rho_k^2 = x_k^2 + y_k^2.$$

We observe that the pretension of the reinforcement, in contrast with the case examined before of longitudinal compressive force, leaves unchanged the first two equations (1.3) referring to transverse bending in the beam in its principal planes, and affects only the third, which refers to the torsion of the beam.

Adding $\bar{m}$ (the reduced torsional moment due to the pretension in the reinforcement) to m (the torsional moment due to the external transverse load) and adding the terms in the second derivative with respect to the torsional angle, we obtain the third equation of the system (1.3) in the form

$$EJ_\omega \theta^{IV} - (GJ_d - R_k \mathfrak{M}_k)\theta'' - m = 0. \tag{1.8}$$

If we denote the expression in brackets of equation (1.8) by

$$\overline{GJ}_d = GJ_d - R_k \mathfrak{M}_k, \tag{1.9}$$

the form of equation (1.8) will be completely analogous to that of the third equation of the system (1.3)

$$EJ_\omega \theta^{IV} - \overline{GJ}_d \theta'' - m = 0. \tag{1.10}$$

We thus arrive at a very important conclusion: the calculation and analysis of the torsional stress and deformation of thin-walled transversely loaded beams with pretensioned reinforcement, can be carried out with equation (1.10), similar to the third equation of the system (1.3) for an ordinary thin-walled beam (without prestressed reinforcement), providing that in the latter the rigidity for pure torsion, GJ_d, is replaced by the new reduced rigidity $\overline{GJ}_d$ (1.9).

Consequently, the theory and design methods of thin-walled beams subjected to an external transverse load apply in their entirety to thin-walled beams with prestressed reinforcement. This opens wide possibilities for theoretical and experimental studies in this field and for their practical application.

Since the differential equations, in the examined case, refer to thin-walled reinforced concrete beams, in which the prestressed reinforcement acts together with the concrete, the principal characteristics of the section, J_x, J_y, J_ω and the moment of inertia for pure torsion J_d should be calculated for the total reduced cross section, including the area of the reinforcement multiplied by the ratio of the elastic moduli of the reinforcing steel and the concrete.

If the initial stress in the beam is caused not only by pretensioned reinforcements, but also by an external compressive longitudinal force, we have to use equations (1.6), replacing in the third equation of this system the rigidity for pure torsion GJ_d by the new reduced rigidity $\overline{GJ}_d$ defined by (1.9).

§ 2. Bending and torsion of a beam under initial longitudinal load

We shall examine some cases of beam loading by longitudinal forces. Let the beam be initially loaded at the ends by an axial longitudinal force P. We shall assume that the beam has two symmetry axes in its cross section. Then the shear center coincides with the section centroid. Under these conditions, obviously $M_x = M_y = 0$; $a_x = a_y = 0$ and the system of equations (1.6) falls into three independent equations of the form

$$\left.\begin{aligned} EJ_y \xi^{IV} \pm P\xi'' &= q_x, \\ EJ_x \eta^{IV} \pm P\eta'' &= q_y, \\ EJ_\omega \theta^{IV} - (GJ_d \mp r^2 P)\theta'' &= m. \end{aligned}\right\} \tag{2.1}$$

In these equations the upper sign of P refers to the case of an external compressive force P and the lower sign to the case of an tensile force.

Each of equations (2.1) can be written either as

$$X^{IV}+\frac{k^2}{l^2}X''=Q, \qquad (2.2)$$

or as

$$X^{IV}-\frac{k^2}{l^2}X''=Q, \qquad (2.3)$$

where X stands for any of the functions ξ, η or θ, depending on which equation is examined; similarly k^2 and Q will have different values for each equation, viz.

$$k^2=\frac{Pl^2}{EJ_y},\quad \frac{Pl^2}{EJ_x},\quad \frac{(GJ_d-r^2P)\,l^2}{EJ_\omega},\quad \frac{(GJ_d+r^2P)\,l^2}{EJ_\omega};$$

$$Q=\frac{q_x}{EJ_y},\quad \frac{q_y}{EJ_x},\quad \frac{m}{EJ_\omega}.$$

Equation (2.3) is entirely similar to the equation of torsion of a thin-walled beam. Hence it is possible to use the previous results in the integration, replacing the coefficients of the equations correspondingly. Thus, for example, if an external torsional moment acts on the compressed beam, its twist is determined by the equation

$$EJ_\omega\theta^{IV}-(GJ_d-r^2P)\,\theta''=m.$$

This equation differs from the torsion equation examined before (II.2.1) only in the coefficient of the second derivative with respect to the torsion angle. In contrast to the previous value $\frac{k^2}{l^2}=\frac{GJ_d}{EJ_\omega}$, the quantity $\frac{k^2}{l^2}$ will be expressed in this case by the equation

$$\frac{k^2}{l^2}=\frac{GJ_d-r^2P}{EJ_\omega}.$$

Consequently, using the solutions obtained before, we have to replace the rigidity in pure torsion GJ_d by a new rigidity (GJ_d-r^2P).

The differential equation (2.2) differs from (2.3) only in the sign of the term with the second derivative and, consequently, its general solution will differ from the general solution of (2.3) in that the particular solutions will be in terms of trigonometric instead of hyperbolic functions of the same argument.

Example 1. We shall examine a compressed beam hinged at one end and free at the other on which a bending moment M_{0y} is acting at the ends (Figure 182). In this case, the equilibrium equation (for $q_y=0$) will have the form

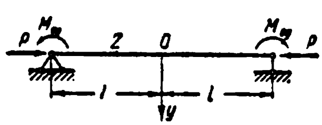


Figure 182

$$EJ_x\eta^{IV}+P\eta''=0. \qquad (2.4)$$

The general solution of the homogeneous equation (2.4) has the form

$$\eta=C_0+C_1z+C_2\cos\frac{kz}{l}+C_3\sin\frac{kz}{l}, \qquad (2.5)$$

where $k=l\sqrt{\frac{P}{EJ_x}}$.

If the origin of the coordinates is taken in the middle of the span,

then, by symmetry, $C_1 = C_3 = 0$ and C_0 and C_2 are determined from the boundary conditions (for $z = l$ $\eta = 0$ and $M_y = M_{0y}$):

$$C_0 = -\frac{M_{0y}}{EJ_x}\frac{l^2}{k^2}, \quad C_2 = \frac{M_{0y}}{EJ_x}\frac{l^2}{k^2\cos k}$$

For the deflection and the bending moment we thus obtain

$$\eta = -\frac{M_{0y}}{EJ_x}\frac{l^2}{k^2}\left(1 - \frac{\cos\frac{kz}{l}}{\cos k}\right),$$

$$M_y = -EJ_x\eta'' = M_{0y}\frac{\cos\frac{kz}{l}}{\cos k}.$$

Calculations, performed for a No 24a I-section beam of 3 m length, give the value $M_{y_{\max}} = 1.56\,M_{0y}$ for the bending moment in the middle of the span, i.e. the longitudinal compressive force increases the design bending moment by 56%. If P is not compressive but tensile, the equation for the bending moment in the middle of the span will be analogous but with the trigonometric functions replaced by hyperbolic ones:

$$M_y = M_{0y}\frac{\operatorname{ch}\frac{kz}{l}}{\operatorname{ch}k}.$$

A comparison of the two analytical expressions for the moments and of their graphs (Figure 183) shows that in the case of an axial tensile force the beam will be always stable, whereas in the case of a compressive force it will be so only for the condition $P \leqslant P_{cr}$, where P_{cr} is determined by (V.3.3) for $\lambda = \frac{\pi}{l}$.

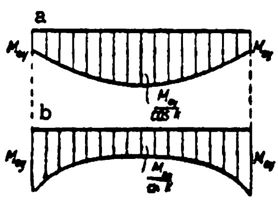


Figure 183

Example 2. We consider the same compressed beam with a uniformly distributed transverse load q acting on it in the plane Oyz.

We shall now have for the deflection η instead of (2.4) the corresponding inhomogeneous equation

$$EJ_x\eta^{IV} + P\eta'' = q, \qquad (2.6)$$

and we obtain the general solution of (2.6) by adding to the solution of the homogeneous equation (2.5) the particular integral of the inhomogeneous equation depending on the load q. We obtain

$$\eta = C_0 + C_1 z + C_2\cos\frac{kz}{l} + C_3\sin\frac{kz}{l} + \frac{q}{2P}z^2. \qquad (2.7)$$

Starting from the solution (2.7) and assuming hinged-end conditions, we obtain (for a span $2l$ and the coordinate origin in the middle of the span) for the deflection and the moment,

$$\eta = \frac{q}{2P}(z^2 - l^2) - \frac{ql^2}{Pk^2}\left(1 - \frac{\cos\frac{k}{l}z}{\cos k}\right),$$

$$M_y = -\frac{ql^2}{Pk^2}\left(1 - \frac{\cos\frac{k}{l}z}{\cos k}\right).$$

These equations are analogous to (II.5.9) (for a thin-walled beam twisted by an external, uniformly distributed torsional moment m) in which θ is replaced by η, B by M_y, m by q, GJ_d by P, the hyperbolic functions by trigonometric functions (taking account of the signs), etc.

As before, it is possible to start here not with the general solution (2.7) but to solve the problem by the method of initial parameters. In our case the basic design quantities are η, η', M and Q. Because of the adopted rule of signs, the bending moment and the transverse force are determined by

$$\left.\begin{aligned} &M = -EJ_x\eta'',\\ &Q = -\int q dz = -\int (EJ_x\eta^{IV} + P\eta'')\,dz = -EJ_x\eta''' - P\eta'. \end{aligned}\right\} \tag{2.8}$$

Since (2.4) and (2.8) are similar to (II.3.1) and (II.2.5), it is obvious that the matrix of the initial parameters is obtained from the matrix of Table 3, § 3, Chapter II, if we replace there θ by η, θ' by η', B by M, H by Q and GJ_d by $-P$.

§ 3. Bending and torsion of beams with pretensioned reinforcement

1. We return to (1.7) which determines the coefficient $\mathfrak{M}_k$ of the reduced torsional moment that accounts for the action of the pretensioned reinforcement. The quantity $\mathfrak{M}_k$ depends on the dimensions of the beam and on the position coordinates of the reinforcement R_k (Figure 181b). By varying the coordinates x_k and y_k we can make $\mathfrak{M}_k$ positive or negative and, in particular, equal to zero. It follows from (1.9) that for $\mathfrak{M}_k = 0$ a prestressed beam does not exert any influence on the variation of $\overline{GJ}_d$, the latter remaining equal to GJ_d. For $\mathfrak{M}_k > 0$ the rigidity of the beam in pure torsion is decreased and for $\mathfrak{M}_k < 0$ it is increased. Assuming $\mathfrak{M}_k = 0$, we obtain the equation of a curve, each of whose points has the property that placing a pretensioned reinforcement there does not change the torsional rigidity of the beam. The equation of this curve is

$$x_k^2 + y_k^2 - \frac{U_x}{J_x} y_k - \frac{U_y}{J_y} x_k - \frac{J_x + J_y}{F} = 0. \tag{3.1}$$

It is completely analogous to (V.8.1) and represents a circle whose radius and position of center are determined from (V.8.2) and (V.8.3),

$$\left.\begin{aligned} &R^2 = \frac{U_x^2}{4J_x^2} + \frac{U_y^2}{4J_y^2} + \frac{J_x + J_y}{F},\\ &k_x = \frac{U_y}{2J_y}, \quad k_y = \frac{U_x}{2J_x}. \end{aligned}\right\} \tag{3.2}$$

For a beam with two symmetry axes $U_x = U_y = 0$ and the radius of the circle equals the radius of gyration $R = \sqrt{\frac{J_x + J_y}{F}}$. Its center coincides with the section centroid. In particular, for a square with side a we have $R = \frac{a}{\sqrt{6}} = 0.408\,a$.

Inside the circle (3.1) $\mathfrak{M}_k > 0$ and outside it $\mathfrak{M}_k < 0$. If we take in equations (1.7) $\mathfrak{M}_k = C$, where C is some constant, we arrive at the equation of a circle whose radius is determined by the relation

$$R_C^2 = \frac{U_x^2}{4J_x^2} + \frac{U_y^2}{4J_y^2} + \frac{J_p}{F} - \mathfrak{M}_k \tag{3.3}$$

(here $J_p = J_x + J_y$ is the polar moment of inertia).

This circle is remarkable in that a pretensioned reinforcement affects the state of stress in a beam equally for any point on the circle in which it may be placed.

Equation (3.3) shows that the highest positive value which $\mathfrak{M}_k$ can attain is

$$\mathfrak{M}_{k\,\max} = \frac{U_x^2}{4J_x^2} + \frac{U_y^2}{4J_y^2} + \frac{J_p}{F} \text{ (whereby } R_C = 0);$$

for $\mathfrak{M}_k = 0$ we have the equality $R_C = R$ (see (3.2)). On further increasing R_C, $\mathfrak{M}_k$ becomes negative. Figure 184 shows a plot of R_C versus $\mathfrak{M}_k$ for a beam of square section with the side a.

It follows from equation (1.9) that it is possible to choose a value for $\mathfrak{M}_k$ for which $\overline{GJ_d} = 0$, i. e. the beam will not show resistance to torsion. For this to occur it is necessary that

$$\overline{GJ_d} = GJ_d - R_k\mathfrak{M}_k = 0,$$

whence

$$\mathfrak{M}_k = \frac{GJ_d}{R_k}.$$

For beams with two symmetry axes we obtain

$$\frac{J_p}{F} - \rho_k^2 = \frac{GJ_d}{R_k},$$

whence

$$\rho_k^2 = x_k^2 + y_k^2 = \frac{J_p}{F} - \frac{GJ_d}{R_k}.$$

If the pretensioned reinforcement passes through the centroid of the section ($\rho_k = 0$), then for the beam not to resist torsion the stress R_k in the reinforcement must be

$$R_k = GF\frac{J_d}{J_p}.$$

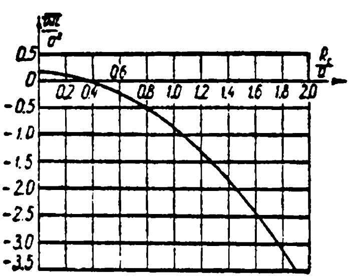


Figure 184

The same effect (no resistance to torsion) can be obtained by correspondingly passing two or several prestressed reinforcements inside and outside the circle (3.2).

2. The theoretical results obtained here and in the previous section, referring to the determination of the reduced torsional rigidity of a prestressed beam, were verified experimentally on small models in the Institute of Mechanics of the Academy

of Sciences of the USSR. One of these models is shown in Figure 185. The model is a thin-walled beam consisting of two intersecting narrow plates in the form of a symmetrical cross. A cylindrical bore, in which a longitudinal metal reinforcing rod can freely move, passes inside the beam along its axis (through the section centroid). An initial tension, whose magnitude is controlled by rotating a support nut, can be applied to this rod

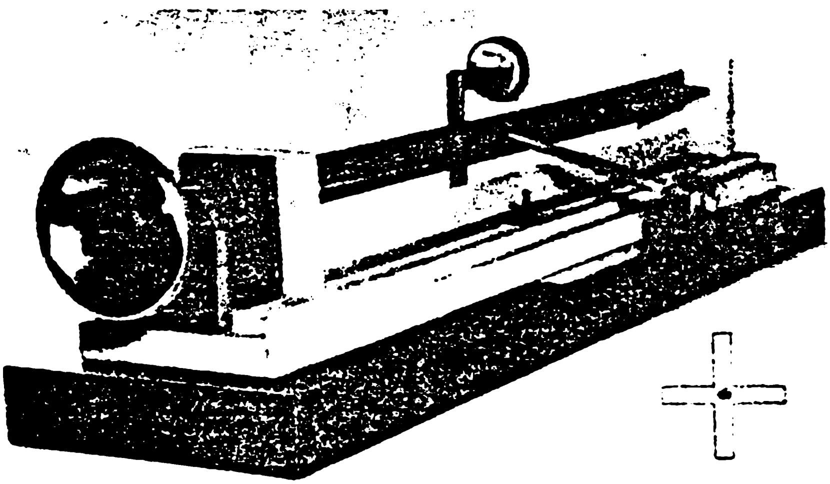


Figure 185

with the help of simple arrangements at the supports. The model is fitted with a lateral horizontal arm in the middle of the span. In addition to transverse bending the load suspended at the end of the arm also twists the model. The torsion angle, measured in the middle section by a needle indicator, will also depend, for a given external torsional moment, on the tension caused inside the beam by the longitudinal reinforcement. When this tension is increased the reduced rigidity decreases according to the above and, consequently, the torsion angle in the middle section is greater for the same torsional moment.

§ 4. Torsion of beams subjected to thermal stresses

We shall now examine the spatial equilibrium problem of a thin-walled beam with free ends having initial thermal stresses and subjected to the action of an external transverse load. These stresses can be given by a one-term bimoment relation*

$$n(z, s) = T\Phi(z)\varphi(s), \tag{4.1}$$

where T is a parameter determining the temperature; $\Phi(z)$ is a function depending on z only and giving the distribution of the normal stresses n

* This will be treated in more detail in Ch. XI.

over the length of the beam; $\varphi(s)$ is the distribution function of the thermal stresses over the cross section of the beam.

In the transition of the beam, due to an external transverse load, into a new deformed state determined by the deflections $\xi=\xi(z)$, $\eta=\eta(z)$ and the torsion angle $\theta=\theta(z)$ the initial thermal stresses in the beam can be reduced to an additional (fictitious) external transverse load in analogy with the approach used above for beams with pretensioned reinforcement. In the present case, since the n determined by (4.1) depends only on the coordinate z, we have for the calculation of the components of this load the following expressions

$$\left.\begin{aligned} \overline{q}_{xt}&=(\xi'\int\limits_F n\,dF)'-[\theta'\int\limits_F n(y-a_y)\,dF]',\\ \overline{q}_{yt}&=(\eta'\int\limits_F n\,dF)'+[\theta'\int\limits_F n(x-a_x)\,dF]',\\ \overline{m}_t&=-[\xi'\int\limits_F n(y-a_y)\,dF]'+[\eta'\int\limits_F n(x-a_x)\,dF]'+\\ &\quad+\{\theta'\int\limits_F n[(x-a_x)^2+(y-a_y)^2]\,dF\}', \end{aligned}\right\} \qquad (4.2)$$

where the primes denote derivatives with respect to z.

Introducing in (4.2) the expressions of n from (4.1) and keeping in mind the orthogonality of the functions $\varphi(s)$ and the generalized coordinates $1, x, y$ following from the law of plane sections (i. e. $\left(\int\limits_F 1\varphi\,dF=\int\limits_F x\varphi\,dF=\int\limits_F y\varphi\,dF=0\right)$, we obtain

$$\overline{q}_{xt}=\overline{q}_{yt}=0,$$

$$\overline{m}_t=T[\Phi(z)\,\theta'(z)]'\int\limits_F\varphi(s)[(x-a_x)^2+(y-a_y)^2]\,dF=KT(\Phi\theta')',$$

where K is the generalized elastic characteristic depending on the function $\varphi(s)$ and on the dimensions of the cross section, and is computed from the formula

$$K=\int\limits_F\varphi[(x-a_x)^2+(y-a_y)^2]\,dF,$$

where a_x, a_y are the coordinates of the shear center.

The basic equation of the theory of restrained torsion now becomes

$$EJ_\omega\theta^{IV}-GJ_d\theta''-KT(\Phi\theta')'-m=0.$$

In the general case this equation has variable coefficients. In many cases, however, the function $\Phi(z)$ is nearly constant, at least in the middle part of the beam. Assuming $\Phi(z)=\text{const}$ we obtain an equation with constant coefficients and the problem is reduced to the determination of the reduced rigidity for pure torsion, which in the present case is

$$\overline{GJ_d}=GJ_d+KT\Phi.$$

§ 5. Stability of beams subjected to initial stresses

1. The general equilibrium equations (1.6) for beams subjected to a given initial state of stress were derived in § 1 of the present chapter and it was mentioned that under certain conditions these equations pass into the stability equations (V.1.10). It was there shown that if the initial stresses are caused by pretensioned reinforcement or are due to the temperature (in those cases where it can be considered constant along the beam) the first two equations of (1.6) do not change but the third does, though it retains its form providing that the rigidity GJ_d of the beam in pure tension is replaced by a new reduced rigidity $\overline{GJ}_d$ which depends on the character of the initial stress,

$$\overline{GJ}_d = GJ_d - R_k\mathfrak{M}_k, \tag{5.1}$$

or

$$\overline{GJ}_d = GJ_d + KT\Phi. \tag{5.2}$$

Taking all this into consideration, we come to the conclusion that all problems connected with the system of stability equations (V.1.10) and also the problems of flexural-torsional vibrations and dynamic stability of thin-walled beams (to be treated in Chapter IX) are easily generalized to include the case of beams in a pretensioned state. For that purpose, it is only necessary to replace the rigidity in pure torsion GJ_d by the reduced rigidity $\overline{GJ}_d$. In the case of a pretensioned reinforcement the latter is calculated from (5.1) and in the case of thermal prestress from expression (5.2) (in those places where the temperature remains constant along the beam). It also follows from these considerations that the prestressed state of a beam influences the magnitude of the critical force causing torsional instability but does not influence the magnitude of the critical Euler forces since the latter do not depend on GJ_d.

2. All the problems connected with prestressed beams, described in the last four sections for thin-walled open sections, can be extended to beams designed with allowance for longitudinal bending moments, and to closed sections. As shown before, for beams of closed section warping under torsion must be expressed in the form $\omega = xy$ and the bimoment of inertia has to be determined from

$$J_\omega = \int_F x^2y^2dF;$$

where we have to take GJ_d as Saint-Venant's torsional rigidity. In particular, for the rectangular box-like closed section with sides d_1, d_2 and thickness δ, examined in Chapter IV, the geometrical characteristics are determined by (V.1.18). When beams are to be designed taking into account the longitudinal bending moments, the contour integrals are replaced by double integrals over the whole area of the cross section.

Chapter VIII

SPATIAL STABILITY OF THIN-WALLED BEAMS WITH CONTINUOUSLY DISTRIBUTED ELASTIC AND RIGID TRANSVERSE CONNECTIONS

§ 1. Stability of beams embedded in an elastic medium

In the previous chapters we examined the stability problem of a thin-walled beam under the assumption that it is free, over the entire span, from external connections restraining to a degree changes in the state of deformation and hence increasing the critical design force.

In engineering design the stability problem of beams enclosed in an elastic medium is of great importance. We encounter such a problem, for example, in stability calculation of a compressed chord of an open bridge. The elastic medium is represented for this chord by the transverse frames connecting the main trusses. If the transverse frames are closely spaced the "elastic medium" can be considered, with a practically sufficient accuracy, as continuously distributed along the beam.

In order to obtain the stability equations we have to consider in the general equilibrium equations (VI.1.2) the functions $\xi(z)$, $\eta(z)$ and $\theta(z)$ as the sought variations of the fundamental displacements and to substitute for the external force factors the expressions $q_x+\bar{q}_x$, $q_y+\bar{q}_y$ and $m+\bar{m}$. Here q_x, q_y and m are determined by (VI.1.13) and (VI.1.27) and represent for a given stress the additional reduced load associated with the additional displacements appearing upon transition from the basic equilibrium shape. The additional reduced load $\bar{q}_x$, $\bar{q}_y$ and $\bar{m}$ is determined by (III.5.3) and results from the continuously distributed elastic connections along the beam. Performing the indicated substitution we obtain

$$\left.\begin{aligned}
&EJ_y\xi^{IV}+k_\xi[\xi-(h_y-a_y)\theta]-[N(\xi'+a_y\theta')]'+(M_x\theta)''=0,\\
&EJ_x\eta^{IV}+k_\eta[\eta+(h_x-a_x)\theta]-[N(\eta'-a_x\theta')]'+(M_y\theta)''=0,\\
&EJ_\omega\theta^{IV}-GJ_d\theta''-[(r^2N+2\beta_yM_x-2\beta_xM_y)\theta']'+\\
&+[q_x^0(e_x-a_x)+q_y^0(e_y-a_y)]\theta+a_x(N\eta')'-a_y(N\xi')'+\\
&\qquad+M_x\xi''+M_y\eta''-k_\xi(h_y-a_y)\xi+k_\eta(h_x-a_x)\eta+\\
&\qquad\qquad+[k_\xi(h_y-a_y)^2+k_\eta(h_x-a_x)^2+k_\theta]\theta=0.
\end{aligned}\right\}\quad(1.1)$$

The differential equations (1.1) are the general stability equations for a thin-walled beam embedded in an elastic medium.

As in the previously examined case of a free beam, we can drop terms containing squares or products of ξ, η and θ since these functions represent small displacements.

The coefficients of equations (1.1) depend on:

1) the elastic constants E and G of the beam material and the geometrical characteristics of the cross section: J_x, J_y, J_d, J_ω, F, a_x, a_y, β_x, β_y;

2) the elastic constants of the medium k_ξ, k_η, k_θ, and the coordinates of the point of contact of this medium with the beam in the plane of the cross section h_x, h_y;

3) the internal force factors $N(z)$, $M_x(z)$ and $M_y(z)$ which characterize the state of stress of the beam and are determined by statical calculations.

4) the components of the external transverse load q_x°, q_y° and the coordinates of its point of application in the plane of the cross section e_x and e_y.

The components of the transverse load $q_x^\circ(z)$ and $q_y^\circ(z)$, like the internal force factors $N(z)$, $M_x(z)$ and $M_y(z)$, are given up to one general parameter. The critical values of this parameter are expressed by the eigenvalues of the boundary-value problem described by the system of homogeneous differential equations (1.1) with homogeneous boundary conditions. The critical design load is the one smallest in absolute value.

In general the integration of equations (1.1) is a very difficult mathematical problem since for an arbitrary external loading these equations can have complicated variable coefficients. This is due to the fact that we derived equations (1.1) for the instability of a beam embedded in an elastic medium from very general assumptions on the external load, the beam cross section and characteristics of the elastic medium. Of the available approximate methods the Bubnov-Galerkin variational method described above may be recommended for the solution of equations (1.1) and the determination of the critical forces.

§ 2. Stability of a beam subjected to an axial longitudinal force

1. General Case. The case of axial compression, which gives rise in a beam with elastic connections only to normal stresses uniformly distributed over the section before onset of instability, is very important in practice. Setting in (1.1) $M_x = M_y = 0$; $q_x^\circ = q_y^\circ = 0$, we obtain the basic stability equations for the case of an external loading consisting only of longitudinal forces which produce an axial normal force $N(z)$ in the cross section,

$$\left.\begin{aligned} &EJ_y\xi^{IV} + k_\xi[\xi - (h_y - a_y)\theta] - [N(\xi' + a_y\theta')]' = 0,\\ &EJ_x\eta^{IV} + k_\eta[\eta - (h_x - a_x)\theta] - [N(\eta' - a_x\theta')]' = 0,\\ &EJ_\omega\theta^{IV} - GJ_d\theta'' - a_y(N\xi')' + a_x(N\eta')' - r^2(N\theta')' -\\ &\quad - k_\xi(h_y - a_y)\xi + k_\eta(h_x - a_x)\eta +\\ &\quad + [k_\xi(h_y - a_y)^2 + k_\eta(h_x - a_x)^2 + k_\theta]\theta = 0. \end{aligned}\right\} \quad (2.1)$$

The normal force $N = N(z)$ in (2.1) is considered (up to one parameter) as a given function of z.

Assuming $N = -P =$ const, we obtain the stability equations for a beam embedded in an elastic medium and loaded at the ends by a longitudinal

compressive force P. These equations can be written

$$\left.\begin{aligned} EJ_y\xi^{IV}+P\xi''+k_\xi\xi+a_yP\theta''-k_\xi(h_y-a_y)\theta=0,\\ EJ_x\eta^{IV}+P\eta''+k_\eta\eta-a_xP\theta''+k_\eta(h_x-a_x)\theta=0,\\ EJ_\omega\theta^{IV}+(r^2P-GJ_d)\theta''+[k_\xi(h_y-a_y)^2+k_\eta(h_x-a_x)^2+\\ +k_\theta]\theta+a_yP\xi''-k_\xi(h_y-a_y)\xi-a_xP\eta''+k_\eta(h_x-a_x)\eta=0.\end{aligned}\right\}\quad(2.2)$$

For constant k_ξ, k_η, k_θ, h_x and h_y, equations (2.2) have constant coefficients and can be integrated in closed analytical form. These equations are valid for a series of practically important problems. We must only show that for $k_\xi=k_\eta=k_\theta=0$, i.e., in the absence of an elastic medium (2.2) reduce to equations (V.3.1) for a free beam.

If the boundary conditions are such that the variations of the displacements ξ, η and θ and the variations of the normal stresses σ (proportional to the second derivatives of the displacements) vanish at the ends of the beam ($z=0$ and $z=l$), the particular solution of (2.2) is

$$\xi=A\sin\frac{n\pi}{l}z,\quad \eta=B\sin\frac{n\pi}{l}z,\quad \theta=C\sin\frac{n\pi}{l}z,\qquad(2.3)$$

where A, B and C are some coefficients and n is an arbitrary positive integer.

Inserting (2.3) in (2.2) and setting the determinant of the coefficients of the constants A, B and C equal to zero, we obtain an equation for the critical value of the force P:

$$\begin{vmatrix} EJ_y\lambda_n^4-\lambda_n^2P+k_\xi & 0 & -a_y\lambda_n^2P-k_\xi(h_y-a_y)\\ 0 & EJ_x\lambda_n^4-\lambda_n^2P+k_\eta & a_x\lambda_n^2P+k_\eta(h_x-a_x)\\ -a_y\lambda_n^2P-k_\xi(h_y-a_y) & a_x\lambda_n^2P+k_\eta(h_x-a_x) & \begin{gathered}EJ_\omega\lambda_n^4-\lambda_n^2(r^2P-GJ_d)+\\ +[k_\xi(h_y-a_y)^2+\\ +k_\eta(h_x-a_x)^2+k_\theta]\end{gathered}\end{vmatrix}=0,\quad(2.4)$$

where

$$\lambda_n=\frac{n\pi}{l}\quad(n=1,\ 2,\ 3,\ \ldots).\qquad(2.5)$$

For any number n of half-waves of the sine function, (2.4) gives three values for the critical force P corresponding to the three degrees of freedom of the beam in the plane Oxy. Note that for any number of half-waves the smallest value of the critical force can be larger than one.

2. Beam with two planes of symmetry. Equations (2.2) refer to a beam having an arbitrary cross section. If the shear center coincides with the section centroid (for example, in a beam with two axes of symmetry) and the point of contact of the beam with the elastic medium is situated at the centroid, then $a_x=a_y=h_x=h_y=0$. In this particular case the system of simultaneous differential equations (2.2) is resolved into three independent equations:

$$\left.\begin{aligned} EJ_y\xi^{IV}+P\xi''+k_\xi\xi=0,\\ EJ_x\eta^{IV}+P\eta''+k_\eta\eta=0,\\ EJ_\omega\theta^{IV}+(r^2P-GJ_d)\theta''+k_\theta\theta=0.\end{aligned}\right\}\qquad(2.6)$$

The first two of these equations, together with the boundary conditions,

determine the bending equilibrium shapes of the beam for which the cross sections undergo only translational displacements. The third equation, with the corresponding boundary conditions, determine the instability of the beam in torsion where the line or shear centers (coinciding here with the line of centroids) remains at rest. If the ends of the beam are hinged the flexural-torsional shapes of the buckling beam are sinusoidal, viz. as $\sin\lambda_n z$ ($n = 1, 2, 3, \ldots$) along the beam. The characteristic equation (2.4) for the critical forces is resolved, like (2.6), into three separate equations,

$$\left.\begin{aligned} EJ_y\lambda_n^4 - P\lambda_n^2 + k_\xi = 0, \\ EJ_x\lambda_n^4 - P\lambda_n^2 + k_\eta = 0, \\ EJ_\omega\lambda_n^4 - (r^2P - GJ_d)\lambda_n^2 + k_\theta = 0. \end{aligned}\right\} \quad (2.7)$$

From (2.7) we obtain the following expressions for the critical forces,

$$\left.\begin{aligned} P_{xn} &= EJ_x\lambda_n^2 + \frac{k_\eta}{\lambda_n^2}, \\ P_{yn} &= EJ_y\lambda_n^2 + \frac{k_\xi}{\lambda_n^2}, \\ P_{\omega n} &= \frac{1}{r^2}\left(EJ_\omega\lambda_n^2 + GJ_d + \frac{k_\theta}{\lambda_n^2}\right). \end{aligned}\right\} \quad (2.8)$$

The first two expressions are identical with the well-known expressions for lateral bending of a beam embedded in an elastic medium. The third gives a new value of the critical force, determined from the condition of torsion.

For given elastic characteristics of the beam and the medium the critical forces, determined by (2.8), depend on the length of the sinusoidal half-waves which correspond to the mode of instability.

To calculating the critical forces from (2.8) we have to determine for each of the three possible forms of instability (two in bending and one in torsion) the number of half-waves by imposing the minimum-value condition of the corresponding critical compressive force. The smallest critical force is chosen for design.

We shall determine the critical forces for an infinitely long beam embedded in an elastic medium. The half-lengths of the sine waves, which in (2.8) give the minimum values for the three critical forces P_x, P_y and P_ω are given by the parameters λ_n (2.5). Setting $\lambda_n = \lambda$, differentiating the critical forces with respect to this parameter (which is permissible, since in the case of an infinitely long beam the critical forces may be regarded as continuous functions of λ) and equating to zero the product $\frac{dP}{d\lambda}$, we obtain

$$EJ_x\lambda - \frac{k_\eta}{\lambda^3} = 0, \quad EJ_y\lambda - \frac{k_\xi}{\lambda^3} = 0, \quad EJ_\omega\lambda - \frac{k_\theta}{\lambda^3} = 0.$$

Whence we obtain for the parameters λ the following values for the three possible forms of instability (two in bending and one in torsion):

$$\lambda_1 = \sqrt[4]{\frac{k_\eta}{EJ_x}}, \quad \lambda_2 = \sqrt[4]{\frac{k_\xi}{EJ_y}}, \quad \lambda_3 = \sqrt[4]{\frac{k_\theta}{EJ_\omega}}.$$

The corresponding half-lengths of the sine waves are

$$l_1 = \pi\sqrt[4]{\frac{EJ_x}{k_\eta}}, \quad l_2 = \pi\sqrt[4]{\frac{EJ_y}{k_\xi}}, \quad l_3 = \pi\sqrt[4]{\frac{EJ_\omega}{k_\theta}}. \quad (2.9)$$

The critical forces P_x, P_y and P_ω, attaining for these wavelengths their smallest values, are given by

$$P_x = 2\sqrt{EJ_x k_\eta}, \quad P_y = 2\sqrt{EJ_y k_\xi}, \quad P_\omega = \frac{1}{r^2}\left(\sqrt{EJ_\omega k_\theta} + GJ_d\right). \quad (2.10)$$

The smallest of these three forces is chosen for design.

It is seen from (2.9) and (2.10) that the wavelengths and the critical forces depend only on the elastic characteristics of the beam and of the medium. It should be noted that (2.10) yield exact values for the critical forces for finite beams, too, if their total length is a multiple of the wavelength as determined by (2.9). In this case the critical forces do not depend on the total length of the beam.

If the total length of the beam is not a half-integral multiple of the wavelength, then in design for stability we choose from the pair of nearest consecutive integers to serve as n the one which when inserted in (2.10) minimizes the corresponding critical force.

3. Beam with one plane of symmetry. If the cross section of the beam has a single axis of symmetry (say, Oy) and the transverse connections constituting the elastic medium lie in the plane of symmetry (here in the plane Oyz), the system (2.2) separates into one equation for $\eta(z)$ and two equations for $\xi(z)$ and $\theta(z)$:

$$\left.\begin{aligned} EJ_x\eta^{IV} + P\eta'' + k_\eta\eta &= 0, \\ EJ_y\xi^{IV} + P\xi'' + k_\xi\xi + a_yP\theta'' - k_\xi(h_y - a_y)\theta &= 0, \\ EJ_\omega\theta^{IV} + (r^2P - GJ_d)\theta'' + [k_\xi(h_y - a_y)^2 + k_\theta]\theta &+ \\ + a_yP\xi'' - k_\xi(h_y - a_y)\xi &= 0 \end{aligned}\right\} \quad (2.11)$$

The first equation refers to instability through bending of the beam in its plane of symmetry Oyz. In this equation the elastic coefficient of the medium is represented by the quantity k_η. The second and third equations form a system which, together with the boundary conditions, determines the flexural-torsional equilibrium shape after onset of instability.

According to (2.11), the determinant (2.4) (in the case of hinged ends) leads to two equations,

$$EJ_x\lambda_n^4 - P\lambda_n^2 + k_\eta = 0,$$

$$\begin{vmatrix} EJ_y\lambda_n^4 - P\lambda_n^2 + k_\xi & -a_yP\lambda_n^2 - k_\xi(h_y - a_y) \\ -a_yP\lambda_n^2 - k_\xi(h_y - a_y) & \begin{gathered} EJ_\omega\lambda_n^4 - \lambda_n^2(r^2P - GJ_d) + \\ + k_\xi(h_y - a_y)^2 + k_\theta \end{gathered} \end{vmatrix} = 0. \quad (2.12)$$

The first of these equations gives an expression for the critical force with which we are familiar (the second of equations (2.7)).

The second equation gives two values for the critical force corresponding, for n half-waves, to two possible flexural-torsional forms of instability. The critical forces determined by these equations depend not only on EJ_y, λ_n and k_ξ, but also on the other elastic and geometrical characteristics: EJ_ω, GJ_d, a_y and h_y, which are exceedingly important in the problem of stability.

These characteristics have in many cases a decisive influence on the magnitude of the critical force.

It is seen from the second equation (2.12) that, for $a_y \neq 0$, instability through "pure" bending (without torsion), as examined in the works of

Yasinskii and Timoshenko, cannot occur for beams embedded in an elastic medium which are subjected to axial compression.

A considerable influence on the mode of instability, and thus on the magnitude of the critical force, is also exerted by the quantity $(h_y - a_y)$, determining the point of contact of the beam with the elastic medium (the intersection of the horizontal reaction forces of the elastic medium with the axis of symmetry).

For $h_y = a_y$, i.e. when the horizontal forces of the elastic medium pass through the shear center, the second equation of (2.1 2) is somewhat simplified, viz.

$$\begin{vmatrix} EJ_y\lambda_n^4 - P\lambda_n^2 + k_\xi & -a_y P\lambda_n^2 \\ -a_y P\lambda_n^2 & EJ_\omega\lambda_n^4 - (r^2P - GJ_d)\lambda_n^2 + k_\theta \end{vmatrix} = 0. \qquad (2.13)$$

We obtain from (2.13) an expression for the critical forces,

$$P = \frac{\lambda_n^2\left[\lambda_n^2(P_\omega + r^2P_y) + r^2k_\xi + k_\theta\right]}{2\lambda_n^4(r^2 - a_y^2)} \pm$$

$$\pm \frac{\sqrt{\lambda_n^4\left[\lambda_n^2(P_\omega + r^2P_y) + r^2k_\xi + k_\theta\right]^2 - 4\lambda_n^4(r^2 - a_y^2)\left[\lambda_n^4P_yP_\omega + \lambda_n^2(k_\xi P_\omega + k_\theta P_y) + k_\xi k_\theta\right]}}{2\lambda_n^4(r^2 - a_y^2)} \qquad (2.14)$$

where

$$P_y = EJ_y\lambda_n^2, \quad P_\omega = EJ_\omega\lambda_n^2 + GJ_d.$$

It is easy to show that the lowest critical force determined by (2.14) and corresponding to flexural-torsional instability will be less than the force obtained from the well-known Yasinskii-Timoshenko equation (not considering torsion). Mechanically, this follows from the fact that by taking into account only the bending instability we tacitly assume transverse connections along the entire beam which stiffen it against torsion. These connections increase the stability of the beam.

§ 3. Stability of a beam subjected to an eccentric longitudinal force

Let the beam have a single plane of symmetry and be subjected to longitudinal forces applied at the ends. We shall consider the beam to be embedded in a continuous elastic medium, exhibiting elastic resistance to bending away from the symmetry plane and to torsion. Since the hypothesis of an inflexible section contour lies at the basis of our theory, we can consider the reaction forces and moments of the medium as applied in the plane of symmetry on a line parallel to the generator of the beam. In other words, we shall assume that the beam–medium contact in the plane of the cross section occurs at a point of the axis of symmetry.

We shall examine the stability of a beam for the case of a compressive force P acting in the plane of symmetry.

If this force does not exceed a certain limit the beam will be under conditions of combined resistance to compression and bending. In this case the state of stress of the beam is characterized only by normal stresses, constant along the beam and varying in the section as a function of the distance from a straight line perpendicular to the axis of symmetry.

For a certain value of the force P plane bending of the beam becomes unstable. The beam passes from one equilibrium shape into another. In general, this transition is characterized by the appearance of bending deformations away from the plane of symmetry and by twisting. The stability equations can be easily obtained from the general equations (1.1).

Assuming in the first and third of equations (1.1)

$$N = -P = \text{const}, \quad M_x = -Pe_y = \text{const}, \quad M_y = 0, \quad q_x^0 = q_y^0 = 0$$

and discarding the second equation, we obtain

$$\left.\begin{aligned} &EJ_y\xi^{IV} + P\xi'' + k_\xi\xi + P(a_y - e_y)\theta'' - k_\xi(h_y - a_y)\theta = 0, \\ &EJ_\omega\theta^{IV} + [P(r^2 + 2\beta_y e_y) - GJ_d]\theta'' + [k_\xi(h_y - a_y)^2 + k_\theta]\theta + \\ &\qquad + P(a_y - e_y)\xi'' - k_\xi(h_y - a_y)\xi = 0. \end{aligned}\right\} \quad (3.1)$$

If the ends of the beam $z = 0$ and $z = l$ are hinged, the displacements ξ and θ and their second derivatives vanish. The eigenfunctions of the system of differential equations (2.1) will then be:

$$\xi = A \sin \lambda_n z, \quad \theta = C \sin \lambda_n z,$$

where $\lambda_n = \frac{n\pi}{l}$ $(n = 1, 2, 3, \ldots)$.

The characteristic equation for the critical forces is

$$\begin{vmatrix} EJ_y\lambda_n^4 - P\lambda_n^2 + k_\xi & -P\lambda_n^2(a_y - e_y) - k_\xi(h_y - a_y) \\ -P\lambda_n^2(a_y - e_y) - k_\xi(h_y - a_y) & \begin{gathered} EJ_\omega\lambda_n^4 - [P(r^2 + 2\beta_y e_y) - GJ_d]\lambda_n^2 + \\ + k_\xi(h_y - a_y)^2 + k_\theta \end{gathered} \end{vmatrix} = 0. \quad (3.2)$$

The critical forces determined by equation (3.2) depend, for given geometrical and elastic characteristics of the beam and the medium and given eccentricity of the force, on the number n of sinusoidal half-waves. Now equation (3.2) is quadratic in P, so that by keeping the number n of half-waves fixed we obtain two values for the critical force P. Of all possible integers n $(n = 1, 2, 3, \ldots)$ we must choose as the design value the one which yields the smallest value for the critical force. For beams embedded in an elastic medium the number of sinusoidal half-waves determining the first equilibrium shape (after instability) is, as a rule, larger than one. This number depends on the elastic coefficients of the medium k_ξ and k_θ. By increasing the coefficients k_ξ and k_θ the number of half-waves is increased.

For an infinitely long beam the critical forces can also be easily determined from (3.2). For this purpose it is necessary to differentiate (3.2) with respect to λ_n^2, taking $P = f(\lambda_n^2)$ as a continuous function of λ_n^2. This is obviously true for the case of an infinitely long beam.

Taking therefore $\frac{dP}{d\lambda_n^2} = 0$, since the smallest value of the critical force corresponds to the minimum of the function $P = f(\lambda_n^2)$, we obtain an additional condition that relates the minimum critical force P to the parameter λ_n. From this condition and from (3.2) we find the value of the smallest critical load and the corresponding wavelength.

Equations (3.1), obtained as a particular case of the more general equations (1.1), represent a generalization of a whole series of problems we have examined before. Thus, for example, for $k_\xi = k_\theta = 0$ equations (3.1) and (3.2) become the stability equations for a free beam with a single axis of symmetry. For $a_y = \beta_y = e_y = 0$ these equations refer to the

stability problem examined in sub-sec. 2 of § 2 for a beam with two planes of symmetry under a longitudinal axial load.

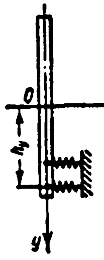


Figure 186

In addition to the problems examined above, equations (3.1) and (3.2) afford to solve a series of new problems concerning the stability of beams in an elastic medium.

We shall examine some of these problems.

1. We easily obtain from (3.2), as a particular case, an equation for the critical forces for a rectangular narrow plate which has elastic connections along the line $y = h_y$, resisting deflections away from the plane Oyz and rotations, which plate is subjected to compression and bending up to instability (Figure 186). For that it is necessary to put in equation (3.2) $a_y = \beta_y = 0$:

$$\begin{vmatrix} EJ_y\lambda_n^4 - P\lambda_n^2 + k_\xi & Pe_y\lambda_n^2 - k_\xi h_y \\ Pe_y\lambda_n^2 - k_\xi h_y & -Pr^2\lambda_n^2 + GJ_d\lambda_n^2 + k_\xi h_y^2 + k_\theta \end{vmatrix} = 0. \quad (3.3)$$

The critical forces determined from (3.3) refer to flexural-torsional instability of the plate. The plate is thereby transformed into a "ruled surface", obtained by rotation of the cross sections with respect to an axis parallel to the generator. This axis is the line of rotation centers and lies in the plane of symmetry at a distance

$$c_y = \frac{Pe_y\lambda_n^2 - k_\xi h_y}{EJ_y\lambda_n^4 - P\lambda_n^2 + k_\xi}$$

from the origin, determined similarly to the rotation center in § 4, Chapter V. The angles of rotation vary along the beam as

$$\theta = C \sin \lambda_n z.$$

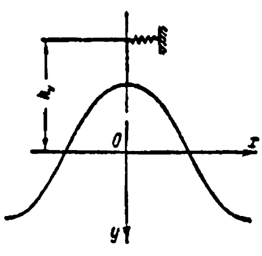


Figure 187

Here the coefficient C remains undetermined. The number n of half-waves if found from the minimum condition for the critical force.

For $k_\xi = k_\theta = 0$ equation (3.3) determines the critical forces for a free plate. The flexural-torsional instability is characterized in this case by a single sine half-wave.

2. Setting in (3.2) k_θ equal to (zero elastic modulus of resistance of the medium to rotation of the beam) we obtain an equation for the critical forces for a beam elastically fixed against deflections from the symmetry plane only (Figure 187). For $EJ_\omega = 0$, $a_y = \beta_y = 0$ and $k_\theta = 0$, equation (3.2) refers to the stability of a plate with hinges at the ends and elastic transverse connections along the line $y = h_y$. The lateral edges of the plate are free. The equation of the critical forces for such a plate is

$$\begin{vmatrix} EJ_y\lambda_n^4 - P\lambda_n^2 + k_\xi & Pe_y\lambda_n^2 - k_\xi h_y \\ Pe_y\lambda_n^2 - k_\xi h_y & -(Pr^2 - GJ_d)\lambda_n^2 + k_\xi h_y^2 \end{vmatrix} = 0. \quad (3.4)$$

3. Setting in (3.2) $P = 0$, $Pe_y = -M_x$ for $e_y \to \infty$, we obtain an equation for the critical values of the moment M_x in pure bending in the plane

of symmetry

$$\begin{vmatrix} EJ_y\lambda_n^4+k_\xi & -M_x\lambda_n^2-k_\xi(h_y-a_y) \\ -M_x\lambda_n^2-k_\xi(h_y-a_y) & EJ_\omega\lambda_n^4+(2\beta_yM_x+GJ_d)\lambda_n^2+ \\ & +k_\xi(h_y-a_y)^2+k_\theta \end{vmatrix}=0. \quad (3.5)$$

4. For a beam with two planes of symmetry and with $h_y=0$ we obtain from equation (3.5) for the critical moment M_x

$$\begin{vmatrix} EJ_y\lambda_n^4+k_\xi & -M_x\lambda_n^2 \\ -M_x\lambda_n^2 & EJ_\omega\lambda_n^4+GJ_d\lambda_n^2+k_\theta \end{vmatrix}=0. \quad (3.6)$$

The number n of sine half-waves is found, as in the previous case, according to the minimum condition for the critical value of M_x.

§ 4. Stability of beams rigidly fixed along a line parallel to the axis

We have examined in the previous section the stability problem of a beam embedded in an elastic medium which resists bending and torsion.

In the stability design of structural elements one often encounters the problem of stability of thin-walled beams having rigid restraints against bending along a generator and elastically fixed against torsion.

This problem could be solved by letting the elastic constants of the medium k_ξ and k_η tend to infinity in the equations of the previous section. Here we solve this problem directly, starting from the general stability equations (VI. 1.28) for a thin-walled beam.

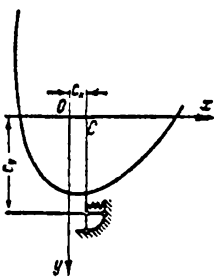


Figure 188

Suppose the beam to be rigidly fixed against deflections along the line $x=c_x$, $y=c_y$ and placed in an elastic medium which resists torsion (Figure 188). By introducing two rigid transverse connections the beam loses two degrees of freedom in the cross section. Such a beam can become unstable through flexural torsion, with the axis of rotation coinciding with the fixed line, i. e. each section of the beam can only rotate about the point with the coordinates $x=c_x$, $y=c_y$.

It is obvious that reaction forces, constituting an additional transverse load on the beam, will appear along the axis of rotation at the onset of instability. We denote the components of these additional loads by q_x^* and q_y^*. The elastic constraint against torsion contributes an additional transverse load $m^*=k_\theta\theta$, where k_θ is the elastic torsional stiffness evinced by the medium.

Recalling the physical meaning of equations (VI.1.28), we can consider this additional transverse load to stand on the right-hand side of these equations instead of zero:

$$\left.\begin{aligned} EJ_y\xi^{IV}-[N(\xi'+a_y\theta')]'+(M_x\theta)''=q_x^*, \\ EJ_x\eta^{IV}-[N(\eta'-a_x\theta')]'+(M_y\theta)''=q_y^*, \end{aligned}\right\} \quad (4.1)$$

$$\begin{gathered}EJ_\omega\theta^{IV}-GJ_d\theta''-[(r^2N+2\beta_yM_x-2\beta_xM_y)\theta']'+\\+[q_x^\circ(c_x-a_x)+q_y^\circ(c_y-a_y)]\theta-a_y(N\xi')'+a_x(N\eta')'+\\+M_x\xi''+M_y\eta''=m^*+q_y^*(c_x-a_x)-q_x^*(c_y-a_y).\end{gathered}\tag{4.1}$$

Boundary conditions expressing the fixity of the point C must be applied to the displacements ξ, η and θ of the beam section

$$\theta(c_x-a_x)+\eta=0,\quad -\theta(c_y-a_y)+\xi=0,$$

whence

$$\eta=-\theta(c_x-a_x),\quad \xi=\theta(c_y-a_y).\tag{4.2}$$

Replacing ξ and η in equations (4.1) by their expressions (4.2), we obtain

$$\left.\begin{gathered}EJ_y(c_y-a_y)\theta^{IV}-(Nc_y\theta')'+(M_x\theta)''=q_x^*,\\-EJ_x(c_x-a_x)\theta^{IV}+(Nc_x\theta')'+(M_y\theta)''=q_y^*,\\EJ_\omega\theta^{IV}-GJ_d\theta''-[(Nr^2+2\beta_yM_x-2\beta_xM_y)\theta']'+\\+[q_x^*(c_x-a_x)+q_y^*(c_y-a_y)]\theta-a_y(c_y-a_y)(N\theta')'-\\-a_x(c_x-a_x)(N\theta')'+M_x(c_y-a_y)\theta''-M_y(c_x-a_x)\theta''=\\=m^*+q_y^*(c_x-a_x)-q_x^*(c_y-a_y).\end{gathered}\right\}\tag{4.3}$$

We can eliminate the quantities q_x^* and q_y^* between these three equations. We then obtain a single differential stability equation for the thin-walled beam (rigidly fixed along a line) corresponding to the single degree of freedom of the cross section of this beam.

For simplicity, we shall consider N, M_x and M_y to be constant along the beam, i.e. we shall examine the case of eccentric compression-extension. For this purpose, assuming that

$$M_x=Ne_y=\text{const},\quad M_y=-Ne_x=\text{const},$$

we can put equations (4.3) in the form

$$\left.\begin{gathered}EJ_y(c_y-a_y)\theta^{IV}-N(c_y-e_y)\theta''=q_x^*,\\-EJ_x(c_x-a_x)\theta^{IV}+N(c_x-e_x)\theta''=q_y^*,\\EJ_\omega\theta^{IV}-GJ_d\theta''-N(r^2+2\beta_ye_y+2\beta_xe_x)\theta''-\\-N(a_y-e_y)(c_y-a_y)\theta''-N(a_x-e_x)(c_x-a_x)\theta''=\\=m^*+q_y^*(c_x-a_x)-q_x^*(c_y-a_y).\end{gathered}\right\}\tag{4.4}$$

Substituting in the last of equations (4.4) the values q_x and q_y^* from the first two, we obtain

$$\begin{gathered}E[J_\omega+J_x(c_x-a_x)^2+J_y(c_y-a_y)^2]\theta^{IV}-GJ_d\theta''-\\-N(r^2+2\beta_xe_x+2\beta_ye_y)\theta''-N(c_x-a_x)(a_x+c_x-2e_x)\theta''-\\-N(c_y-a_y)(a_y+c_y-2e_y)\theta''+k_\theta\theta=0.\end{gathered}\tag{4.5}$$

We now observe that the quantity

$$J_\omega+J_x(c_x-a_x)^2+J_y(c_y-a_y)^2=J_{\omega_C}\tag{4.6}$$

is equal to the sectorial rigidity of the section $\int_F\omega_C^2dF$ (with respect to the

point $x=c_x$, $y=c_y$). This is a consequence of the following general expression of the change in the sectorial moment of inertia when the pole is shifted from the shear center to the point C,

$$J_{\omega_C}=J_{\omega}+J_xc_x^2+J_yc_y^2+k^2F,$$

where c_x and c_y are the components of the displacement vector of the pole; k is a coefficient depending on c_x, c_y as well as on the position (on the section contour) of the new origin of sectorial areas. This new origin can always be chosen so that k vanishes. In this case J_{ω_C} will have a minimum value and the corresponding diagram of sectorial areas will be orthogonal to the diagram of the distribution of material over the section. It is easy to verify the last statement by using expression (I.4.3) for the transformation of the sectorial area under translation of the pole and the origin.

The quantity J_{ω_C} representing, for $k=0$, the rigidity of the beam section for bending with torsion but no longitudinal force, is determined in the same way as J_{ω}, but with the pole at the point c_x, c_y and not at the shear center.

Inserting the expression of J_{ω_C} and replacing the quantities r, β_x and β_y by their values from (V.1.6), we put (4.5) in the form

$$EJ_{\omega_C}\theta^{IV}-GJ_d\theta''-N\left[\frac{J_x+J_y}{F}+c_x^2+c_y^2+\right.$$
$$\left.+e_x\left(\frac{U_y}{J_y}-2c_x\right)+e_y\left(\frac{U_x}{J_x}-2c_y\right)\right]\theta''+k_\theta\theta=0.$$

Hence, expressing the deformation in the form $\theta=C\sin\lambda_nz$, we find the critical values $N=N_{cr}$ for a beam fixed along the lines $x=c_x$, $y=c_y$ and restrained against rotations at the ends (for $z=0$ and $z=l$, $\theta=\theta''=0$):

$$N_{cr}=-\frac{EJ_{\omega_C}\lambda_n^2+GJ_d+\frac{k_\theta}{\lambda_n^2}}{\frac{J_x+J_y}{F}+c_x^2+c_y^2+e_x\left(\frac{U_y}{J_y}-2c_x\right)+e_y\left(\frac{U_x}{J_x}-2c_y\right)},\qquad (4.7)$$

where, as before (see (2.5)) $\lambda_n=\frac{n\pi}{l}$.

The critical forces, determined by (4.7) for a given beam length, depend on the number n of half waves. For design we choose the integer n, for which the absolute value of the critical force is smallest.

The half-wavelength that minimizes the critical compressive force $P_{cr}=-N_{cr}$ of an infinitely long beam is determined by

$$l=\pi\sqrt{\frac{EJ_{\omega_C}}{k_\theta}}.\qquad (4.8)$$

In complete analogy with the derivation of (2.9), we obtain (4.8) by equating to zero the derivative with respect to λ_n of the right-hand side of (4.7). Analogously, we obtain an expression for the smallest critical force.

The relation (4.7) permits the design of the beam for stability under very general assumptions as to the point of application of the longitudinal force on the axis of symmetry of the section and as to the connections restraining the beam against displacements in the plane of the cross section.

We can obtain from this relation, as particular cases, the final form of the solutions of certain problems in spatial stability of beams and plates.

We shall examine several problems of practical interest.

a) Axial compression of a beam. Setting (4.7) $e_x = e_y = 0$, we obtain

$$P_{cr} = \frac{EJ_{\omega_C}\lambda_n^2 + GJ_d + \frac{k_\theta}{\lambda_n^2}}{\frac{J_x + J_y}{F} + c_x^2 + c_y^2}. \tag{4.9}$$

b) Pure bending. Assuming $N = 0$, $\lim(Ne_y) = M_x$, $e_y \to \infty$, $e_x = 0$ we obtain from (4.7) (multiplying first by the denominator of the right-hand side) an expression for the moment under which the beam loses its fundamental bending shape in the plane Oyz:

$$M_{x\,cr} = \frac{EJ_{\omega_C}\lambda_n^2 + GJ_d + \frac{k_\theta}{\lambda_n^2}}{2c_y - \frac{U_x}{J_x}}. \tag{4.10}$$

c) A beam without elastic connections against torsion. Here the elastic coefficient k_θ of the medium has to be taken equal to zero. The relation (4.7) assumes the form

$$P_{cr} = \frac{EJ_{\omega_C}\lambda_n^2 + GJ_d}{\frac{J_x + J_y}{F} + c_x^2 + c_y^2 + e_x\left(\frac{U_y}{J_y} - 2c_x\right) + e_y\left(\frac{U_x}{J_x} - 2c_y\right)}. \tag{4.11}$$

The instability shape for this examined medium is a sinusoid with one half-wave extending along the whole of the beam.

The critical force determined by (4.11) or (4.7) depends, for given elastic and geometrical characteristics of the beam and given point of contact with the rigid transverse connections in the plane of the cross section, on the eccentricity of the force P. For certain values of the eccentricity e, determined from the equation

$$\frac{J_x + J_y}{F} + c_x^2 + c_y^2 + e_x\left(\frac{U_y}{J_y} - 2c_x\right) + e_y\left(\frac{U_x}{J_x} - 2c_y\right) = 0, \tag{4.12}$$

this force becomes infinite. If the left-hand side of (4.12) is negative, the critical force N_{cr} reaches positive values which indicates the possibility of instability of plane bending under eccentric extension (Figure 189).

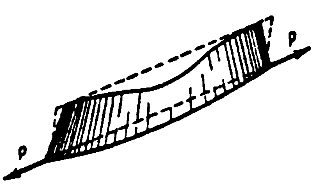


Figure 189

Equation (4.12) determines the limit of stable equilibrium of a beam under eccentric compression or extension. For given geometrical characteristics of the beam this equation relates in a simple algebraic form the coordinates e_x, e_y and c_x, c_y of the point of application of the force N and of the center of rotation, respectively.

Fixing the position of one of these points, we can determine from (4.7) the locus of the other point for infinite values of the critical force.

It is easy to see that for a given center of rotation the locus of the points of application of the forces for infinite critical force will be a straight

line. For given eccentricity the locus of the centers of rotation for which the critical force become infinite will be a circle. For finite values of the critical force the loci of each point for a fixed position of the other point will have the same form.

d) Lateral buckling of a beam. Equation (4.11) gives the critical force P_{cr} corresponding to instability in which the beam twists about the given line $x = c_x$, $y = c_y$.

Varying the coordinates c_x and c_y, we shall obtain various values for the critical force. For $c_x = \infty$ or $c_y = \infty$, (4.11) reduces, according to (4.6), to Euler's relation (V.3.3):

$$P_{x\,cr} = EJ_x\lambda^2, \quad P_{y\,cr} = EJ_y\lambda^2.$$

The instability corresponding to the critical Euler force will occur through bending, regardless of the position of the point of application on the axis of symmetry, since the axis of rotation is at infinity.

e) Twisting of a beam about the shear center. Assuming in expression (4.7) $c_x = a_x$, $c_y = a_y$, we obtain

$$P_{cr} = \frac{EJ_{\omega_C}\lambda_n^2 + GJ_d + \frac{k_\theta}{\lambda_n^2}}{r^2 + 2\beta_x e_x + 2\beta_y e_y}. \tag{4.13}$$

Expression (4.13) gives the critical force P_{cr} for a beam subjected to compression and bending in the plane of symmetry and which buckles through twisting about the line of shear centers.

f) Rectangular narrow plate. Equation (4.7) can be used to design separate elements of a thin-walled beam (plates) for local stability.

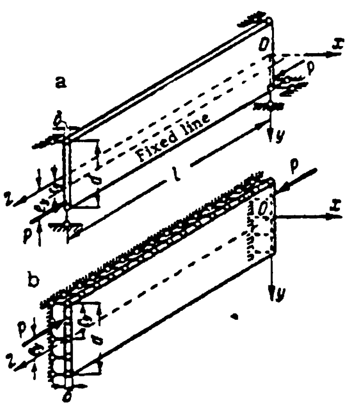


Figure 190

Setting $k_\theta = 0$, $U_x = U_y = 0$, $J_x = 0$, we obtain an expression for the critical force for the case of a plate hinged at the ends $z = 0$ and $z = l$, fixed along the line $x = 0$, $y = c_y$ and loaded by a longitudinal force acting in the plane of the plate at a distance e_y from the middle of the section (Figure 190a):

$$P_{cr} = \frac{P_y c_y^2 + GJ_d}{\frac{d^2}{12} + c_y(c_y - 2e_y)}, \tag{4.14}$$

since in this case, according to (4.6), $EJ_{\omega_C}\lambda_n^2 = EJ_y\lambda_n^2 c_y^2 = P_y c_y^2$, where P_y is the Euler force calculated from

$$P_y = EJ_y\lambda_n^2 = \frac{Ed\delta^3}{12}\frac{\pi^2}{l^2};$$

d and δ are respectively the width and thickness of the plate.

Upon torsional instability the plate assumes the form of a ruled surface. The cross sections of the plate which remain plane rotate about the given rotation axis.

Setting in (4.14) $c_y = -\frac{a}{2}$, $J_d = \frac{1}{3}d\delta^3$, we obtain an expression for the

critical force for a plate with one free and three independently hinged edges (Figure 190b):

$$P_{\text{кр}} = \frac{3P_y d + 4G\delta^3}{4d + 12e_y}. \tag{4.15}$$

The critical force as determined by (4.15) depends on the eccentricity e_y of this force.

For $e_y = -\frac{d}{3}$ the denominator of the right-hand side of (4.15) vanishes and the critical force becomes infinite.

For $e_y = 0$ we obtain the critical force in axial compression,

$$P_{cr} = \frac{3}{4} P_y + G\frac{\delta^3}{d}. \tag{4.16}$$

The relations (4.15) and (4.16), derived for narrow plates with a ratio of length to width $\frac{l}{d} > 4$, yield critical force values very nearly the same as the exact values obtained by Timoshenko, who allowed for bending of the plate in two directions /181/.

§ 5. Application of the method of virtual displacements

When a continuous longitudinal load is arbitrarily distributed along the beam, a normal force $N = N(z)$ (i.e., depending on the position of the section along the beam) appears in the cross section. Equations (1.1) will have in this case variable coefficients. For a beam with two planes of symmetry, fixed along the axis of symmetry and subjected to a longitudinal load in the same axis, the equations of spatial stability have the form·

$$\left.\begin{aligned} EJ_y\xi^{\text{IV}} - (N\xi')' + k_\xi \xi = 0, \\ EJ_x\eta^{\text{IV}} - (N\eta')' + k_\eta \eta = 0, \\ EJ_\omega\theta^{\text{IV}} - [(r^2 N + GJ_d)\theta']' + k_\theta \theta = 0. \end{aligned}\right\} \tag{5.1}$$

The middle terms of these equations have variable coefficients, depending on the form of the function $N(z)$.

If the external longitudinal load has a constant value q and acts in the direction of the positive Oz axis, and at the initial section $z = 0$ the normal force vanishes: $N(0) = 0$; we find from the equilibrium condition for a portion of the beam that $N(z)$ must be

$$N(z) = -qz.$$

In this case equations (5.1) become

$$\left.\begin{aligned} EJ_y\xi^{\text{IV}} + q(z\xi')' + k_\xi \xi = 0, \\ EJ_x\eta^{\text{IV}} + q(z\eta')' + k_\eta \eta = 0, \\ EJ_\omega\theta^{\text{IV}} + [(r^2 qz - GJ_d)\theta']' + k_\theta \theta = 0. \end{aligned}\right\} \tag{5.2}$$

Equations (5.2) and the boundary conditions completely describe the instability of a heavy beam embedded in an elastic medium whose shear center coincides with the sectional center of gravity.

If the load varies linearly along the beam, as is the case, for example, in compressed chords of open-girder bridges loaded along the

entire span by a uniformly distributed transverse load, the function $N(z)$ is represented by a parabola.

Indeed, in a beam of length $2l$, having hinges at the ends (the origin chosen at the middle of the beam) the bending moment due to a vertical uniformly distributed load q is determined from the formula

$$M(z)=\frac{q}{2}(l^2-z^2).$$

We shall examine the truss of a bridge with an upper chord and a lower chord, connected by bars. The upper chord is compressed and the lower is extended. For the compressed upper chord of the truss we obtain

$$N(z)=-\frac{q}{2h}(l^2-z^2), \tag{5.3}$$

where h is the distance between the chords and q is the load per unit length on one truss.

Substituting (5.3) in equations (5.1) we find,

$$\left.\begin{aligned} EJ_y\xi^{IV}+\frac{q}{2h}[(l^2-z^2)\xi']'+k_\xi\xi=0,\\ EJ_x\eta^{IV}+\frac{q}{2h}[(l^2-z^2)\eta']'+k_\eta\eta=0,\\ EJ_\omega\theta^{IV}+\left\{\left[\frac{r^2q}{2h}(l^2-z^2)-GJ_d\right]\theta'\right\}'+k_\theta\theta=0.\end{aligned}\right\} \tag{5.4}$$

In equations (5.4) the resistance of the individual bars is simulated by an elastic medium represented by the coefficients k_ξ, k_η and k_θ. In the stability calculation of a compressed chord of an open bridge these coefficients are determined by the elasticity of the truss bars. Considering that these bars form for the compressed chord a rigid basis in the vertical plane Oyz, i. e. neglecting (at the onset of instability) the elongation of the bars and proceeding directly from the hinged joints of the bars to the compressed chord, we obtain for the elastic constants of the medium k_η and k_θ the values:

$$k_\eta=\infty,\quad k_\theta=0.$$

The coefficient k_ξ in the first of equations (5.4) is determined by the resistance of the bars to the bending of the chord in the horizontal plane.

For $k_\eta=\infty$ the second equation is eliminated since in this limiting case the deflections η vanish. The remaining two equations become

$$EJ_y\xi^{IV}+\frac{q}{2h}[(l^2-z^2)\xi']'+k_\xi\xi=0,$$

$$EJ_\omega\theta^{IV}+\left\{\left[\frac{r^2q}{2h}(l^2-z^2)-GJ_d\right]\theta'\right\}'=0.$$

If the compressed chord of the bridge has one (vertical) axis of symmetry in the cross section and the transverse frames exhibit elastic resistance to bending in the horizontal plane and to torsion, the stability equations are

$$EJ_y\xi^{IV}-(N\xi')'+k_\xi\xi+a_y(N\theta')'+k_{\xi\theta}\theta=0,$$

$$EJ_\omega\theta^{IV}-GJ_d\theta''-r^2(N\theta')'+k_\theta\theta+a_y(N\xi')'+k_{\theta\xi}\xi=0.$$

In these equations the former elastic constants of the medium are k_ξ, k_θ, while $k_{\xi\theta}$ and $k_{\theta\xi}$ denote the new reduced elastic constants of the medium consisting, respectively, of the elastic reactions per unit length of the transverse frames per unit rotation $\theta = 1$ of an elementary transverse strip of the upper chord and the moments of the transverse frames per unit displacement $\xi = 1$ of the same transverse strip. Thus, the coefficients $k_{\theta\xi}$ and $k_{\xi\theta}$ express mutual reaction forces of the elastic medium and are equal to one another.

In the case of a uniformly distributed load we already have (5.3) for the longitudinal force $N(z)$ of the compressed chord of the bridge.

The stability equations will have the form

$$\left.\begin{aligned} &EJ_y\xi^{\mathrm{IV}} - \frac{q}{2h}[(l^2 - z^2)\,\xi']' + k_\xi\xi + \\ &\qquad + a_y\frac{q}{2h}[(l^2 - z^2)\,\theta']' + k_{\xi\theta}\theta = 0, \\ &EJ_\omega\theta^{\mathrm{IV}} - GJ_d\theta'' - r^2\frac{q}{2h}[(l^2 - z^2)\,\theta']' + k_\theta\theta + \\ &\qquad + a_y\frac{q}{2h}[(l^2 - z^2)\,\xi']' + k_{\theta\xi}\xi = 0. \end{aligned}\right\} \tag{5.5}$$

The critical value of the transverse load, under which the compressed chord may buckle in the flexural-torsional shape, is determined by the homogeneous differential equations (5.5) and by the boundary conditions. The exact determination of the critical force from equations (5.5) and the boundary conditions is a very complicated mathematical problem since these equations have variable coefficients. In practice, with a proper choice of the form of instability, the Bubnov-Galerkin method described in detail in § 6, Chapter VI ensures sufficient accuracy.

As an example, we shall examine a beam with hinged ends*. For the instability shape we prescribe the functions

$$\xi = \sum A_n \sin\lambda_n z, \quad \theta = \sum B_n \sin\lambda_n z. \tag{5.6}$$

As shown before, these functions satisfy the condition of hinge constraints.

The work equations for the displacements of the m-th term of the series (5.6) are

$$\left.\begin{aligned} &\sum A_n \int_0^l [EJ_y\lambda_n^4 \sin\lambda_n z - (N\lambda_n \cos\lambda_n z)' + k_\xi \sin\lambda_n z] \sin\lambda_m z\, dz + \\ &\qquad + \sum B_n \int_0^l [a_y(N\lambda_n \cos\lambda_n z)' + k_{\xi\theta} \sin\lambda_n z] \sin\lambda_m z\, dz = 0, \\ &\sum A_n \int_0^l [k_{\theta\xi} \sin\lambda_n z + a_y(N\lambda_n \cos\lambda_n z)'] \sin\lambda_m z\, dz + \\ &+ \sum B_n \int_0^l [EJ_\omega\lambda_n^4 \sin\lambda_n z + GJ_d\lambda_n^2 \sin\lambda_n z - r^2(N\lambda_n \cos\lambda_n z)' + \\ &\qquad + k_\theta \sin\lambda_n z] \sin\lambda_m z\, dz = 0. \end{aligned}\right\} \tag{5.7}$$

* These boundary conditions correspond to a single-span girder bridge having hinge supports at the ends.

Setting in (5.7) $m = \overline{1, 2}, 3, \ldots$, we obtain a system of homogeneous algebraic equations in the coefficients A_n and B_n $(n = 1, 2, 3, \ldots)$ of the series (5.6).

Equating the determinant of this system to zero, we obtain the characteristic equation for the determination of the critical forces. The degree of the characteristic equation in the required critical force equals twice the number of terms of the series. The determination of the critical force from the characteristic equation of the system (5.6) for a large number of terms necessitates much computational work.

In order to simplify the calculation it is possible in the present case to use the method of successive approximations. In the first approximation we retain only the first term in (5.6). Let

$$\xi = A_n \sin \lambda_n z, \quad \theta = B_m \sin \lambda_m z. \tag{5.8}$$

We shall assume that n and m in (5.8) can have different values in the general case. The work equations for the displacements (5.8) are

$$\left.\begin{aligned}
&A_n \int_0^l [EJ_y \lambda_n^4 \sin \lambda_n z - (N\lambda_n \cos \lambda_n z)' + k_\xi \sin \lambda_n z] \sin \lambda_n z \, dz + \\
&\qquad + B_m \int_0^l [a_y (N\lambda_m \cos \lambda_m z)' + k_{\xi\theta} \sin \lambda_m z] \sin \lambda_n z \, dz = 0, \\
&A_n \int_0^l [k_{\theta\xi} \sin \lambda_n z + a_y (N\lambda_n \cos \lambda_n z)'] \sin \lambda_m z \, dz + \\
&\quad + B_m \int_0^l [EJ_\omega \lambda_m^4 \sin \lambda_m z + GJ_d \lambda_m^2 \sin \lambda_m z - r^2 (N\lambda_m \cos \lambda_m z)' + \\
&\qquad\qquad + k_\theta \sin \lambda_m z] \sin \lambda_m z \, dz = 0.
\end{aligned}\right\} \tag{5.9}$$

Assuming that the product terms vanish in equations (5.9), i. e. neglecting to a first approximation the work done by the reaction of the elastic medium and using the functions (5.3), we obtain

$$\left.\begin{aligned}
&(EJ_y \lambda_n^4 + k_\xi) \int_0^l \sin^2 \lambda_n z \, dz - \frac{q}{2h} \lambda_n^2 \int_0^l (lz - z^2) \sin^2 \lambda_n z \, dz - \\
&\qquad - \frac{q}{2h} \lambda_n \int_0^l (l - 2z) \sin \lambda_n z \cos \lambda_n z \, dz = 0, \\
&(EJ_\omega \lambda_m^4 + GJ_d \lambda_m^2 + k_\theta) \int_0^l \sin^2 \lambda_m z \, dz - \\
&\qquad - \frac{r^2 q}{2h} \lambda_m^2 \int_0^l (lz - z^2) \sin^2 \lambda_m z \, dz - \\
&\qquad\qquad - \frac{r^2 q}{2h} \lambda_m \int_0^l (l - 2z) \sin \lambda_m z \cos \lambda_m z \, dz = 0.
\end{aligned}\right\} \tag{5.10}$$

The first equation of (5.10) refers to instability through bending. The second refers to the torsional mode. Expanding the integrals and solving these equations for the critical load q, we obtain

$$q_1 = \frac{12\left(EJ_y \frac{n^4\pi^4}{l^4} + k_\xi\right)h}{n^2\pi^2 + 9}. \tag{5.11}$$

$$q_2 = \frac{12\left(EJ_\omega \frac{n^4\pi^4}{l^4} + GJ_d \frac{n^2\pi^2}{l^2} + k_\theta\right)h}{r^2(n^2\pi^2+9)}. \tag{5.11}$$

Formulas (5.11) give the critical forces as functions of the parameters n and m, determining the number of sinusoidal half-waves for instability through bending and through torsion separately. With the help of these expressions we can now determine the extreme values of the parameters n and m, for which q_1 and q_2 are minimal. Instability through combined bending and torsion is thus determined. Introducing in (5.9) the values of n and m from the minimum condition for q_1 and q_2 and equating the determinant of these equations to zero, we obtain

$$\begin{vmatrix} \displaystyle\int_0^l [EJ_y\lambda_n^4 \sin\lambda_n z - (N\lambda_n \cos\lambda_n z)' + k_\xi \sin\lambda_n z] \sin\lambda_n z\, dz & \displaystyle\int_0^l [a_y (N\lambda_m \cos\lambda_m z)' + k_{\xi\theta} \sin\lambda_m z] \sin\lambda_n z\, dz \\ \displaystyle\int_0^l [a_y (N\lambda_n \cos\lambda_n z)' + k_{\theta\xi} \sin\lambda_n z] \sin\lambda_m z\, dz & \displaystyle\int_0^l [EJ_\omega\lambda_m^4 \sin\lambda_m z + GJ_d\lambda_m^2 \sin\lambda_m z - r^2(N\lambda_m \cos\lambda_m z)' + k_\theta \sin\lambda_m z] \sin\lambda_m z\, dz \end{vmatrix} = 0 \tag{5.12}$$

Equation (5.12) is quadratic in q. The smaller root of this equation gives the critical design load. The instability along the beam will be composite, consisting of bending according to a sinusoidal function with n half-waves and torsion according to a sinusoidal function with m half-waves.

Taking into consideration the mixed terms in n and m of the series (5.6) on both sides and setting up the work equations, we obtain more accurate values for the critical force. It should be noted that the convergence of these series is very good. This is due to the fact that the present method is essentially based on the expansion of the determinant of the infinite system in the neighborhood of those values of n and m which yield the smallest values for the forces q_1 and q_2, determined from (5.11).

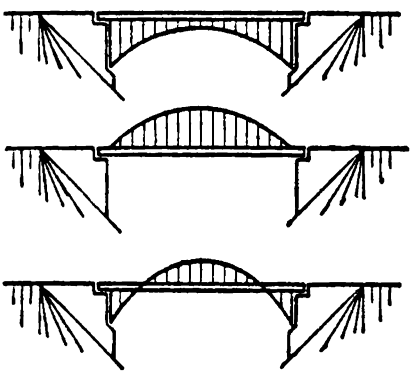


Figure 191

§ 6. Spatial stability of arch bridges

1. We shall apply the described method to the stability problem of arch bridges constructed around girder (stiffening girder) consisting of a thin-walled beam and two compressed arches. Such bridges are called in the technical literature bridges of the Langer type and are constructed according to one of the three schematic diagrams shown in Figure 191. In particular, we shall assume the *stiffening girder* to have an I-section, to possess two symmetry axes and to be supported by two arches placed in the planes of the flanges. The cross section of the bridge is shown in Figure 192.

The stability equations of such a bridge, subjected to a horizontal wind drag, are obtained from the stability equations of beams in plane bending (VI.3.3) if we add terms accounting for the influence of the compressed arches. This influence is manifested in the form of a reduced additional load transmitted to the roadway part by each arch and resulting from the transition of the bridge structure at instability into a new deformed state. The magnitude of this reduced additional load transmitted by one arch is equal to $(-Hv'')$ where H is the thrust of each arch; v are the vertical displacements of the arch axes, equal to the deflections of the flanges of the I-beam in its plane.

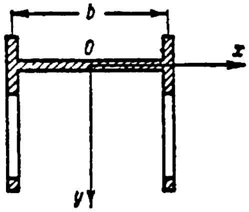


Figure 192

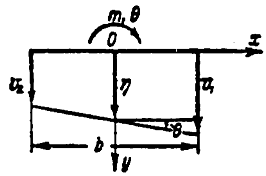


Figure 193

The projection of the additional reduced load, transmitted by both arches, on the direction of the axis Oy is

$$q_y^0 = -H(v_1'' + v_2''),$$

where v_1 and v_2 are the vertical displacements of the first and second arch (Figure 193).

Since the sum of the vertical displacements of both arches is equal to twice the displacement of the centroid of the cross section of the roadway part, the reduced vertical load can be given in the form

$$q_y^0 = -2H\eta''.$$

The additional reduced torsional moment will be

$$m^0 = H\frac{b}{2}(v_2'' - v_1''). \tag{6.1}$$

But from Figure 193 it appears that

$$\frac{v_1 - v_2}{b} = \tan\theta \approx \theta.$$

Therefore, (6.1) finally becomes

$$m^0 = -H\frac{b^2}{2}\theta''.$$

Introducing these additional loads in equations (VI.3.3) and allowing for the fact that in this case a horizontal external load is acting, we obtain the differential equations for the spatial stability of arch bridges:

$$\left.\begin{aligned} EJ_x\eta^{\mathrm{IV}} + 2H\eta'' + (M_y\theta)'' = 0, \\ M_y\eta'' + EJ_\omega\theta^{\mathrm{IV}} - GJ_d\theta'' + H\frac{b^2}{2}\theta'' + q_xe_x\theta = 0. \end{aligned}\right\} \tag{6.2}$$

The homogeneous differential equations (6.2), together with the corresponding homogeneous boundary conditions, determine an infinite number of values q_x, the smallest of which will give the critical load.

2. In order to obtain the differential equations for investigating the deformations and stresses of a suspension bridge under the loads q_y and m and taking into account the given initial state of stress characterized by the quantities q_x, M_y, H, the right-hand sides of (6.2) must be taken as different from zero:

$$\left.\begin{aligned} EJ_x\eta^{IV}+2H\eta''+(M_y\theta)''=q_y, \\ M_y\eta''+EJ_\omega\theta^{IV}-GJ_d\theta''+H\frac{b^2}{2}\theta''+q_xe_x\theta=m. \end{aligned}\right\} \qquad (6.3)$$

Thus, (6.3) are the equilibrium equations of a beam under combined loading (in addition to the external transverse load the beam is under a given state of stress due to the lateral action of the wind and the braces of the arches). It is possible to obtain from this system the stability equations (6.2) as well as the previously obtained equations of resistance.

3. If the span construction has the form of a thin-walled beam of closed section, equations (6.2) and (6.3) are also applicable. The difference will only consist in that the characteristics of the cross section J_ω and J_d refer to torsion and in the method of their calculation. How these characteristics are calculated for closed sections was explained in detail in § 1, Chapter V.

4. The exact determination of the critical load from equations (6.2) and the boundary conditions requires much computational work. The problem is considerably simplified if we use the principle of virtual displacements as in § 6, Chapter VI. Assuming

$$\left.\begin{aligned} \eta(z)=B\psi(z), \\ \theta(z)=C\chi(z) \end{aligned}\right\} \qquad (6.4)$$

and using the results given in § 6, Chapter VI we obtain an equation for the critical load, viz.

$$\begin{vmatrix} \int_0^l [EJ_x(\psi'')^2-2H(\psi')^2]\,dz & q_x\int_0^l \overline{M}_y\psi''\chi\,dz \\ q_x\int_0^l \overline{M}_y\psi''\chi\,dz & \int_0^l \left[EJ_\omega(\chi'')^2+\left(GJ_d-H\frac{b^2}{2}\right)(\chi')^2\right]dz+q_xe_x\int_0^l \chi^2\alpha\,dz \end{vmatrix}=0, \qquad (6.5)$$

where $\alpha=\alpha(z)$ is a function expressing the distribution of the load q_x along the span*; $\overline{M}_y=\overline{M}_y(z)$ is the bending moment due to a load $q_x=1$.

The stability equations (6.5) are symmetrical about the principal diagonal (from the upper left to the lower right corner). This accords with Betti's theorem of reciprocity. The symmetry is due to the values of all the critical loads determined by (6.5) being real.

5. As an example, we shall examine the stability of an arch bridge subjected to a horizontal wind load which we shall consider uniformly

* We have previously introduced the parameter q_x of the transverse wind load. This load will, therefore, appear in the new notation as $q_x\alpha(z)$. Instability sets in when the parameter q_x reaches the critical value.

distributed along the bridge. It is assumed that the span structure is supported at the abutments by hinges. In this case

$$\left.\begin{aligned} a(z) &= 1, \\ \overline{M}_y &= \frac{z(l-z)}{2}, \\ \psi(z) &= \sin\lambda z, \\ \chi(z) &= \sin\lambda z, \end{aligned}\right\} \tag{6.6}$$

where $\lambda = \frac{n\pi}{l}$ (n is an integer).

The functions $\psi(z)$ and $\chi(z)$, determined by (6.6), satisfy all the boundary conditions of the problem.

Evaluating the integrals in the stability equation (6.5), we obtain an equation for the critical load

$$\begin{vmatrix} EJ_x\lambda^4 - 2H\lambda^2 & \frac{q_x}{4}\left(1+\frac{\lambda^2 l^2}{3}\right) \\ \frac{q_x}{4}\left(1+\frac{\lambda^2 l^2}{3}\right) & EJ_\omega\lambda^4 + \left(GJ_d - H\frac{b^2}{2}\right)\lambda^2 + q_x e_x \end{vmatrix} = 0. \tag{6.7}$$

From equation (6.7), for a given H, two values of the critical load can be found for each number n of half-waves. We note that the obtained values of q_x will be approximate since the functions $\sin\lambda z$ are not integrals of equations (6.2).

The number n of half-waves must be chosen such that the critical load will be a minimum. Instability with $n=1$ is connected with extension or compression of the arches. Since the resistance of the arches to extension is considerably larger than that of the stiffening girder and the arches to bending, this form of instability is excluded in practice.

Therefore, n in equation (6.7) can take integral values from 2 to infinity. The minimum critical load is usually obtained for $n=2$.

By varying in the obtained expressions for q_x some quantity, for example H, it is possible to study the influence of the latter on the value of the critical load.

The stability equation (6.7) can be solved not only for q_x, but also for any other quantity, like J_x. In this case we determine the minimum value of J_x which ensures the stability of the bridge for the given lateral load q_x and other given conditions.

6. By studying the dependence of q_x on the thrust H it can be shown that by increasing the thrust the critical load is reduced and can even vanish for a certain value of H. In this case the bridge can become unstable under the action of a vertical load only.

Setting in equation (6.7) $q_x = 0$, we see that it is resolved into two independent equations,

$$EJ_x\lambda^4 - 2H\lambda^2 = 0, \tag{6.8}$$

$$EJ_\omega\lambda^4 + GJ_d\lambda^2 - H\frac{b^2}{2}\lambda^2 = 0. \tag{6.9}$$

Equation (6.8) determines the critical value of the thrust for instability through bending in the vertical plane,

$$H_{cr} = \frac{n^2\pi^2}{2l^2}EJ_x.$$

The vertical, uniformly distributed, critical load on both arches

which corresponds to this thrust is

$$q_{y\,cr}=\frac{8n^2\pi^2}{l^4}EJ_xf,$$

where f is the rise of each arch*.

Equation (6.9) determines the critical value of the thrust for spatial instability through flexural torsion of the roadway part,

$$H_{cr}=\frac{2}{b^2}\left(\frac{n^2\pi^2}{l^2}EJ_\omega+GJ_d\right).$$

In this case the corresponding critical vertical load will be

$$q_{y\,cr}=\frac{32f}{b^2l^2}\left(\frac{n^2\pi^2}{l^2}EJ_\omega+GJ_d\right).$$

According to a previous remark the minimum load is obtained for $n=2$ and is determined by one of the expressions

$$q_{y\,cr}=\frac{32\pi f}{l^4}EJ_x \tag{6.10}$$

or

$$q_{y\,cr}=\frac{32f}{b^2l^2}\left(\frac{4\pi^2}{l^2}EJ_\omega+GJ_d\right).$$

We must choose the smaller of the two values $q_{y\,cr}$ obtained.

If we neglect the rigidity for pure torsion GJ_d, it can be shown that expressions (6.10) yield the same result for an I-section, since then $J_\omega=\frac{b^2}{4}J_x$.

§ 7. Spatial stability of suspension bridges

1. The stability equations of suspension bridges with a stiffening girder in the form of a thin-walled beam of symmetrical section (Figure 194),

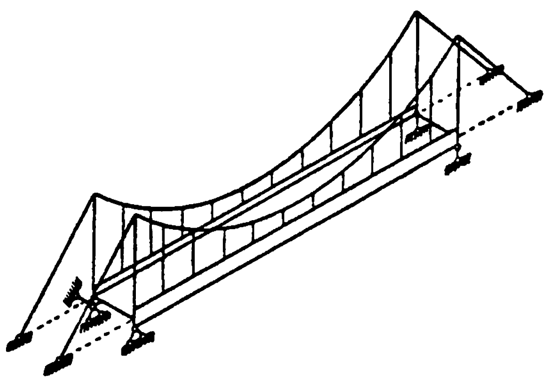


Figure 194

* The rise of an arch of length l subjected to a uniformly distributed transverse load of magnitude q is determined, as is known from structural mechanics, by the formula

$$f=\frac{ql^2}{8H}.$$

subjected to a horizontal wind drag q_x, is obtained from the equations for the arch bridge (see § 6) by changing the sign of the thrust H. As a result we shall have:

$$\left.\begin{aligned} EJ_x\eta^{IV} - 2H\eta'' + (M_y\theta)'' = 0, \\ M_y\eta'' + EJ_\omega\theta^{IV} - GJ_d\theta'' - H\frac{b^2}{2}\theta'' + q_x e_x\theta = 0. \end{aligned}\right\} \quad (7.1)$$

Applying, as in the previous section, the principle of virtual displacements to the determination of the critical load from equations (7.1), we obtain the stability equation

$$\begin{vmatrix} \int_0^l [EJ_x(\psi'')^2 + 2H(\psi')^2]\,dz & q_x\int_0^l \overline{M}_y\psi''\chi\,dz \\ q_x\int_0^l \overline{M}_y\psi''\chi\,dz & \int_0^l \left[EJ_\omega(\chi'')^2 + \left(GJ_d + H\frac{b^2}{2}\right)(\chi')^2\right]dz + q_x e_x\int_0^l \chi^2\alpha\,dz \end{vmatrix} = 0. \quad (7.2)$$

Since equation (7.2) has a symmetrical structure, all the values obtained for the critical load are also real.

2. As an example, we shall examine the collapse of the Tacoma suspension bridge (Wash. USA) under the statical action of a horizontal wind load. We shall consider the load to be uniformly distributed along the span,

$$\alpha(z) = 1.$$

We shall also take it that the roadway part has hinged ends. We then have

$$\left.\begin{aligned} &\overline{M}_y = \frac{z(l-z)}{2}, \\ &\psi(z) = \sin\lambda z, \quad \chi(z) = \sin\lambda z, \quad \eta(z) = B\sin\lambda z, \quad \theta(z) = C\sin\lambda z. \end{aligned}\right\} \quad (7.3)$$

Expressions (7.3) for $\lambda = \frac{n\pi}{l}$ and $n = 1, 2, 3, \ldots$ satisfy all the boundary conditions of the problem.

The stability equation is obtained from equation (6.6) on replacing $+H$ by $-H$:

$$\begin{vmatrix} EJ_x\lambda^4 + 2H\lambda^2 & \frac{q_x}{4}\left(1 + \frac{\lambda^2 l^2}{3}\right) \\ \frac{q_x}{4}\left(1 + \frac{\lambda^2 l^2}{3}\right) & EJ_\omega\lambda^4 + \left(GJ_d + H\frac{b^2}{2}\right)\lambda^2 + q_x e_x \end{vmatrix} = 0. \quad (7.4)$$

We write the solution of (7.4) in the form

$$q_{x\,cr} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where

$$\begin{aligned} a &= \frac{1}{16}\left(1 + \frac{\lambda^2 l^2}{3}\right)^2, \\ b &= -(EJ_x\lambda^4 + 2H\lambda^2)\,e_x, \\ c &= -(EJ_x\lambda^4 + 2H\lambda^2)\left[EJ_\omega\lambda^4 + \left(GJ_d + H\frac{b^2}{2}\right)\lambda^2\right]. \end{aligned}$$

We shall now go over to actual numerical data. We have the following values /87/ for the characteristics of the cross section of the roadway part, schematically shown in Figure 195:

$$J_x = 0.0539\ \mathrm{m}^4,$$
$$J_\omega = 1.903\,\mathrm{m}^6,$$
$$J_d = 1.146 \cdot 10^{-7}\,\mathrm{m}^4,$$

the span of the bridge is $l = 853.4$ m; the chain thrust is $H = 5{,}650$ ton.

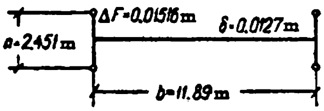


Figure 195

We adopt the following values of the elastic moduli of the materials of the roadway part (steel),

$$E = 2.1 \cdot 10^7 \frac{\text{ton}}{\text{m}^2}, \quad G = 0.8 \cdot 10^7 \frac{\text{ton}}{\text{m}^2}.$$

Considering the wind pressure only on the wind side and neglecting the lee side, we put

$$e_x = \frac{b}{2} = 5.944\,\mathrm{m}.$$

Not counting the value $n = 1$ for reasons given in the previous section, we find by calculation that instability takes place with two half waves. The minimum critical load corresponding to this form of instability is

$$q_{xкp} = 0.90 \frac{\text{ton}}{\text{m}}.$$

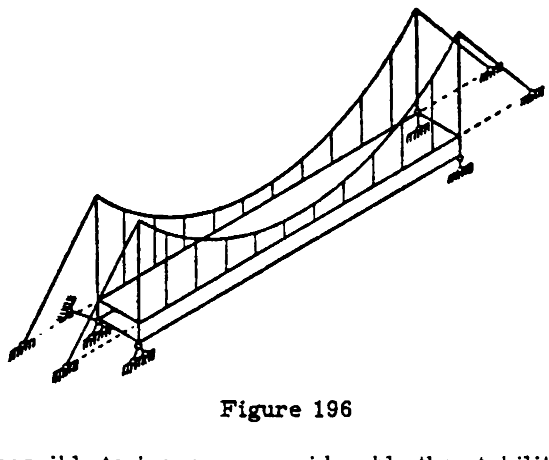


Figure 196

3. It is possible to increase considerably the stability of suspension bridges with the same expenditure in material if the span structure of the roadway part is executed not in the form of a thin-walled open section but in the form of a box-like section (Figure 196). The torsional rigidity increases considerably in the latter case and the minimum critical load is correspondingly increased.

4. It is also possible to attain increased stability of a suspension bridge with a roadway part in the form of an open section by special structural arrangements which increase the torsional rigidity (see Chapter III). The most effective are transverse bimoment connections preventing the

warping of the cross section. Such connections can be braces, lattice truss girders, skew diaphragms. In order to confer upon the structure a larger flexural-torsional rigidity, the transverse bimoment connections should be placed closest to the ends of the span structure.

§ 8. Application of the theory to the stability design of airfoils

1. We shall examine the stability problem in plane bending of a beam of rectangular box-like section (caisson), shown in Figure 197. We shall assume that the cross section does not deform in its plane. The supporting structure of an airplane wing is of this type.

The stability equations (VI.3.3) for plane bending under the action of a transverse load in the plane Oxz will have the form

$$\left.\begin{aligned} EJ_x\eta^{\mathrm{IV}}+(M_y\theta)''=0,\\ EJ_\omega\theta^{\mathrm{IV}}-GJ_d\theta''+q_xe_x\theta+M_y\eta''=0,\end{aligned}\right\} \tag{8.1}$$

where q_x is the resultant frontal pressure on a unit transverse strip of the wing surface, e_x is the eccentricity of the point of application of the resultant frontal pressure q_x.

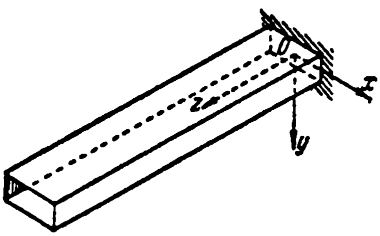


Figure 197

For an airplane wing moving in a stationary air current with the velocity v the transverse load q_x can be taken to be proportional to the square of the velocity,

$$q_x=-kv^2, \tag{8.2}$$

where k is an aerodynamical [drag] coefficient depending on the form of the airplane wing (streamlining) and on other factors.

The bending moment M_y for uniform distribution of q_x along the wing of the airplane and for boundary conditions corresponding to a rigid fixing of the end $z=0$, the end $z=l$ being free, is given by

$$M_y=\frac{q_x(l-z)^2}{2}, \tag{8.3}$$

where l is the wing span. Introducing (8.2), the relation (8.3) becomes

$$M_y=-\frac{kv^2(l-z)^2}{2}.$$

The system of differential equations (8.1), together with the homogeneous boundary conditions associated with them, thus affords the determination of the critical velocity v of the airplane in a stationary air current in which the plane-bent wing may become unstable through flexural-torsional deformations.

Assuming, as before (see (6.4)),

$$\eta=B\psi(z),\quad \theta=C\chi(z),$$

choosing the functions $\phi(z)$ and $\chi(z)$ so that they will satisfy the boundary conditions of the problem and taking these same functions $\phi(z)$ and $\chi(z)$ as the virtual displacements in the variational method described before, we can write the characteristic equation of the present homogeneous boundary problem, viz.

$$\begin{vmatrix} \int_0^l EJ_x(\phi'')^2\,dz & q_x\int_0^l \overline{M}_y\chi\phi''\,dz \\ q_x\int_0^l \overline{M}_y\chi\phi''\,dz & \int_0^l [EJ_\omega(\chi'')^2 + GJ_d(\chi')^2]\,dz + q_x e_x\int_0^l \alpha\chi^2\,dz \end{vmatrix} = 0. \quad (8.4)$$

Here, as in § 6, $\alpha = \alpha(z)$ is a function expressing the distribution of the transverse load q_x along the generator; $\overline{M}_y = \overline{M}_y(z)$ is the bending moment due to the load $q_x = 1$.

For $\alpha(z) = 1$ $\overline{M}_y = \frac{(l-z)^2}{2}$.

It is possible to solve equation (8.4) not only for q_x, but also for any other quantity entering this equation in terms of the other given quantities. For example, it is possible to find the minimum value of the bimoment of inertia J_ω which ensures, for given values of q_x, e_x, EJ_x and GJ_d, the stability of the caisson.

2. In examining the problem of strength and stability of an airplane wing it may be assumed that the latter is rigidly built-in into the fuselage and free at the other end. Accordingly, it is possible to take, up to an arbitrary factor, the functions of the statical deflection of a cantilever due to a uniformly distributed transverse load as the functions $\phi(z)$ and $\chi(z)$ required for the approximate determination of the critical load. Generally speaking, the rigidity of this cantilever must vary according to the same function as the corresponding rigidities of the airplane wing. However, as a first-order approximation, the deflection curves of a beam of constant section may also be used for wings of variable section /75/.

Let $\phi(z)$ and $\chi(z)$ satisfy the differential equation $Z^{\mathrm{IV}}(z) = 1$ and the boundary conditions

$$Z(0) = Z'(0) = 0, \quad Z''(l) = Z'''(l) = 0.$$

We can set then, up to a constant factor,

$$\phi(z) = \chi(z) = z^4 - 4lz^3 + 6l^2z^2. \quad (8.5)$$

The definite integrals in the determinant (8.4) are easily calculated,

$$\int_0^l (\phi'')^2\,dz = \frac{144}{5}l^5, \quad \int_0^l \overline{M}_y\chi\phi''\,dz = \frac{4}{15}l^9,$$

$$\int_0^l (\chi')^2\,dz = \frac{72}{7}l^7, \quad \int_0^l \chi^2\,dz = \frac{104}{45}l^9,$$

and after some transformations the expanded determinant (8.4) takes the form

$$v^4 - \frac{936\,EJ_x e_x}{kl^4}v^2 - \frac{5832\,EJ_x}{7k^2l^8}(14\,EJ_\omega + 5GJ_dl^2) = 0. \quad (8.6)$$

Equation (8.6) determines the critical velocity of the airplane.

It is also possible to employ the eigenfunctions of the natural bending oscillations for the functions $\varphi(z)$ and $\chi(z)$. These functions satisfy the differential equation $Z^{IV}=\mu^4 Z$ and the corresponding boundary conditions given in § 13 of Chapter V.

3. In calculating the geometrical characteristics of the wing section, entering the coefficients of the differential equations (8.1), we proceed from the caisson form of cross section constituting the supporting structure of the wing. As for the streamlined skin, this external form of the wing is taken into account by the aerodynamic coefficient k. In calculating the geometrical characteristics of the caisson we take the warping function to be not sectorial as in open profiles, but axial

$$\omega = xy.$$

The bimoment of inertia is calculated, therefore, from the expression

$$J_\omega = \int_F \omega^2\, dF = \int_F x^2 y^2\, dF,$$

and the moment of inertia J_d is determined by the methods of the theory of pure torsion. In particular, for a profile having the form of a closed rectangle with the sides d_1 and d_2 and thickness δ, the geometrical characteristics are determined from (V.1.18).

4. We can also take into consideration an initial stress in the caisson appearing as a result of thermal action or as a result of the initial tension in the frame. On the basis of what was discussed in Chapter VII, we have to use in this case the reduced rigidity $\overline{GJ}_d$, calculated by one of the following expressions, instead of the rigidity GJ_d:

a) In the case of initial thermal effects by (VII.5.2),

$$\overline{GJ}_d = GJ_d + KT\Phi,$$

b) In the case of initial tension in the frame by (VII.1.9),

$$\overline{GJ}_d = GJ_d - R_k \mathfrak{M}_k.$$

We can lessen or increase the reduced rigidity $\overline{GJ}_d$ by varying the position of the pretensioned reinforcement (see § 3, Chapter VII).

5. The above theory of the stability of structures of the airplane-wing type is summed up in the homogeneous differential equations (8.1) and the associated homogeneous boundary conditions. The parameter in these equations is the airplane velocity in a steady air current. When this velocity is less than critical the determinant of the system (8.1) does not vanish. The homogeneous equations (8.1) with the homogeneous boundary conditions (i. e. without an external load) will then have only vanishing solutions.

But when the structure is subjected to a vertical transverse load, such as the lift of an airplane wing, equations (8.1) will be inhomogeneous and their right-hand sides will contain quantities depending on the load. The inhomogeneous system corresponding to (8.1) will then have nonzero solutions. The terms q_x and M_y (proportional to the square of the velocity v) on the right-hand sides of these inhomogeneous equations indicate that the total stiffness of the structure depends here not only on the elastic and geometrical characteristics of the caisson but also on the velocity of

the airplane. When this velocity approaches its critical value the flexural-torsional stiffness of the structure is reduced on the whole.

6. The supporting structure of the airplane wing can be also constructed as a thin-walled box-like profile of variable section. The stability equations for plane bending (8.1) are accordingly generalized and read

$$(EJ_x\eta'')'' + (M_y\theta'') = 0,$$
$$M_y\eta'' + (EJ_\omega\theta'')'' - (GJ_d\theta')' + q_x e_x\theta = 0.$$

The characteristic equation (8.4) retains its form also for a beam of variable cross section, differing only in that the characteristics of the cross section J_x, J_ω, J_d are certain given functions of the coordinate z.

§ 9. Stability of a circular cylindrical shell with reinforcing beams /54/

Our theories of thin-walled beams and of the orthotropic cylindrical, longitudinally moment-less [membrane-like] shell, permit the solution of the combined contact problem of strength, stability and vibrations for a thin-walled spatial system consisting of a cylindrical shell with reinforcing longitudinal beams. Applying the general variational method to the design of a centrally compressed shell (Figure 198) and representing the components of the warping on the circular arc of the middle surface cross section by the trigonometric functions $\sin n\beta$, we obtain for each term of the eigenfunction expansion of the solution $W = W(z)$ the ordinary sixth-order differential equation with constant coefficients,

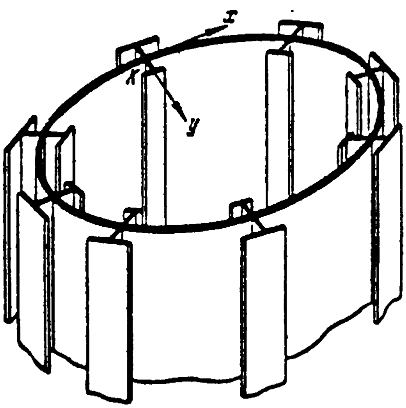


Figure 198

$$(AJ_1 - S^2)\,W^{VI} + \left(2B_2S - AK - B_1J_1 + AK_1\frac{p}{E}\right)W^{IV} + \left(AJ_2 + B_1K - B_2^2 - B_1K_1\frac{p}{E}\right)W'' - B_1J_2W = 0. \quad (9.1)$$

The quantities in the coefficients of this equation are calculated according to Table 44.

The upper line of the table gives the index of the expansion term of the two-dimensional eigenfunction $\Phi(z, \beta) = \sum_{n=1}^{4} W(z)\cos n\beta$ as a trigonometric polynomial in terms of the angular coördinate $\beta = s/R$. The extreme left column gives the quantities to be determined. We have adopted in these expressions the following notations: h is the thickness of the shell; R is the radius of its middle surface; F is the area of the cross section of a thin-walled beam; J_x and J_y are the moments of inertia of the beam section with respect to axes passing through the contact point K of the beam with the shell.

Table 44

	$n=1$	$n=2$	$n=3$	$n=4$
A	$2\pi Rh+8F$	$2\pi Rh+8F$	$2\pi Rh+8F$	$2\pi Rh+16F$
B_1	$\frac{\pi h}{2R}$	$\frac{2\pi h}{R}$	$\frac{9\pi h}{2R}$	$\frac{8\pi h}{R}$
B_2	$\frac{\pi h}{2}$	πh	$\frac{3\pi h}{2}$	$2\pi h$
J_1	$8(J_y+J_x)$	$8\left(J_y+4J_x-\frac{6}{R}J_{\omega x}+\frac{9}{R^2}J_\omega\right)$	$8\left(J_y+9J_x-\frac{16}{R}J_{\omega x}+\frac{64}{R^2}J_\omega\right)$	$256J_x$
J_2	0	$72\frac{\pi}{R^3}J+12\pi\frac{q}{G}$	$1152\frac{\pi}{R^3}J+72\pi\frac{q}{G}$	$7200\frac{\pi}{R^3}J+240\pi\frac{q}{G}$
S	$8S_x$	$16S_x$	$24S_x$	$64S_x$
K	$\frac{\pi Rh}{2}$	$\frac{\pi Rh}{2}+\frac{36}{R^2}J_d$	$\frac{\pi Rh}{2}+\frac{256}{R^2}J_d$	$\frac{\pi Rh}{2}$
K_1	$4\pi Rh+16F$	$10\pi Rh+40F+48\frac{S_x}{R}+72\frac{J_y+J_x}{R^2}$	$20\pi Rh+80F+128\frac{S_x}{R}+512\frac{J_y+J_x}{R^2}$	$34\pi Rh+256F$

The axes Ky, Kx are chosen for the beam in such a way that the axis Kx is directed along the tangent to the sectional circle of the shell and Ky along the normal to it (Figure 199).

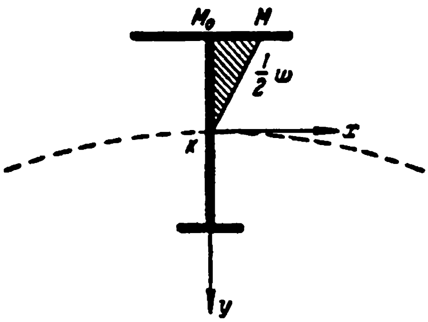


Figure 199

The generalized coordinates Γ, $x(s)$, $y(s)$, $\omega(s)$ will in general not be orthogonal if the origin of the axes $x(s)$ and $y(s)$ and the pole of the sectorial area $\omega(s)$ are placed at the contact point K. J_ω is the sectorial moment of inertia of the beam section; S_x is the statical moment of the beam section with respect to the axis Kx; $J_{\omega y}$ is the sectorial product of inertia; J_d is the moment of inertia of the beam for pure torsion; G is the shear modulus [rigidity]; q is the internal pressure. J is the reduced moment of inertia of the shell per unit length for a longitudinal section; if the shell is reinforced by transverse ribs this moment has to include the average contribution of the moment of inertia per unit length of the transverse sections of the ribs.

For an arbitrary symmetrical section with the axis of symmetry Ky the quantities J_x, J_y, S_x, J_ω, $J_{\omega x}$ are calculated from the expressions:

$$\left.\begin{aligned} &J_x=\int\limits_F y^2\,dF, \quad J_y=\int\limits_F x^2\,dF, \quad S_x=\int\limits_F y\,dF,\\ &J_\omega=\int\limits_F \omega^2\,dF, \quad J_{\omega x}=\int\limits_F x\omega\,dF. \end{aligned}\right\} \tag{9.2}$$

These integrals are evaluated over the whole cross section of the beam. If $\delta=\delta(s)$ is the thickness of the beam wall, then $dF=\delta\,ds$.

The quantity p in equation (9.1) is the required critical stress for the examined thin-wall spatial structure under axial compression. E is [Young's] modulus of elasticity.

Setting in equation (9.1)

$$W(z)=\sin\frac{\pi z}{\lambda},$$

where λ is the sinusoidal half-wave length in the longitudinal direction, we obtain an equation for the required critical stress,

$$\begin{aligned} -(AJ_1-S^2)+\left(AK_1\frac{p}{E}-AK-B_1J_1-2B_2S\right)\gamma+\\ +\left(B_1K_1\frac{p}{E}-AJ_2-B_1K+B_2^2\right)\gamma^2-B_1J_2\gamma^3=0, \end{aligned} \tag{9.3}$$

where $\gamma=\lambda^2$. This cubical equation and the associated minimum condition $\frac{\partial p}{\partial\gamma}=0$ determine, for given values of A, J_1, S^2, ..., J_ω three real values of the required longitudinal critical stress p for each of the four examined periodical deformations of the shell cross section. The smallest value of p_{cr} must be chosen as design stress. To this stress will correspond well-defined half-wave lengths in the transverse and longitudinal directions.

If we set $q=0$ in the expression for $J_{\bullet}$, we shall have a general solution for the stability of the shell in the absence of internal pressure q. For $p=Aq$ and $q<0$ we shall have a general solution for the stability problem of a shell subjected to external pressure.

If the shell is reinforced by longitudinal beams, each of which only resists extension or compression, as, for example, in reinforced concrete structures, we have to set in the above expressions for each beam $J_x=J_y=J_\omega=J_{\omega x}=S_x=J_d=0$.

When there are no transverse ribs, the quantity J depends only on the thickness h of the shell and is computed from the formula $J=\frac{h^3}{12}$. For $J=0$ we shall have the solution of the problem of a thin-walled momentless [membrane] shell reinforced by longitudinal ribs. If the shell is also reinforced by transverse ribs, the quantity J can be determined by the following expression, if these ribs are closely spaced,

$$J=\frac{J_p}{a},$$

where J_p is the moment of inertia of the area of the combined section of the rib and shell in the longitudinal direction; a is the distance apart of the transverse ribs.

Chapter IX

GENERAL THEORY OF FLEXURAL-TORSIONAL VIBRATIONS AND DYNAMIC STABILITY

§ 1. Differential equations of free vibrations

The general theory of thin-walled beams developed in the present work along with the problems of strength and stability allows also the solution, in a general form, of the problem of spatial vibrations of beams with inflexible sectional contours.

Our starting point in the treatment of problems of beam vibration is d'Alembert's principle; we consider formally the dynamical problem as a statical problem with the interial forces added to the elastic forces of the beam.

As was shown in Chapter I, the statical equilibrium problem of a beam, in the absence of external shear forces along the lateral edges, leads to the differential equations

$$\left.\begin{aligned} EF\zeta'' + \int_L p_z\,ds &= 0, \\ EJ_y\xi^{IV} - \int_L \frac{\partial p_z}{\partial z}\,x\,ds - q_x &= 0, \\ EJ_x\eta^{IV} - \int_L \frac{\partial p_z}{\partial z}\,y\,ds - q_y &= 0, \\ EJ_\omega\theta^{IV} - GJ_d\theta'' - \int_L \frac{\partial p_z}{\partial z}\,\omega\,ds - m_A &= 0, \end{aligned}\right\} \tag{1.1}$$

where $p_z = p_z(z, s)$ is the projection of the surface load on the generator of the cylindrical surface; $q_x(z)$ and $q_y(z)$ are the projections of the transverse load per unit beam length; $m_A(z)$ is the external twisting moment due to the transverse loads q_x and q_y, with respect to the shear center.

Passing to the vibration problem of a beam we must take the displacements ζ, ξ, η, θ as functions of two variables: the abscissa z and the time t.

We shall now determine the inertial forces per unit area of the middle surface of the beam. In vibration of the beam any point M of the middle surface will be in motion. The displacement of this point is a vector, depending on the position of the point on the surface (the coordinates z, s) and the time t. The components of this vector on the x, y, z-axes are

given by (I.3.4) and (I.3.16), viz.

$$\left.\begin{aligned}\xi_M &= \xi-(y-a_y)\theta,\\ \eta_M &= \eta+(x-a_x)\theta,\\ \zeta_M &= \zeta-\xi' x-\eta' y-\theta'\omega,\end{aligned}\right\} \tag{1.2}$$

where ζ, ξ, η and θ are functions of the two variables z and t. Knowing the components of the total displacement of an arbitrary point, we can determine from these components the inertial forces per unit area of the middle surface. To do this it is necessary to multiply the mass of a parallelepiped with the basis $dz\,ds=1$ and height equal to the thickness of the beam wall δ, by the acceleration in the corresponding direction. We obtain

$$\left.\begin{aligned}p_x &= -\frac{\gamma\delta}{g}\frac{\partial^2\xi_M}{\partial t^2},\\ p_y &= -\frac{\gamma\delta}{g}\frac{\partial^2\eta_M}{\partial t^2},\\ p_z &= -\frac{\gamma\delta}{g}\frac{\partial^2\zeta_M}{\partial t^2},\end{aligned}\right\} \tag{1.3}$$

where γ is the specific gravity of the beam material, $\delta=\delta(s)$ is the thickness of the beam wall, g is the gravity acceleration.

Introducing now in (1.3) the displacements (1.2) of the point M, we obtain

$$\left.\begin{aligned}p_x &= -\frac{\gamma\delta}{g}\left[\frac{\partial^2\xi}{\partial t^2}-(y-a_y)\frac{\partial^2\theta}{\partial t^2}\right],\\ p_y &= -\frac{\gamma\delta}{g}\left[\frac{\partial^2\eta}{\partial t^2}+(x-a_x)\frac{\partial^2\theta}{\partial t^2}\right],\\ p_z &= -\frac{\gamma\delta}{g}\left(\frac{\partial^2\zeta}{\partial t^2}-\frac{\partial^3\xi}{\partial z\partial t^2}x-\frac{\partial^3\eta}{\partial z\partial t^2}y-\frac{\partial^3\theta}{\partial z\partial t^2}\omega\right).\end{aligned}\right\} \tag{1.4}$$

The components p_x, p_y, p_z determine the vector of the inertial surface force at an arbitrary point M of the middle surface of the beam. From these components we can now determine the force factors q_x, q_y, m_A, entering in equations (1.1). Keeping in mind the orthogonality of the functions 1, $x(s)$, $y(s)$ and $\omega(s)$, viz.

$$\left.\begin{aligned}&\int_F x\,dF=\int_F y\,dF=\int_F xy\,dF=0,\\ &\int_F \omega\,dF=\int_F \omega x\,dF=\int_F \omega y\,dF=0,\end{aligned}\right\} \tag{1.5}$$

and in view of the relations (1.4), we obtain

$$\left.\begin{aligned}q_x &= \int_L p_x\,ds=-\left(\frac{\partial^2\xi}{\partial t^2}+a_y\frac{\partial^2\theta}{\partial t^2}\right)\frac{\gamma F}{g},\\ q_y &= \int_L p_y\,ds=-\left(\frac{\partial^2\eta}{\partial t^2}-a_x\frac{\partial^2\theta}{\partial t^2}\right)\frac{\gamma F}{g},\\ m_A &= \int_L [p_y(x-a_x)-p_x(y-a_y)]\,ds=\\ &= -\left(a_y\frac{\partial^2\xi}{\partial t^2}-a_x\frac{\partial^2\eta}{\partial t^2}+r^2\frac{\partial^2\theta}{\partial t^2}\right)\frac{\gamma F}{g}.\end{aligned}\right\} \tag{1.6}$$

Upon introducing the expression for p_z from the third of relations (1.4) in the definite integrals of (1.1) and using the orthogonality relations (1.5), these integrals become

$$\left.\begin{aligned}
&\int\limits_L p_z ds = -\frac{\gamma F}{g}\frac{\partial^2\zeta}{\partial t^2}, \qquad \int\limits_L \frac{\partial p_z}{\partial z} x ds = \frac{\gamma J_y}{g}\frac{\partial^4\xi}{\partial z^2\partial t^2},\\
&\int\limits_L \frac{\partial p_z}{\partial z} y ds = \frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2}, \qquad \int\limits_L \frac{\partial p_z}{\partial z}\omega ds = \frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2}.
\end{aligned}\right\} \tag{1.7}$$

Substituting in (1.1) instead of the load terms their expressions for the case of vibrations of a beam (1.6) and (1.7), we obtain the following differential equations

$$\left.\begin{aligned}
&EF\frac{\partial^2\zeta}{\partial z^2} - \frac{\gamma F}{g}\frac{\partial^2\zeta}{\partial t^2} = 0,\\
&EJ_y\frac{\partial^4\xi}{dz^4} - \frac{\gamma J_y}{g}\frac{\partial^4\xi}{\partial z^2\partial t^2} + \frac{\gamma F}{g}\frac{\partial^2\xi}{\partial t^2} + \frac{a_y\gamma F}{g}\frac{\partial^2\theta}{\partial t^2} = 0,\\
&EJ\,\frac{\partial^4\eta}{\partial z^4} - \frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2} + \frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2} - \frac{a_x\gamma F}{g}\frac{\partial^2\theta}{\partial t^2} = 0,\\
&\frac{a_y\gamma F}{g}\frac{\partial^2\xi}{\partial t^2} - \frac{a_x\gamma F}{g}\frac{\partial^2\eta}{\partial t^2} + EJ_\omega\frac{\partial^4\theta}{\partial z^4} - GJ_d\frac{d^2\theta}{dz^2} -\\
&\qquad - \frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2} + \frac{r^2\gamma F}{g}\frac{\partial^2\theta}{\partial t^2} = 0.
\end{aligned}\right\} \tag{1.8}$$

Equations (1.8) refer to the free vibrations of a thin-walled beam with an open inflexible section contour of arbitrary shape. The first equation determines, independently of the other three and together with the boundary and initial conditions, the longitudinal vibrations of the beam. The remaining three equations form, in the general case, a symmetrical system which, together with the boundary and initial conditions, determines the transverse flexural-torsional vibrations of the beam. In the particular case when $a_x = a_y = 0$, i. e. when the shear center coincides with the centroid of the section (as, for example, when the section has two axes of symmetry) the system of simultaneous equations (1.8) is resolved into the independent differential equations

$$\left.\begin{aligned}
&EF\frac{\partial^2\zeta}{\partial z^2} - \frac{\gamma F}{g}\frac{\partial^2\zeta}{\partial t^2} = 0,\\
&EJ_y\frac{\partial^4\xi}{\partial z^4} - \frac{\gamma J_y}{g}\frac{\partial^4\xi}{\partial z^2\partial t^2} + \frac{\gamma F}{g}\frac{\partial^2\xi}{\partial t^2} = 0,\\
&EJ_x\frac{\partial^4\eta}{\partial z^4} - \frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2} + \frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2} = 0,\\
&EJ_\omega\frac{\partial^4\theta}{\partial z^4} - GJ_d\frac{\partial^2\theta}{\partial z^2} - \frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2} + \frac{r^2\gamma F}{g}\frac{\partial^2\theta}{\partial t^2} = 0.
\end{aligned}\right\} \tag{1.9}$$

The first of equations (1.9) refers, as the general case discussed above, to the longitudinal vibrations of the beam. The second and third of equations (1.9) refer to pure transverse flexural vibrations of a beam in the principal planes. These equations differ from the usual equations given in courses of structural dynamics in that the middle terms represent the

components of the transverse inertial loads due to the rotations of the cross section about the principal axes*.

The last equation of (1.9) refers to the rotation of the beam element about a longitudinal axis passing through the shear center (in the present case, through the centroid) since the coordinates of the shear center a_x and a_y vanish here.

It follows from the above analysis that pure transverse flexural vibrations of beams (without torsion) are possible only when the shear center coincides with the section centroid. In general the vibrations of thin-walled beams (such as ribbed arch-shells and ribbed hipped systems) will be flexural-torsional, determined by the system of the last three equations (1.8) and the corresponding boundary and initial conditions.

§ 2. Integration of the equations of vibration for beams

Equations (1.8) are linear and homogeneous, having constant coefficients which depend on the physical and geometrical characteristics of the beam. In addition to the derivatives of the required functions ζ, ξ, η and θ, with respect to the abscissa z, these equations also contain second derivatives with respect to the time t. It follows that the examined vibrations of the beam will be harmonic. Using Poisson's method of separation of variables, i.e. putting

$$\left.\begin{aligned}\zeta(z, t) &= \sum_{n=1,2,3,\dots} \varkappa_n(z)\sin k_n t,\\ \xi(z, t) &= \sum_{n=1,2,3,\dots} \varphi_n(z)\sin k_n t,\\ \eta(z, t) &= \sum_{n=1,2,3,\dots} \psi_n(z)\sin k_n t,\\ \theta(z, t) &= \sum_{n=1,2,3,\dots} \chi_n(z)\sin k_n t,\end{aligned}\right\} \tag{2.1}$$

substituting in (1.8) and reducing by the factor $\sin k_n t$, we obtain for the n-th terms

$$\left.\begin{aligned}EF\varkappa_n'' + \frac{\gamma F}{g}k_n^2\varkappa_n &= 0,\\ EJ_y\varphi_n^{\mathrm{IV}} + \frac{\gamma J_y}{g}k_n^2\varphi_n'' - \frac{\gamma F}{g}k_n^2\varphi_n - \frac{a_y\gamma F}{g}k_n^2\chi_n &= 0,\\ EJ_x\psi_n^{\mathrm{IV}} + \frac{\gamma J_x}{g}k_n^2\psi_n'' - \frac{\gamma F}{g}k_n^2\psi_n + \frac{a_x\gamma F}{g}k_n^2\chi_n &= 0,\\ \frac{a_x\gamma F}{g}k_n^2\psi - \frac{a_y\gamma F}{g}k_n^2\varphi_n + EJ_\omega\chi_n^{\mathrm{IV}} - &\\ -\left(GJ_d - \frac{\gamma J_\omega}{g}k_n^2\right)\chi_n'' - \frac{r^2\gamma F}{g}k_n^2\chi_n &= 0.\end{aligned}\right\} \tag{2.2}$$

The differential equations (2.2) are ordinary and homogeneous. The coefficients of these equations form a symmetrical matrix and depend on

* In Timoshenko's book "Theory of Vibrations in Engineering", the two-term equation

$$EJ_y\frac{\partial^4\xi}{\partial z^4} + \frac{\gamma F}{g}\frac{\partial^2\xi}{\partial t^2} = 0,$$

is given along with an equation of the form of the second of equations (1.9). The middle term is usually neglected in the theory of vibrations of beams because of its smallness.

the physical and geometrical characteristics of the beam and on the square of the vibration frequency k_n^2.. The symmetrical structure of (2.2) and (1.8) is due to Betti's principle of reciprocity, which applies to all the linear theory of dynamic elasticity. For homogeneous boundary conditions the homogeneous differential equations (2.2) determine the eigenvalues and eigenfunctions of the present boundary-value problem. Since equations (2.2) have a symmetrical structure, the eigenvalues will be real. These numbers, as known from the theory of linear integral equations with symmetrical regular kernels, have a discrete point-spectrum. We thus obtain for the given homogeneous differential equations (2.2) and the homogeneous self-adjoint boundary conditions an infinite set of eigenvalues, determined by the corresponding characteristic transcendental equations. These eigenvalues determine (by the first equation (2.2)) all the frequencies of the longitudinal and (by the system of the last three equations (2.2)) flexural-torsional vibrations of a thin-walled beam with a rigid cross-sectional contour.

We shall examine the problem of vibrations in beams whose end sections are hinged with respect to the displacements ξ, η and θ, i.e. the end sections are fixed against the deflections ξ and η and the torsional angle θ and can be freely displaced away from the plane of the section. These boundary conditions correspond to a beam or a cylindrical shell resting on diaphragms at the end sections which are rigid in their plane and flexible in respect of displacements from their plane. These conditions are close to the real conditions of shells supported on curved edges. Choosing the origin of the coordinate at one of the beam ends and denoting the span by l (the distance between the supporting diaphragms) we can write the boundary conditions in the following form

$$\left.\begin{aligned}&\text{for } z=0 \quad \xi=\eta=\theta=0, \quad \frac{\partial^2\xi}{\partial z^2}=\frac{\partial^2\eta}{\partial z^2}=\frac{\partial^2\theta}{\partial z^2}=0;\\ &\text{for } z=l \quad \xi=\eta=\theta=0, \quad \frac{\partial^2\xi}{\partial z^2}=\frac{\partial^2\eta}{\partial z^2}=\frac{\partial^2\theta}{\partial z^2}=0.\end{aligned}\right\} \tag{2.3}$$

The vanishing of the second derivatives of the displacements ξ, η and θ follows from the condition of no normal stresses σ at the ends of the beam. These stresses, as shown before, are linear in the second derivatives of the deflections ξ and η and the torsional angle θ.

The boundary values (2.3) of the required functions φ_n, ψ_n and χ_n in equations (2.2), representing the coefficients in expressions (2.1), are here

$$\left.\begin{aligned}&\text{for } z=0 \quad \varphi_n=\psi_n=\chi_n=0, \quad \varphi_n''=\psi_n''=\chi_n''=0;\\ &\text{for } z=l \quad \varphi_n=\psi_n=\chi_n=0, \quad \varphi_n''=\psi_n''=\chi_n''=0.\end{aligned}\right\} \tag{2.4}$$

The conditions (2.4) refer to flexural-torsional vibrations of a beam with hinged ends. We shall not examine the longitudinal vibrations, expressed by the first of equations (2.2), as we have obtained for these vibrations the well-known wave equation.

Since equations (2.2) contain the required functions and their even-order derivatives and since the coefficients of these equations do not depend on the integration variable, the eigenfunctions for the flexural-torsional vibrations of a beam with the boundary conditions (2.4) must be

$$\left.\begin{aligned}\varphi_n &= A_n \sin\lambda_n z,\\ \psi_n &= B_n \sin\lambda_n z,\\ \chi_n &= C_n \sin\lambda_n z,\end{aligned}\right\} \tag{2.5}$$

where $\lambda_n = \frac{n\pi}{l}$; n is an arbitrary positive integer; A_n, B_n, C_n are arbitrary quantities depending on the index of the expansion term.

Introducing expressions (2.5) in the differential equations (2.2) and cancelling everywhere the common factor $\sin\lambda_n z$, we obtain

$$\left.\begin{aligned}&\left(EJ_y\lambda_n^4 - \frac{\gamma J_y}{g}\lambda_n^2 k_n^2 - \frac{\gamma F}{g}k_n^2\right)A_n - \frac{a_y\gamma F}{g}k_n^2 C_n = 0,\\ &\left(EJ_x\lambda_n^4 - \frac{\gamma J_x}{g}\lambda_n^2 k_n^2 - \frac{\gamma F}{g}k_n^2\right)B_n + \frac{a_x\gamma F}{g}k_n^2 C_n = 0,\\ &\qquad -\frac{a_y\gamma F}{g}k_n^2 A_n + \frac{a_x\gamma F}{g}k_n^2 B + \\ &+\left[EJ_\omega\lambda_n^4 + GJ_d\lambda_n^2 - (J_\omega\lambda_n^2 + r^2F)\frac{\gamma}{g}k_n^2\right]C_n = 0.\end{aligned}\right\} \tag{2.6}$$

As we see, the functions (2.5) satisfy the differential equations (2.2) under the condition that the coefficients A_n, B_n and C_n satisfy the linear homogeneous algebraic equations (2.6). Therefore the functions (2.5), forming a complete system and being orthogonal in the interval $(0, l)$, are the eigenfunctions of the examined boundary problem. The frequencies k_n of the flexural-torsional vibrations of the beam are determined from the condition that the determinant of the system (2.6) must vanish:

$$\begin{vmatrix} EJ_y\lambda_n^4 - (J_y\lambda_n^2 + F)\frac{\gamma}{g}k_n^2 & 0 & -\frac{a_y\gamma F}{g}k_n^2 \\ 0 & EJ_x\lambda_n^4 - (J_x\lambda_n^2 + F)\frac{\gamma}{g}k_n^2 & \frac{a_x\gamma F}{g}k_n^2 \\ -\frac{a_y\gamma F}{g}k_n^2 & \frac{a_x\gamma F}{g}k_n^2 & \begin{array}{c}EJ_\omega\lambda_n^4 + GJ_d\lambda_n^2 - \\ -(J_\omega\lambda_n^2 + r^2F)\frac{\gamma}{g}k_n^2\end{array} \end{vmatrix} = 0. \tag{2.7}$$

Equation (2.7) is cubical in the squared frequency k_n^2. The roots of this equation are real. To each value of the quantity λ_n (or, which is the same, the value n), characterizing together with expressions (2.5) the flexural-torsional vibrations, there correspond three values of the vibration frequency of the beam. To these frequencies correspond three modes of transverse vibrations of the beam. Each of these modes is characterized by the cross sections rotating about an axis parallel to the generator of the beam. The coordinates of the center of rotation in the plane of the cross section are

$$\left.\begin{aligned}c_{nx} &= \frac{1 - \frac{\gamma}{Eg}\frac{k_n^2}{\lambda_n^2}}{1 - \frac{\gamma}{Eg}\frac{k_n^2}{\lambda_n^2}\left(1 + \frac{F}{J_x\lambda_n^2}\right)}a_x,\\ c_{ny} &= \frac{1 - \frac{\gamma}{Eg}\frac{k_n^2}{\lambda_n^2}}{1 - \frac{\gamma}{Eg}\frac{k_n^2}{\lambda_n^2}\left(1 + \frac{F}{J_y\lambda_n^2}\right)}a_y.\end{aligned}\right\} \tag{2.8}$$

Formulas (2.8) are obtained from the first two expressions (1.2) by using the condition of fixity of the point with the coordinates (c_x, c_y), from the first two equations (2.6) and from (2.1) and (2.5).

Thus, to each sinusoidal vibration mode (to each λ_n) there correspond three centers of rotation and three vibration modes in the cross section.

In the particular case $a_x = a_y = 0$, i.e. when the shear center coincides with the section centroid, we obtain the following expressions for the roots of equation (2.7):

$$\left.\begin{aligned} k_{nx}^2 &= \frac{EJ_x\lambda_n^4}{J_x\lambda_n^2 + F}\frac{g}{\gamma}, \\ k_{ny}^2 &= \frac{EJ_y\lambda_n^4}{J_y\lambda_n^2 + F}\frac{g}{\gamma}, \\ k_{n\omega}^2 &= \frac{EJ_\omega\lambda_n^4 + GJ_d\lambda_n^2}{J_\omega\lambda_n^2 + r^2F}\frac{g}{\gamma}. \end{aligned}\right\} \quad (2.9)$$

The first two formulas determine the frequencies of the flexural vibrations of the beam in which its cross sections in the plane $z = \text{const}$ only undergo translational displacements, parallel to the principal axes of the section. These expressions differ from the ones usually given in various courses in the terms $J_x\lambda_n^2$, $J_y\lambda_n^2$ in the denominator, which are due to the rotational inertial forces of the beam element $dz = 1$ with respect to the principal axes of the section.

Neglecting these inertial forces, we obtain expressions for the frequencies of the transverse vibrations identical with those given, for example, in Timoshenko's book /179/:

$$k_{nx}^2 = \frac{EJ_x\lambda_n^4}{F}\frac{g}{\gamma},$$

$$k_{ny}^2 = \frac{EJ_y\lambda_n^4}{F}\frac{g}{\gamma}.$$

The last formula of (2.9) seems to have been published first by the author. This formula determines the frequencies of the torsional vibrations of a thin-walled beam of open section and rigid contour. In these vibrations the cross sections rotate about the shear center, which coincides here with the center of gravity section. If we neglect, as in the case of flexural vibrations, the inertial forces due to the longitudinal sectorial displacements of the material points of the beam, the third expression (2.9) assumes the simpler form

$$k_{n\omega}^2 = \frac{EJ_\omega\lambda_n^4 + GJ_d\lambda_n^2}{J_x + J_y}\frac{g}{\gamma}.$$

We note that this simplification of (2.9) results in a small error only for the thin-walled beams used in building and aircraft structures. In the case of ribbed arch-shells and hipped roofs with nondeformable contour (with transverse stiffening ribs), neglect of the longitudinal inertial forces may lead to large errors. Therefore, in determining the vibration frequencies of thin-walled structures with very complex cross sections, it is necessary to proceed from the more exact formulas (2.9) for sections with two axes of symmetry and from equation (2.7) for arbitrary asymmetrical open sections.

We shall now examine a beam having a single axis of symmetry in cross section. Denoting this axis by Oy, we have $a_x = 0$. Equation (2.7) is resolved, in this case, into two equations, one of which will be of the first degree and the other of the second degree in the square of the frequency

$$\left.\begin{aligned} &EJ_x\lambda_n^4 - (J_x\lambda_n^2 + F)\frac{\gamma}{g}k_n^2 = 0, \\ &\begin{vmatrix} EJ_y\lambda_n^4 - (J_y\lambda_n^2 + F)\frac{\gamma}{g}k_n^2 & -\frac{a_y\gamma F}{g}k_n^2 \\ -\frac{a_y\gamma F}{g}k_n^2 & \begin{array}{l} EJ_\omega\lambda_n^4 + GJ_d\lambda_n^2 - \\ -(J_\omega\lambda_n^2 + r^2F)\frac{\gamma}{g}k_n^2 \end{array} \end{vmatrix} = 0. \end{aligned}\right\} \qquad (2.10)$$

The first equation gives the frequency of the flexural vibrations of the beam in the plane of symmetry Oyz. The second equation gives, for a given number, n, of sinusoidal half-waves two real positive values of the frequency of flexural-torsional vibration of the beam.

§ 3. Vibration of beams under a longitudinal force

We shall examine the problem of vibrations of thin-walled elastic beams loaded by a (generally eccentric) longitudinal compressive or tensile force P at the ends. Let the coordinates of the point of application of the compressive force P in the plane of the cross section of the beam be e_x, e_y. We shall assume that this force is the resultant of elementary longitudinal forces, linearly distributed over the end sections of the beam. The normal stresses n at an arbitrary point of the beam in equilibrium are then determined by

$$n = -\frac{P}{F} - \frac{Pe_x}{J_y}x - \frac{Pe_y}{J_x}y.$$

We shall derive the equations of motion of an elementary transverse strip of the beam dz loaded at the sections $z =$const and $z + dz =$const by longitudinal forces $n\delta$ ($\delta = \delta(s)$ is the thickness of the beam wall). We obtain these equations by generalizing equations (1.8), adding to the elastic forces the inertial forces expressed by the left-hand sides of the equations, the reduced transverse loads q_{xP}, q_{yP} and the torsional moments m_{AP} arising from the given longitudinal forces due to the change in the state of deformation of the middle surface of the beam during vibration. We have the following expressions for these forces (see § 1, Chapter V) from the stability problem of a beam with small displacements ξ, η, and θ,

$$\begin{aligned} q_{xP} &= -[P\xi'' - P(e_y - a_y)\theta''], \\ q_{yP} &= -[P\eta'' + P(e_x - a_x)\theta''], \\ m_{AP} &= -[-P(e_y - a_y)\xi'' + P(e_x - a_x)\eta'' + \\ &\quad + P(r^2 + 2\beta_x e_x + 2\beta_y e_y)\theta''], \end{aligned}$$

where r^2, β_x and β_y are the geometrical characteristics of the beam section, defined by (V.1.6) and (V.1.7).

Adding the loads q_{xP}, q_{yP} and the moments m_{AP} to the left-hand sides of equations (1.8) and discarding the first equation (referring to longitudinal vibrations) we obtain

$$\left.\begin{aligned}
&EJ_y\frac{\partial^4\xi}{\partial z^4}-\frac{\gamma J_y}{g}\frac{\partial^4\xi}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\xi}{\partial t^2}+P\frac{\partial^2\xi}{\partial z^2}+\frac{\gamma Fa_y}{g}\frac{\partial^2\theta}{\partial t^2}-\\
&\qquad\qquad-P(e_y-a_y)\frac{\partial^2\theta}{\partial z^2}=0,\\
&EJ_x\frac{\partial^4\eta}{\partial z^4}-\frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2}+P\frac{\partial^2\eta}{\partial z^2}-\frac{\gamma Fa_x}{g}\frac{\partial^2\theta}{\partial t^2}+\\
&\qquad\qquad+P(e_x-a_x)\frac{\partial^2\theta}{\partial z^2}=0,\\
&\frac{\gamma Fa_y}{g}\frac{\partial^2\xi}{\partial t^2}-P(e_y-a_y)\frac{\partial^2\xi}{\partial z^2}-\frac{\gamma Fa_x}{g}\frac{\partial^2\eta}{\partial t^2}+P(e_x-a_x)\frac{\partial^2\eta}{\partial z^2}+\\
&\quad+EJ_\omega\frac{\partial^4\theta}{\partial z^4}-\frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2}+\frac{\gamma Fr^2}{g}\frac{\partial^2\theta}{\partial t^2}-GJ_d\frac{\partial^2\theta}{\partial z^2}+\\
&\qquad\qquad+P(r^2+2\beta_xe_x+2\beta_ye_y)\frac{\partial^2\theta}{\partial z^2}=0.
\end{aligned}\right\}\qquad(3.1)$$

Equations (3.1) are linear and homogeneous, with coefficients dependent on the elastic and geometrical characteristics of the beam and on the magnitude of the longitudinal force P and the coordinates e_x, e_y of its point of application in the plane of the cross section*.

These equations and boundary conditions determine the flexural-torsional shapes and vibration frequencies of a beam under a longitudinal force. It is plain from (3.1) that the free vibrations of loaded beams will be harmonic. Assuming, as before (see (2.1)),

$$\left.\begin{aligned}
\xi(z,t)&=\sum_{n=1,2,3,\ldots}\varphi_n(z)\sin k_nt,\\
\eta(z,t)&=\sum_{n=1,2,3,\ldots}\psi_n(z)\sin k_nt,\\
\theta(z,t)&=\sum_{n=1,2,3,\ldots}\chi_n(z)\sin k_nt
\end{aligned}\right\}\qquad(3.2)$$

and introducing (3.2) in (3.1) we obtain

$$\left.\begin{aligned}
&EJ_y\varphi_n^{\mathrm{IV}}+\left(\frac{\gamma J_y}{g}k_n^2+P\right)\varphi_n''-\frac{\gamma F}{g}k_n^2\varphi_n-\frac{\gamma Fa_y}{g}k_n^2\chi_n-\\
&\qquad\qquad-P(e_y-a_y)\chi_n''=0,\\
&EJ_x\psi_n^{\mathrm{IV}}+\left(\frac{\gamma J_x}{g}k_n^2+P\right)\psi_n''-\frac{\gamma F}{g}k_n^2\psi_n+\frac{\gamma Fa_x}{g}k_n^2\chi_n+\\
&\qquad\qquad+P(e_x-a_x)\chi_n''=0,\\
&EJ_\omega\chi_n^{\mathrm{IV}}+\left[P(r^2+2\beta_xe_x+2\beta_ye_y)+\frac{\gamma J_\omega}{g}k_n^2-GJ_d\right]\chi_n''-\\
&-\frac{\gamma Fr^2}{g}k_n^2\chi_n-\frac{\gamma Fa_y}{g}k_n^2\varphi_n-P(e_y-a_y)\varphi_n''+\frac{\gamma Fa_x}{g}k_n^2\psi_n+\\
&\qquad\qquad+P(e_x-a_x)\psi_n''=0.
\end{aligned}\right\}\qquad(3.3)$$

* If we consider the longitudinal force P in equations (3.1) as a given function of t: $P=P(t)$, we obtain equations of the general theory of spatial dynamic stability of thin-walled beams. For $P(t)=P_0\cos(pt)$ we shall have a system of three equations with periodic coefficients. Special problems in this theory are examined in the works of V.V. Bolotin and I.I. Gol'denblatt /19, 67/.

The system of ordinary differential equations (3.3), together with the homogeneous boundary conditions, determines the eigenvalues and eigenfunctions of the boundary-value problem.

In the particular case of the boundary conditions (2.4) corresponding to a beam with freely hinged supports, the eigenfunctions will be $\sin \frac{n\pi}{l} z$. Introducing the expressions for the required functions φ_n, ψ_n and χ_n (2.5) in (3.3), we obtain a system of algebraic equations for the coefficients A_n, B_n and C_n

$$\left.\begin{aligned}
&\left[EJ_y\lambda_n^4-\left(\frac{\gamma J_y}{g}k_n^2+P\right)\lambda_n^2-\frac{\gamma F}{g}k_n^2\right]A_n- \\
&\qquad -\left[\frac{\gamma F a_y}{g}k_n^2-P(e_y-a_y)\lambda_n^2\right]C_n=0, \\
&\left[EJ_x\lambda_n^4-\left(\frac{\gamma J_x}{g}k_n^2+P\right)\lambda_n^2-\frac{\gamma F}{g}k_n^2\right]B_n+ \\
&\qquad +\left[\frac{\gamma F a_x}{g}k_n^2-P(e_x-a_x)\lambda_n^2\right]C_n=0, \\
&-\left[\frac{\gamma F a_y}{g}k_n^2-P(e_y-a_y)\lambda_n^2\right]A_n+\left[\frac{\gamma F a_x}{g}k_n^2-P(e_x-a_x)\lambda_n^2\right]B_n+ \\
&+\left\{EJ_\omega\lambda_n^4-\left[P(r^2+2\beta_x e_x+2\beta_y e_y)-GJ_d+\frac{\gamma J_\omega}{g}k_n^2\right]\lambda_n^2-\right. \\
&\qquad \left.-\frac{\gamma F r^2}{g}k_n^2\right\}C_n=0.
\end{aligned}\right\} \quad (3.4)$$

Equations (3.4) are linear, homogeneous and with coefficients depending on the geometrical and physical characteristics of the beam, on the magnitude of the longitudinal force and the coordinates of its point of application, on the parameter λ_n characterizing the half-wavelength of $\sin \lambda_n z$ and on the vibration frequency k_n^2 corresponding to the n-th sinusoidal mode of the deformation along the beam. Equating the determinant of the system (3.4) to zero, we obtain the following equation for the vibration frequencies,

$$\begin{vmatrix}
P_{ny}\left(1-\frac{k_n^2}{k_{ny}^2}\right)-P & 0 & -\frac{\gamma F a_y}{g}\frac{k_n^2}{\lambda_n^2}+P(e_y-a_y) \\
0 & P_{nx}\left(1-\frac{k_n^2}{k_{nx}^2}\right)-P & \frac{\gamma F a_x}{g}\frac{k_n^2}{\lambda_n^2}-P(e_x-a_x) \\
\begin{matrix}-\frac{\gamma F a_y}{g}\frac{k_n^2}{\lambda_n^2}+\\ +P(e_y-a_y)\end{matrix} & \begin{matrix}\frac{\gamma F a_x}{g}\frac{k_n^2}{\lambda_n^2}-\\ -P(e_x-a_x)\end{matrix} & \begin{matrix}P_{n\omega}\left(1-\frac{k_n^2}{k_{n\omega}^2}\right)r^2-\\ -P(r^2+2\beta_x e_x+2\beta_y e_y)\end{matrix}
\end{vmatrix}=0. \quad (3.5)$$

We have adopted here for brevity the notations (V.3.3) and (V.3.4) as well as (2.9),

$$\begin{aligned}
&P_{nx}=EJ_x\lambda_n^2, && k_{nx}^2=\frac{EJ_x\lambda_n^4}{J_x\lambda_n^2+F}\frac{g}{\gamma}, \\
&P_{ny}=EJ_y\lambda_n^2, && k_{ny}^2=\frac{EJ_y\lambda_n^4}{J_y\lambda_n^2+F}\frac{g}{\gamma}, \\
&P_{n\omega}=\frac{1}{r^2}(EJ_\omega\lambda_n^2+GJ_d), && k_{n\omega}^2=\frac{EJ_\omega\lambda_n^4+GJ_d\lambda_n^2}{J_\omega\lambda_n^2+Fr^2}\frac{g}{\gamma},
\end{aligned}$$

where

$$\lambda_n = \frac{n\pi}{l} \quad (n = 1, 2, 3, \ldots).$$

Equation (3.5) determines for each sine wave $\sin \lambda_n z$ three frequencies corresponding to three flexural-torsional vibration modes of the beam. The coordinates of the centers of rotation of the beam cross sections are given by

$$c_{nx} = \frac{\left(1 - \frac{k_n^2}{k_{nx}^2} + \frac{\gamma F}{g} \frac{k_n^2}{P_{nx}\lambda_n^2}\right) a_x - \frac{P}{P_{nx}} e_x}{1 - \frac{k_n^2}{k_{nx}^2} - \frac{P}{P_{nx}}},$$

$$c_{ny} = \frac{\left(1 - \frac{k_n^2}{k_{ny}^2} + \frac{\gamma F}{g} \frac{k_n^2}{P_{ny}\lambda_n^2}\right) a_y - \frac{P}{P_{ny}} e_y}{1 - \frac{k_n^2}{k_{ny}^2} - \frac{P}{P_{ny}}}.$$

We shall examine some particular cases.

1) Assuming in equation (3.5) $e_x = e_y = 0$, we obtain an equation for the frequencies of a beam under an axial longitudinal compression P,

$$\begin{vmatrix} P_{ny}\left(1 - \frac{k_n^2}{k_{ny}^2}\right) - P & 0 & -\left(\frac{\gamma F}{g}\frac{k_n^2}{\lambda_n^2} + P\right) a_y \\ 0 & P_{nx}\left(1 - \frac{k_n^2}{k_{nx}^2}\right) - P & \left(\frac{\gamma F}{g}\frac{k_n^2}{\lambda_n^2} + P\right) a_x \\ -\left(\frac{\gamma F}{g}\frac{k_n^2}{\lambda_n^2} + P\right) a_y & \left(\frac{\gamma F}{g}\frac{k_n^2}{\lambda_n^2} + P\right) a_x & \left[P_{n\omega}\left(1 - \frac{k_n^2}{k_{n\omega}^2}\right) - P\right] r^2 \end{vmatrix} = 0. \quad (3.6)$$

2) Changing the sign of the force in equation (3.6), we obtain an equation for the frequencies for a beam under axial tension P

$$\begin{vmatrix} P_{ny}\left(1 - \frac{k_n^2}{k_{ny}^2}\right) + P & 0 & -\left(\frac{\gamma F}{g}\frac{k_n^2}{\lambda_n^2} - P\right) a_y \\ 0 & P_{nx}\left(1 - \frac{k_n^2}{k_{nx}^2}\right) + P & \left(\frac{\gamma F}{g}\frac{k_n^2}{\lambda_n^2} - P\right) a_x \\ -\left(\frac{\gamma F}{g}\frac{k_n^2}{\lambda_n^2} - P\right) a_y & \left(\frac{\gamma F}{g}\frac{k_n^2}{\lambda_n^2} - P\right) a_x & \left[P_{n\omega}\left(1 - \frac{k_n^2}{k_{n\omega}^2}\right) + P\right] r^2 \end{vmatrix} = 0.$$

3) Assuming in equation (3.5) $P = e_x = 0$ and the product $Pe_y = -M_x = $ $= \text{const}$ to be finite, we obtain an equation for the frequencies of a beam subjected to bending moments M_x acting in the end plane Oyz and causing normal stresses $n = \frac{M_x}{J_x} y$ in the beam (in the static state),

$$\begin{vmatrix} P_{ny}\left(1 - \frac{k_n^2}{k_{ny}^2}\right) & 0 & -\left(\frac{\gamma F a_y}{g}\frac{k_n^2}{\lambda_n^2} + M_x\right) \\ 0 & P_{nx}\left(1 - \frac{k_n^2}{k_{nx}^2}\right) & \frac{\gamma F a_x}{g}\frac{k_n^2}{\lambda_n^2} \\ -\left(\frac{\gamma F a_y}{g}\frac{k_n^2}{\lambda_n^2} + M_x\right) & \frac{\gamma F a_x}{g}\frac{k_n^2}{\lambda_n^2} & P_{n\omega}\left(1 - \frac{k_n^2}{k_{n\omega}^2}\right) r^2 + 2\beta_y M_x \end{vmatrix} = 0.$$

4) For $a_x = a_y = \beta_x = \beta_y = 0$, (3.5) determines all the vibration frequencies for a beam with two axes of symmetry in the cross section under a longitudinal force P applied at an arbitrary point (e_x, e_y) of the section

$$\begin{vmatrix} P_{ny}\left(1-\frac{k_n^2}{k_{ny}^2}\right)-P & 0 & Pe_y \\ 0 & P_{nx}\left(1-\frac{k_n^2}{k_{nx}^2}\right)-P & -Pe_x \\ Pe_y & -Pe_x & \left[P_{n\omega}\left(1-\frac{k_n^2}{k_{n\omega}^2}\right)-P\right]r^2 \end{vmatrix} = 0.$$

5) For $a_x = a_y = e_x = e_y = 0$, (3.5) gives for the vibration frequencies the following values

$$\left.\begin{aligned} \bar{k}_{nx}^2 &= \left(1-\frac{P}{P_{nx}}\right)k_{nx}^2, \\ \bar{k}_{ny}^2 &= \left(1-\frac{P}{P_{ny}}\right)k_{ny}^2, \\ \bar{k}_{n\omega}^2 &= \left(1-\frac{P}{P_{n\omega}}\right)k_{n\omega}^2. \end{aligned}\right\} \tag{3.7}$$

Expressions (3.7) refer to vibrations in beams axially loaded by the force P and with a shear center coinciding with the centroid of the section.

The first two expressions determine the frequencies of the flexural vibrations of beams in the principal central planes. The last expression determines the frequencies of the torsional vibrations with a center of rotation coinciding with the section centroid. It is evident from (3.7) that by increasing the compressive force P the vibration frequencies are reduced.

6) If the longitudinal force approaches a critical value, determined in general by the now familiar equation

$$\begin{vmatrix} P_{ny}-P & 0 & P(e_y-a_y) \\ 0 & P_{nx}-P & -P(e_x-a_x) \\ P(e_y-a_y) & -P(e_x-a_x) & P_{n\omega}-P(r^2+2\beta_x e_x+2\beta_y e_y) \end{vmatrix} = 0,$$

the vibration frequency tends to zero. The vibration period will be infinite.

In addition to the particular cases examined here, equation (3.5) allows also the solution of other problems of vibrations of beams under a longitudinal force. For example it is possible to determine from this equation the value of the force P for a given sine wave and vibration frequency k_n^2, viz. by varying the parameter P, the beam is adjusted to a definite mode ("tuned").

The solution of this problem may be of practical value in the acoustics of thin-walled elastic beams in flexural-torsional vibration.

The theory discussed here of spatial vibrations in beams under longitudinal forces and moments at the ends can be employed in an experimental verification by an acoustic method of the theory of spatial stability of beams. On the basis of this theory it is also easy to solve generally the problem of forced flexural-torsional vibrations of thin-walled beams. This problem is of great practical importance in designing airplane structures.

§ 4. Action of a time-dependent load

The differential equations of free vibrations (1.8) obtained in § 1 may be used for the design of thin-walled beams and also of thin-walled ribbed shells and hipped systems in roof structures of large spans, in the case of forces arbitrarily varying with time.

If we replace the zero on the right-hand side of (1.8) by the values of the corresponding components of the disturbing forces*, the equations of motion become

$$\left.\begin{aligned}
&EJ_y\frac{\partial^4\xi}{\partial z^4}-\frac{\gamma J_y}{g}\frac{\partial^4\xi}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\xi}{\partial t^2}+\frac{a_y\gamma F}{g}\frac{\partial^2\theta}{\partial t^2}=q_x(z,t),\\
&EJ_x\frac{\partial^4\eta}{\partial z^4}-\frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2}-\frac{a_x\gamma F}{g}\frac{\partial^2\theta}{\partial t^2}=q_y(z,t),\\
&\frac{a_y\gamma F}{g}\frac{\partial^2\xi}{\partial t^2}-\frac{a_x\gamma F}{g}\frac{\partial^2\eta}{\partial t^2}+EJ_\omega\frac{\partial^4\theta}{\partial z^4}-GJ_d\frac{\partial^2\theta}{\partial z^2}-\\
&\qquad\qquad-\frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2}+\frac{r^2\gamma F}{g}\frac{\partial^2\theta}{\partial t^2}=m_A(z,t).
\end{aligned}\right\}\tag{4.1}$$

In general, these equations form a simultaneous system of differential equations in the unknown functions $\xi=\xi(z,t)$, $\eta=\eta(z,t)$ and $\theta=\theta(z,t)$ and determine spatial flexural-torsional vibrations.

If the beam has a vertical axis of symmetry (the axis Oy) then the system (4.1) is resolved into two independent groups, viz.

$$\left.\begin{aligned}
EJ_x\frac{\partial^4\eta}{\partial z^4}-\frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2}&=q_y(z,t),\\
EJ_y\frac{\partial^4\xi}{\partial z^4}-\frac{\gamma J_y}{g}\frac{\partial^4\xi}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\xi}{\partial t^2}+\frac{a_y\gamma F}{g}\frac{\partial^2\theta}{\partial t^2}&=q_x(z,t),\\
\frac{a_y\gamma F}{g}\frac{\partial^2\xi}{\partial t^2}+EJ_\omega\frac{\partial^4\theta}{\partial z^4}-GJ_d\frac{\partial^2\theta}{\partial z^2}-\frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2}+\frac{r^2\gamma F}{g}\frac{\partial^2\theta}{\partial t^2}&=m_A(z,t).
\end{aligned}\right\}\tag{4.2}$$

The first of equations (4.2), containing only partial derivatives of the function $\eta=\eta(z,t)$, represents the forced vertical purely flexural vibrations.

The second and third of equations (4.2), together with the boundary conditions, determine the spatial flexural-torsional vibrations.

If the beam has two axes of symmetry, the system (4.1) assumes the form

$$\left.\begin{aligned}
EJ_x\frac{\partial^4\eta}{\partial z^4}-\frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2}&=q_y(z,t),\\
EJ_y\frac{\partial^4\xi}{\partial z^4}-\frac{\gamma J_y}{g}\frac{\partial^4\xi}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\xi}{\partial t^2}&=q_x(z,t),\\
EJ_\omega\frac{\partial^4\theta}{\partial z^4}-GJ_d\frac{\partial^2\theta}{\partial z^2}-\frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2}+\frac{r^2\gamma F}{g}\frac{\partial^2\theta}{\partial t^2}&=m_A(z,t).
\end{aligned}\right\}\tag{4.3}$$

The first two equations (4.3) refer to flexural vibrations in the vertical and horizontal planes. The last equation refers to torsional vibrations in which the cross section rotates about the shear center which coincides, in the present case, with the section centroid.

We shall now examine the forced vibrations of a ribbed shell having

* We refer, as before, to the first of equations (1.8), describing longitudinal vibrations.

a vertical axis of symmetry in the cross section and whose curved edges rest on diaphragms which are rigid with respect to motion in their plane and flexible with respect to motion away from it.

The problem thus leads mathematically to the integration of a system in linear inhomogeneous differential equations with given boundary and initial conditions.

The boundary conditions of the problem have the form:

$$\left.\begin{aligned} &\text{for } z=0 \quad \xi=\eta=\theta=0, \quad \frac{\partial^2\xi}{\partial z^2}=\frac{\partial^2\eta}{\partial z^2}=\frac{\partial^2\theta}{\partial z^2}=0, \\ &\text{for } z=l \quad \xi=\eta=\theta=0, \quad \frac{\partial^2\xi}{\partial z^2}=\frac{\partial^2\eta}{\partial z^2}=\frac{\partial^2\theta}{\partial z^2}=0. \end{aligned}\right\} \tag{4.4}$$

Under these conditions the solution of the system (4.2) can be given as

$$\left.\begin{aligned} \xi(z,t) &= \sum_{n=1}^{\infty} \xi_n(t)\sin\lambda_n z, \\ \eta(z,t) &= \sum_{n=1}^{\infty} \eta_n(t)\sin\lambda_n z, \\ \theta(z,t) &= \sum_{n=1}^{\infty} \theta_n(t)\sin\lambda_n z, \end{aligned}\right\} \tag{4.5}$$

where $\lambda_n=\frac{n\pi}{l}$, n is a positive integer, l is the span of the shell. It is easy to see that each term of the expansion (4.5) satisfies the boundary conditions (4.4).

We shall assume that the external load can be expanded in a Fourier series,

$$\left.\begin{aligned} q_x(z,t) &= f(t)\sum_{n=1}^{\infty} q_{xn}\sin\lambda_n z, \\ q_y(z,t) &= f(t)\sum_{n=1}^{\infty} q_{yn}\sin\lambda_n z, \\ m_A(z,t) &= f(t)\sum_{n=1}^{\infty} m_{An}\sin\lambda_n z, \end{aligned}\right\} \tag{4.6}$$

where $f(t)$ is a function common to all the components, which determines the temporal variation of the external forces; q_{xn}, q_{yn}, m_{An} are the coefficients of the series, determined by well-known expressions for the coefficients of Fourier series.

We now insert (4.5) and (4.6) in the basic differential equations (4.2). Cancelling the common factor $\sin\lambda_n z$, we obtain for the n-th term the ordinary inhomogeneous differential equations

$$\left.\begin{aligned} EJ_x\lambda_n^4\eta_n(t)+\frac{\gamma}{g}(J_x\lambda_n^2+F)\,\eta_n''(t) &= q_{yn}f(t), \\ EJ_y\lambda_n^4\xi_n(t)+\frac{\gamma}{g}(J_y\lambda_n^2+F)\,\xi_n''(t)+\frac{a_y\gamma F}{g}\theta_n''(t) &= q_{xn}f(t), \\ \frac{a_y\gamma F}{g}\xi_n''(t)+(EJ_\omega\lambda_n^4+GJ_d\lambda_n^2)\,\theta_n(t)+\frac{\gamma}{g}(J_\omega\lambda_n^2+r^2F)\,\theta_n''(t) &= m_{An}f(t). \end{aligned}\right\} \tag{4.7}$$

The general solution of the homogeneous form of (4.7)* can be

* The homogeneous form of equations (4.7) is obtained by putting zero on the right-hand sides of the inhomogeneous form of (4.7).

obtained in the form of simple harmonic vibrations with the frequency k_n and phase μ

$$\left.\begin{aligned} \eta_n(t) &= A\sin(k_n t+\mu),\\ \xi_n(t) &= B\sin(k_n t+\mu),\\ \theta_n(t) &= C\sin(k_n t+\mu). \end{aligned}\right\} \tag{4.8}$$

Inserting (4.8) in the homogeneous form of equations (4.7) and cancelling the common factor $\sin(k_n t+\mu)$, we obtain

$$\left.\begin{aligned} \left[EJ_x\lambda_n^4-\frac{\gamma}{g}(J_x\lambda_n^2+F)k_n^2\right]A=0,\\ \left[EJ_y\lambda_n^4-\frac{\gamma}{g}(J_y\lambda_n^2+F)k_n^2\right]B-\frac{a_y\gamma F}{g}k_n^2C=0,\\ -\frac{a_y\gamma F}{g}k_n^2B+\left[EJ_\omega\lambda_n^4+GJ_d\lambda_n^2-\frac{\gamma}{g}(J_\omega\lambda_n^2+r^2F)k_n^2\right]C=0. \end{aligned}\right\} \tag{4.9}$$

The functions (4.8) satisfy the homogeneous differential equations (4.7) if the coefficients A, B and C satisfy the linear homogeneous algebraic equations (4.9) obtained above.

The natural vibration frequency of the shell is obtained by equating to zero the determinant of the coefficients of the constants A, B and C. We obtain for these frequencies two independent equations,

$$\left.\begin{aligned} EJ_x\lambda_n^4-\frac{\gamma}{g}(J_x\lambda_n^2+F)k_n^2=0,\\ \begin{vmatrix} EJ_y\lambda_n^4-\frac{\gamma}{g}(J_y\lambda_n^2+F)k_n^2 & -\frac{a_y\gamma F}{g}k_n^2\\ -\frac{a_y\gamma F}{g}k_n^2 & EJ_\omega\lambda_n^4+GJ_d\lambda_n^2-\frac{\gamma}{g}(J_\omega\lambda_n^2+r^2F)k_n^2 \end{vmatrix}=0. \end{aligned}\right\} \tag{4.10}$$

The first of these is linear and the other a quadratic equation in the squared frequency k_n. Thus, to each value of $\lambda_n=\frac{n\pi}{l}$ $(n=1, 2, 3, \ldots)$, characterizing according to (4.5) the form of the vibrations, there correspond three free vibration frequencies of the shell k_{n0}, k_{n1} and k_{n2}. The first frequency corresponds to vertical flexural vibration and the second and third to combined flexural-torsional vibrations.

Having found the frequencies from (4.10), the general integral of the homogeneous system (4.7) can be written

$$\left.\begin{aligned} \eta_n(t) &= A\sin(k_{n0}t+\mu_0),\\ \xi_n(t) &= B_1\sin(k_{n1}t+\mu_1)+B_2\sin(k_{n2}t+\mu_2),\\ \theta_n(t) &= \rho_1B_1\sin(k_{n1}t+\mu_1)+\rho_2B_2\sin(k_{n2}t+\mu_2). \end{aligned}\right\} \tag{4.11}$$

Here ρ_1 and ρ_2 are coefficients of the relation between the arbitrary constants B and C of (4.8).

Using the second of equations (4.9), we obtain

$$\rho_1=\frac{EJ_y\lambda_n^4\frac{g}{\gamma}-(J_y\lambda_n^2+F)k_{n1}^2}{a_yFk_{n1}^2},\quad \rho_2=\frac{EJ_y\lambda_n^4\frac{g}{\gamma}-(J_y\lambda_n^2+F)k_{n2}^2}{a_yFk_{n2}^2}.$$

The general integral (4.11) of the homogeneous system (4.7) can also

be given in another form,

$$\left.\begin{aligned}\eta_n(t)&=A\sin k_{n0}t+E\cos k_{n0}t,\\ \xi_n(t)&=B_1\sin k_{n1}t+D_1\cos k_{n1}t+B_2\sin k_{n2}t+D_2\cos k_{n2}t,\\ \theta_n(t)&=\rho_1(B_1\sin k_{n1}t+D_1\cos k_{n1}t)+\rho_2(B_2\sin k_{n2}t+D_2\cos k_{n2}t).\end{aligned}\right\} \tag{4.12}$$

We now proceed to find the particular integral of the inhomogeneous equations (4.7). It can be found by the method of variation of constants or by the method of initial conditions*. We shall follow the latter.

We express the six arbitrary constants of the integral (4.12) of the homogeneous system (4.7) in terms of the initial displacements $\eta_n(0)$, $\xi_n(0)$ and $\theta_n(0)$ and the initial velocities $\eta_n'(0)$, $\xi_n'(0)$ and $\theta_n'(0)$. From equations (4.12) for $\eta_n(t)$, $\xi_n(t)$ and $\theta_n(t)$ and their derivatives we obtain six equations for the determination of the six constants, viz.

$$\left.\begin{aligned}E&=\eta_n(0), & k_{n0}A&=\eta_n'(0),\\ D_1+D_2&=\xi_n(0), & k_{n1}B_1+k_{n2}B_2&=\xi_n'(0),\\ \rho_1D_1+\rho_2D_2&=\theta_n(0), & \rho_1k_{n1}B_1+\rho_2k_{n2}B_2&=\theta_n'(0).\end{aligned}\right\} \tag{4.13}$$

The solution of (4.13) yields

$$\left.\begin{aligned}A&=\frac{\eta_n'(0)}{k_{n0}}, & E&=\eta_n(0),\\ B_1&=\frac{\xi_n'(0)\rho_2-\theta_n'(0)}{k_{n1}(\rho_2-\rho_1)}, & D_1&=\frac{\xi_n(0)\rho_2-\theta_n(0)}{\rho_2-\rho_1},\\ B_2&=\frac{\theta_n'(0)-\xi_n'(0)\rho_1}{k_{n2}(\rho_2-\rho_1)}, & D_2&=\frac{\theta_n(0)-\xi_n(0)\rho_1}{\rho_2-\rho_1}.\end{aligned}\right\} \tag{4.14}$$

We may replace the arbitrary constants in the general integral (4.12) by other quantities having a definite physical meaning. Introducing expressions (4.14) in (4.12), we obtain

$$\left.\begin{aligned}\eta_n(t)&=\eta_n(0)\cos k_{n0}t+\frac{\eta_n'(0)}{k_{n0}}\sin k_{n0}t,\\ \xi_n(t)&=\frac{1}{\rho_2-\rho_1}\Big[(\rho_2\cos k_{n1}t-\rho_1\cos k_{n2}t)\,\xi_n(0)+\\ &+(\cos k_{n2}t-\cos k_{n1}t)\,\theta_n(0)+\left(\frac{\rho_2}{k_{n1}}\sin k_{n1}t-\frac{\rho_1}{k_{n2}}\sin k_{n2}t\right)\xi_n'(0)+\\ &+\left(\frac{1}{k_{n2}}\sin k_{n2}t-\frac{1}{k_{n1}}\sin k_{n1}t\right)\theta_n'(0)\Big],\\ \theta_n(t)&=\frac{1}{\rho_2-\rho_1}\Big[\rho_1\rho_2(\cos k_{n1}t-\cos k_{n2}t)\,\xi_n(0)+\\ &+(\rho_2\cos k_{n2}t-\rho_1\cos k_{n1}t)\,\theta_n(0)+\\ &+\rho_1\rho_2\left(\frac{1}{k_{n1}}\sin k_{n1}t-\frac{1}{k_{n2}}\sin k_{n2}t\right)\xi_n'(0)+\\ &+\left(\frac{\rho_2}{k_{n2}}\sin k_{n2}t-\frac{\rho_1}{k_{n1}}\sin k_{n1}t\right)\theta_n'(0)\Big]\end{aligned}\right\} \tag{4.15}$$

If the initial conditions are known, it is possible to determine with the help of (4.15) the values of the functions $\xi_n(t)$, $\eta_n(t)$, $\theta_n(t)$ entering (4.5). Therefore, for the case of free vibrations we can determine from (4.5) the displacements and velocities of an arbitrary, infinitely small shell element.

* In its application to vibration problems we shall call the method of initial parameters the method of initial conditions.

Suppose the displacements and velocities of an arbitrary shell element are initially zero and at the instant t_1 the velocities $\eta_n'(t_1)$, $\xi_n'(t_1)$ and $\theta_n'(t_1)$ are imparted to the element. The motion due to the resulting velocities is given by the expressions

$$\left.\begin{aligned}
\eta_n(t) &= \frac{\eta_n'(t_1)}{k_{n0}} \sin k_{n0}(t-t_1),\\
\xi_n(t) &= \frac{1}{\rho_2-\rho_1}\left\{\left[\frac{\rho_2}{k_{n1}}\sin k_{n1}(t-t_1) - \frac{\rho_1}{k_{n2}}\sin k_{n2}(t-t_1)\right]\xi_n'(t_1)+\right.\\
&\quad\left.+\left[\frac{1}{k_{n2}}\sin k_{n2}(t-t_1) - \frac{1}{k_{n1}}\sin k_{n1}(t-t_1)\right]\theta_n'(t_1)\right\}\\
\theta_n(t) &= \frac{\rho_1\rho_2}{\rho_2-\rho_1}\left\{\left[\frac{1}{k_{n1}}\sin k_{n1}(t-t_1) - \frac{1}{k_{n2}}\sin k_{n2}(t-t_1)\right]\xi_n'(t_1)+\right.\\
&\quad\left.+\left[\frac{1}{\rho_1 k_{n2}}\sin k_{n2}(t-t_1) - \frac{1}{\rho_2 k_{n1}}\sin k_{n1}(t-t_1)\right]\theta_n'(t_1)\right\}.
\end{aligned}\right\} \tag{4.16}$$

Imparting these velocities amounts to the action of an instantaneous force. If we take this instantaneous force as known we can express the velocities $\eta_n'(t_1)$, $\xi_n'(t_1)$ and $\theta_n'(t_1)$ in terms of it. If the force, whose components are $q_{xn}f(t_1)$, $q_{yn}f(t_1)$ and $m_{An}f(t_1)$, is acting during a very short time Δt_1, it imparts to the element the following accelerations: $\eta_n''(t_1)$, $\xi_n''(t)$ and $\theta_n''(t)$. These accelerations are easily determined from (4.7) if, according to the previous considerations, we assume there $t=t_1$, $\eta_n(t_1)=\xi_n(t_1)=$ $=\theta_n(t_1)=0$.

The solution of the equations thus obtained give the following values for the accelerations,

$$\eta_n''(t_1) = \frac{g}{\gamma}\frac{q_{yn}}{J_x\lambda_n^2+F}f(t_1),$$

$$\xi_n''(t_1) = \frac{g}{\gamma}\frac{q_{xn}(J_\omega\lambda_n^2+r^2F)-m_{An}a_yF}{(J_y\lambda_n^2+F)(J_\omega\lambda_n^2+r^2F)-a_y^2F^2}f(t_1),$$

$$\theta_n''(t_1) = \frac{g}{\gamma}\frac{m_{An}(J_y\lambda_n^2+F)-q_{xn}a_yF}{(J_y\lambda_n^2+F)(J_\omega\lambda_n^2+r^2F)-a_y^2F^2}f(t_1).$$

Consequently, the velocities will be

$$\left.\begin{aligned}
\eta_n'(t_1) &= \frac{g}{\gamma}\frac{q_{yn}}{J_x\lambda_n^2+F}f(t_1)\,\Delta t_1,\\
\xi_n'(t_1) &= \frac{g}{\gamma}\frac{q_{xn}(J_\omega\lambda_n^2+r^2F)-m_{An}a_yF}{(J_y\lambda_n^2+F)(J_\omega\lambda_n^2+r^2F)-a_y^2F^2}f(t_1)\,\Delta t_1,\\
\theta_n'(t_1) &= \frac{g}{\gamma}\frac{m_{An}(J_y\lambda_n^2+F)-q_{xn}a_yF}{(J_y\lambda_n^2+F)(J_\omega\lambda_n^2+r^2F)-a_y^2F^2}f(t_1)\,\Delta t_1.
\end{aligned}\right\} \tag{4.17}$$

Inserting the values of the velocities (4.17) in (4.16), we obtain the functions $\xi_n(t)$, $\eta_n(t)$ and $\theta_n(t)$ entering the expansion (4.5) for the arbitrary time t, at which the action of the force ceased. If several pulses will be conferred at different times, the expressions should be summed. If the pulses are continuous in time, these sums become integrals. The obtained expressions will be the particular integrals of the inhomogeneous equations (4.7).

§ 5. Spatial flexural-torsional vibrations of suspension bridges

We obtain the differential equations of spatial vibrations of suspension bridges having two axes of symmetry in the cross section from equations (VIII.7.1) for the spatial stability of these bridges if we add to the equations the inertial and the aerodynamic forces. Adding the inertial forces in the same way as § 1, we obtain

$$\left.\begin{aligned} EJ_x\frac{\partial^4\eta}{\partial z^4} - 2H\frac{\partial^2\eta}{\partial z^2} - \frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2} + \frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2} + \frac{\partial^2}{\partial z^2}(M_y\theta) = \bar{q}_y,\\ M_y\frac{\partial^2\eta}{\partial z^2} + EJ_\omega\frac{\partial^4\theta}{\partial z^4} - GJ_d\frac{\partial^2\theta}{\partial z^2} - H\frac{b^2}{2}\frac{\partial^2\theta}{\partial z^2} - \frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2} + \frac{\gamma F r^2}{g}\frac{\partial^2\theta}{\partial t^2} = \bar{m}.\end{aligned}\right\} \tag{5.1}$$

Here:

$\bar{q}_y$ is the lift per unit length of the bridge,

$$\bar{q}_y = C_L(\theta)\frac{r}{2}Av^2, \tag{5.2}$$

where C_L is the coefficient of the lift as a function of the torsional angle θ (Figure 200); r is the air density; A is the horizontal projection of a unit length of the bridge; v is the wind velocity;

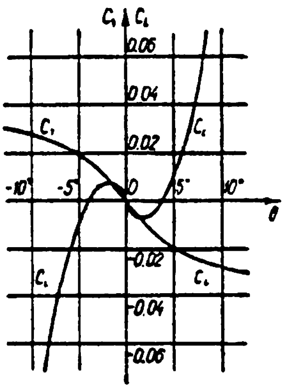


Figure 200

$\bar{m}$ is the torsional moment per unit length of the bridge caused by the wind and calculated from

$$m = C_T(\theta)\,ub^2\frac{v^2}{2g}, \tag{5.3}$$

where b is the width of the bridge between the suspensions; g is the specific gravity of the air; u is the gravity acceleration; C_T is the torsion coefficient, being a function of the torsional angle θ; for small angles θ, C_T can be expressed by

$$C_T = -k\theta \tag{5.4}$$

(k is a constant).

In the following we shall neglect the lift, which has slight influence on the final results.

Introducing in the right-hand sides of (5.1) the torsional moment (5.3) and using the expression (5.4), we obtain the differential equations of dynamic stability in the form

$$\left.\begin{aligned} EJ_x\frac{\partial^4\eta}{\partial z^4} - 2H\frac{\partial^2\eta}{\partial z^2} - \frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2} + \frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2} + \frac{\partial^2}{\partial z^2}(M_y\theta) = 0,\\ M_y\frac{\partial^2\eta}{\partial z^2} + EJ_\omega\frac{\partial^4\theta}{\partial z^4} - GJ_d\frac{\partial^2\theta}{\partial z^2} - H\frac{b^2}{2}\frac{\partial^2\theta}{\partial z^2} - \\ - \frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2} + \frac{\gamma F r^2}{g}\frac{\partial^2\theta}{\partial t^2} + kub^2\frac{v^2}{2g}\theta = 0.\end{aligned}\right\} \tag{5.5}$$

Since only second order derivatives with respect to time enter equations (5.5), the vibrations will be simple harmonic

$$\left.\begin{aligned} \eta(z,t) = \eta(z)\sin\omega t,\\ \theta(z,t) = \theta(z)\sin\omega t,\end{aligned}\right\} \tag{5.6}$$

where ω is the angular frequency.

Inserting (5.6) in equations (5.5) and cancelling the common factor $\sin \omega t$, we obtain a system of ordinary differential equations,

$$\left.\begin{aligned} &EJ_x\eta^{IV} - 2H\eta'' + \frac{\gamma J_x}{g}\omega^2\eta'' - \frac{\gamma F}{g}\omega^2\eta + (M_y\theta)'' = 0, \\ &M_y\eta'' + EJ_\omega\theta^{IV} - \left(GJ_d + H\frac{b^2}{2}\right)\theta'' + \frac{\gamma J_p}{g}\omega^2\theta'' - \\ &\qquad - \frac{\gamma Fr^2}{g}\omega^2\theta + kab^2\frac{v^2}{2g}\theta = 0, \end{aligned}\right\} \tag{5.7}$$

where differentiation is with respect to the variable z.

In the following we shall assume that the wind pressure is uniformly distributed along the bridge. In this case

$$M_y = \frac{q}{2}z(l - z).$$

We shall also assume that the vibrations of the bridge have two half-waves along the main span. This can be justified in the same way as we did in § 7 of Chapter VIII, where we examined the statical stability of suspension bridges.

We assume

$$\eta(z) = B\sin\frac{2\pi}{l}z, \quad \theta(z) = C\sin\frac{2\pi}{l}z.$$

The application of the principle of virtual displacements to the integration of the differential equations (5.7) leads to a system of two homogeneous algebraic equations

$$\left.\begin{aligned} &\left[EJ_x\left(\frac{2\pi}{l}\right)^4 + 2H\left(\frac{2\pi}{l}\right)^2 - \frac{\gamma J_x}{g}\omega^2\left(\frac{2\pi}{l}\right)^2 - \frac{\gamma F}{g}\omega^2\right]\frac{l}{2}B - \\ &\qquad - \frac{a_1 v^2 l}{8}\left(1 + \frac{2^2\pi^2}{3}\right)C = 0, \\ &-\frac{a_1 v^2 l}{8}\left(1 + \frac{2^2\pi^2}{3}\right)B + \left[EJ_\omega\left(\frac{2\pi}{l}\right)^4 + \left(GJ_d + H\frac{b^2}{2}\right)\left(\frac{2\pi}{l}\right)^2 - \right. \\ &\qquad \left. - \frac{\gamma J_p}{g}\omega^2\left(\frac{2\pi}{l}\right)^2 - \frac{\gamma Fr^2}{g}\omega^2 + kub^2\frac{v^2}{2g}\right]\frac{l}{2}C = 0. \end{aligned}\right\} \tag{5.8}$$

In deriving equations (5.8) we took into consideration that the relation between the wind velocity v and the pressure q can be written in the approximate form

$$q = a_1 v^2.$$

Equating to zero the determinant of the coefficients of B and C, we obtain a biquadratic equation for the velocity v

$$av^4 - bv^2 - c = 0. \tag{5.9}$$

The coefficients of this equation depend on various physical, geometrical and aerodynamic quantities, including ω^2.

Equation (5.9) together with the extremum condition for the velocity

$$\frac{\partial(v^2)}{\partial(\omega^2)} = 0,$$

allows the determination of the critical wind velocity which will cause collapse of the suspension bridge.

We shall apply the above procedure to the Tacoma suspension bridge mentioned before. The form of the flexural-torsional deformations of the roadway part of the bridge is shown in Figures 201 and 202. The description of the bridge structure and of the disaster itself are given in the work of F. D. Dmitriev /87/.

We have the following basic data:

$$
\begin{aligned}
l &= 853\ \mathrm{m}, & H &= 5.65\cdot 10^{6}\ \mathrm{kg},\\
b &= 11.9\ \mathrm{m}, & E &= 2.1\cdot 10^{10}\ \mathrm{kg/m^2},\\
F &= 0.654\ \mathrm{m^2}, & G &= 0.8\cdot 10^{10}\ \mathrm{kg/m^2},\\
Fr^2 &= J_p = J_x + J_y = 10.7\ \mathrm{m^4}, & \gamma &= 7.85\cdot 10^{3}\ \mathrm{kg/m^3},\\
J_x &= 0.122\ \mathrm{m^4}, & \varepsilon &= 1.29\ \mathrm{kg/m^3},\\
J_y &= 10.6\ \mathrm{m^4}, & g &= 9.81\ \mathrm{m/sec^2},\\
J_d &\approx 0, & a_1 &= 0.144\ \mathrm{kg\,sec^2/m^4},\\
J_\omega &= 4.31\ \mathrm{m^6}, & k &= 0.267.
\end{aligned}
$$

We took k as the [trigonometrical] tangent of the inclination angle of the tangent to the curve C_T (Figure 200) at the origin of coordinates, since we have to do with very small vibrations. If these vibrations are not stable the amplitude increases with time which ultimately leads to structural failure.

The calculations performed gave the following value for the critical wind velocity:

$$v_{cr} = 60.9\ \mathrm{km/hr} = 16.9\ \mathrm{m/sec}.$$

To this velocity there corresponds a pressure

$$q = 41.4\ \mathrm{kg/m}.$$

The actual wind velocity on the day of the disaster was $v = 67\ \mathrm{km/hr} = 18.6\ \mathrm{m/sec}$. As we see, the error in the calculation of the critical velocity as compared with the real one does not exceed 10%. This error is mainly due to somewhat different boundary conditions at the support of the roadway part of the bridge which had elastic jamming and not hinges at the ends. This increased rigidity of the design system led to a calculated critical velocity smaller than the velocity at which the bridge was wrecked.

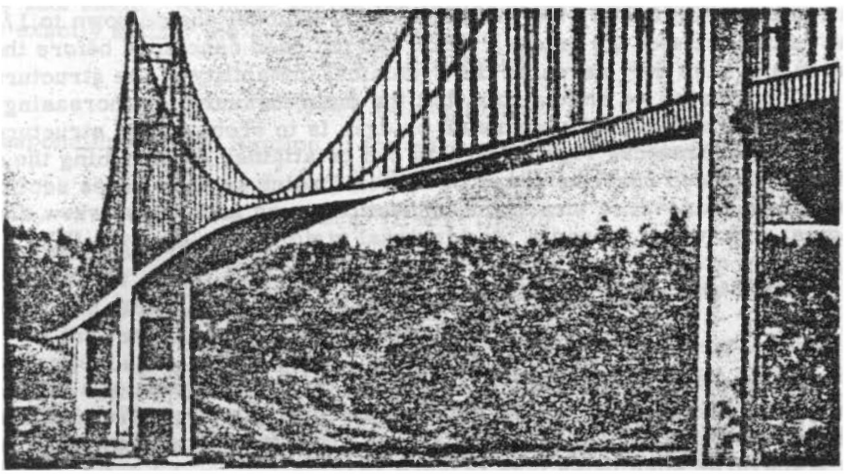


Figure 201

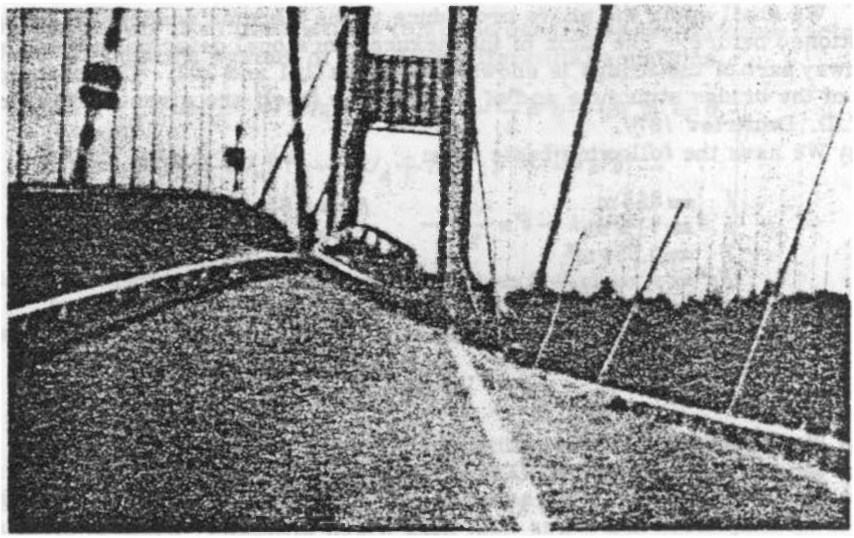


Figure 202

If we compare the statical horizontal load q_{st} which the bridge could sustain without becoming unstable with the dynamic wind pressure q_{dyn} corresponding to the velocity at the disaster, we see that

$$q_{st} \gg q_{dyn},$$

i. e. the bridge could sustain a very small dynamic horizontal load under vibrations while the critical static horizontal load was rather large.

It is necessary to emphasize especially the fact that the disaster happened as a result of flexural-torsional vibration of the sort treated in our theory. Only vertical purely flexural vibrations did not cause disaster even at the 36 cycles per minute attained on the day of disaster; the frequency then fell to 14 cycles per minute immediately before the disaster, and then the vibration became flexural-torsional which precipitated collapse. This is also confirmed by the experiments on a model scaled down to 1/100 of the natural size of the bridge, which had not been concluded before the disaster and have duly revealed the dynamical instability of the structure.

In conclusion we should note that the main measure for increasing the dynamical stability of suspension bridges is to erect a span structure of high flexural-torsional rigidity. This can be attained by designing the span section in a closed box-like shape or by reinforcing an open cross section by additional transverse bimoment connections such as braces, skew diaphragms, etc, which increase the torsional rigidity (see Chapter III).

§ 6. Free vibrations and aerodynamic stability of airfoil-type structures

1. We shall first examine the problem of free vibration of a rigid, rectangular, box-like section having two axes of symmetry. We shall assume that the box-like shell is rigidly fixed along one transverse edge, the other transverse edge being free (Figure 197).

The differential equations of free transverse vibrations of an open-section beam with two axes of symmetry is given by equations (1.9),

$$\left.\begin{aligned} EJ_y \frac{\partial^4 \xi}{\partial z^4} - \frac{\gamma J_y}{g} \frac{\partial^4 \xi}{\partial z^2 \partial t^2} + \frac{\gamma F}{g} \frac{\partial^2 \xi}{\partial t^2} = 0, \\ EJ_x \frac{\partial^4 \eta}{\partial z^4} - \frac{\gamma J_x}{g} \frac{\partial^4 \eta}{\partial z^2 \partial t^2} + \frac{\gamma F}{g} \frac{\partial^2 \eta}{\partial t^2} = 0, \\ EJ_\omega \frac{\partial^4 \theta}{\partial z^4} - GJ_d \frac{\partial^2 \theta}{\partial z^2} - \frac{\gamma J_\omega}{g} \frac{\partial^4 \theta}{\partial z^2 \partial t^2} + \frac{\gamma r^2 F}{g} \frac{\partial^2 \theta}{\partial t^2} = 0. \end{aligned}\right\} \tag{6.1}$$

These equations are also valid for a thin-walled closed shell of rigid profile. The difference will be only in the method of calculating the geometrical characteristics J_d and J_ω (see Chapter VIII, § 6, sub-sec. 3). The geometrical characteristics of a shell of wall thickness δ and rectangular cross section with sides d_1 and d_2 are calculated by (V.1.18),

$$\left.\begin{aligned} &F = 2\delta(d_1 + d_2), \quad J_x = \frac{1}{6}\delta d_1^2(d_1 + 3d_2), \quad J_y = \frac{1}{6}\delta d_2^2(d_2 + 3d_1), \\ &J_d = \frac{4}{F}\delta^2 d_1^2 d_2^2, \quad J_\omega = \frac{1}{48} F d_1^2 d_2^2, \quad Fr^2 = J_x + J_y. \end{aligned}\right\} \tag{6.2}$$

The system (6.1) consists of three independent equations. It is possible therefore to determine independently the frequencies of free transverse vibrations of the shell corresponding to pure flexural vibrations in the planes Oxz and Oyz and to pure torsional vibrations. To determine these frequencies we use the approximate variational method of Bubnov-Galerkin. We have

$$\left.\begin{aligned} \xi(z) &= A \sin k_x t \cdot \varphi(z), \\ \eta(z) &= B \sin k_y t \cdot \psi(z), \\ \theta(z) &= C \sin k_\theta t \cdot \chi(z). \end{aligned}\right\} \tag{6.3}$$

In (6.3) k_x is the required angular vibration frequency in the plane Oxz, k_y is the vibration frequency in the plane Oyz and k_θ is the frequency of the torsional vibrations.

The functions $\varphi(z)$, $\psi(z)$ and $\chi(z)$ represent the transverse vibrations of a box-like shell. We must assign the initial values of these functions. They exactly satisfy the boundary conditions:

$$\varphi(0) = \psi(0) = \chi(0) = 0,$$

$$\varphi'(0) = \psi'(0) = \chi'(0) = 0,$$

corresponding to rigid fixation of the edge $z = 0$ and the conditions

$$\varphi''(l) = \psi''(l) = \chi''(l) = 0,$$

$$\varphi'''(l) = \psi'''(l) = \chi'''(l) = 0,$$

which correspond to the free edge $z = l$. Expressions (6.3) are generally not integrals of equations (6.1), so that when (6.3) are inserted in (6.1), the left-hand sides of (6.1) do not vanish.

We thus stipulate that equations (6.1) be satisfied in integral form according to Lagrange's principle of virtual displacements; as virtual displacements serve the functions φ, ψ and χ, like in § 13, Chapter V. To achieve this, we substitute (6.3) in (6.1), multiply by the virtual displacements φ, ψ and χ respectively, and integrate over the whole length of the shell. After integration by parts and cancellation of A, B, C we are

led to the following equations in k_x, k_y and k_θ:

$$EJ_y\int_0^l(\varphi'')^2dz-\frac{\gamma J_y}{g}k_x^2\int_0^l(\varphi')^2dz-\frac{\gamma F}{g}k_x^2\int_0^l\varphi^2dz=0,$$

$$EJ_x\int_0^l(\psi'')^2dz-\frac{\gamma J_x}{g}k_y^2\int_0^l(\psi')^2dz-\frac{\gamma F}{g}k_y^2\int_0^l\psi^2dz=0,$$

$$EJ_\omega\int_0^l(\chi'')^2dz+\left(GJ_d-\frac{\gamma J_\omega}{g}k_\theta^2\right)\int_0^l(\chi')^2dz-\frac{\gamma Fr^2}{g}k_\theta^2\int_0^l\chi^2dz=0.$$

Whence we obtain expressions for the frequencies,

$$\left.\begin{aligned}
k_x^2&=\frac{EJ_y\int_0^l(\varphi'')^2dz}{J_y\int_0^l(\varphi')^2dz+F\int_0^l\varphi^2dz}\,\frac{g}{\gamma},\\
k_y^2&=\frac{EJ_x\int_0^l(\psi'')^2dz}{J_x\int_0^l(\psi')^2dz+F\int_0^l\psi^2dz}\,\frac{g}{\gamma},\\
k_\theta^2&=\frac{EJ_\omega\int_0^l(\chi'')^2dz+GJ_d\int_0^l(\chi')^2dz}{J_\omega\int_0^l(\chi')^2dz+Fr^2\int_0^l\chi^2dz}\,\frac{g}{\gamma}.
\end{aligned}\right\}\tag{6.4}$$

Taking as the functions $\varphi(z)$, $\psi(z)$ and $\chi(z)$ the flexure functions of the cantilever (VIII.8.5)

$$\varphi(z)=\psi(z)=\chi(z)=z^4-4lz^3+6l^2z^2$$

and calculating the definite integrals in (6.4), we obtain the expressions for the frequencies

$$k_x^2=\frac{1134EJ_y}{(405J_y+91l^2F)l^4}\,\frac{g}{\gamma},$$

$$k_y^2=\frac{1134EJ_x}{(405J_x+91l^2F)l^4}\,\frac{g}{\gamma},$$

$$k_\theta^2=\frac{81\,(14EJ_\omega+5l^2GJ_d)}{(405J_\omega+91l^2Fr^2)\,l^4}\,\frac{g}{\gamma}.$$

If we use (6.2), the expressions for the natural frequencies become

$$k_x^2=E\frac{g}{\gamma}\,\frac{378d_2^2\,(d_2+3d_1)}{l^4\,[135d_2^2\,(d_2+3d_1)+364l^2\,(d_1+d_2)]},$$

$$k_y^2=E\frac{g}{\gamma}\,\frac{378d_1^2\,(d_1+3d_2)}{l^4\,[135d_1^2\,(d_1+3d_2)+364l^2\,(d_1+d_2)]},$$

$$k_\theta^2=E\frac{g}{\gamma}\,\frac{162d_1^2d_2^2\left[7\,(d_1+d_2)^2+120\dfrac{G}{E}\,l^2\right]}{l^4\,(d_1+d_2)^2\,[405d_1^2\,d_2^2+364l^2\,(d_1+d_2)^2]}.$$

These formulas show that the torsional rigidity of a beam-shell, which is proportional to the shear modulus G, influences the vibration frequency only in restrained torsion. The flexural vibrations do not depend

on this rigidity. It follows that the initial stresses, thermal or mechanical, due to an initial tension of the longitudinal elements (stringers, spars), in the shell of the caisson, can exert through the reduced rigidity $\overline{GJ}_d$ (see (VII.5.2) and (VII.1.9)) a considerable influence on the frequency of the torsional vibrations as well.

2. We shall now examine the problem of aerodynamic stability of the same box-like shell with a rigid profile, situated in a steady air stream (Figure 197).

The vibration equations of the shell are obtained in this case from (6.1) if we add to them the aerodynamic loads, allowing for frontal drag and for lift. Proceeding exactly as in § 8, Chapter VIII but dealing now with vibrations, we shall have a system of two differential equations since the first equation (6.1) for frontal drag of the wing in the plane Oxz differs from the remaining two,

$$\left.\begin{aligned} EJ_x\frac{\partial^4\eta}{\partial z^4}-\frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2}+\frac{\partial^2}{\partial z^2}(M_y\theta)=\bar{q}_y,\\ M_y\frac{\partial^2\eta}{\partial z^2}+EJ_\omega\frac{\partial^4\theta}{\partial z^4}-GJ_d\frac{\partial^2\theta}{\partial z^2}-\frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2}+\frac{\gamma Fr^2}{g}\frac{\partial^2\theta}{\partial t^2}+q_xe_x\theta=\bar{m}.\end{aligned}\right\}\tag{6.5}$$

The right-hand sides of these equations represent respectively the normal force and the torsional moment resulting from the lift of airfoil-type structures. These quantities are determined by expressions (5.2) and (5.3). Since we are examining small vibrations, assuming

$$\bar{q}_y=-k_\eta(v)\,\theta(z,t),\quad \bar{m}=-k_\theta(v)\,\theta(z,t),\tag{6.6}$$

where $k_\eta(v)$ and $k_\theta(v)$ are aerodynamic coefficients which depend on the form and geometrical dimensions of the streamlined profile and also (linearly) on the velocity of the airplane in the steady air flow.

The resultant of the forces of the frontal pressure q_x also depends, as was shown in § 5, on the airplane velocity v and can be given, for example, by

$$q_x=-kv^2,$$

where k is similarly an aerodynamic coefficient depending on the form and geometrical dimensions of the streamlined profile and some other factors.

The bending moment M_y for a uniformly distributed load q_x is determined from (VIII.8.3).

Using (6.6), we write the system (6.5) in the following form

$$\left.\begin{aligned} &EJ_x\frac{\partial^4\eta}{\partial z^4}-\frac{\gamma J_x}{g}\frac{\partial^4\eta}{\partial z^2\partial t^2}+\frac{\gamma F}{g}\frac{\partial^2\eta}{\partial t^2}+\frac{\partial^2}{\partial z^2}(M_y\theta)+k_\eta(v)\theta=0,\\ &M_y\frac{\partial^2\eta}{\partial z^2}+EJ_\omega\frac{\partial^4\theta}{\partial z^4}-GJ_d\frac{\partial^2\theta}{\partial z^2}-\frac{\gamma J_\omega}{g}\frac{\partial^4\theta}{\partial z^2\partial t^2}+\\ &\qquad+\frac{\gamma Fr^2}{g}\frac{\partial^2\theta}{\partial t^2}+[q_xe_x+k_\theta(v)]\theta=0.\end{aligned}\right\}\tag{6.7}$$

In order to determine the vibration frequency of the shell we assume

$$\eta(z,t)=\eta(z)\sin\omega t,\quad \theta(z,t)=\theta(z)\sin\omega t,\tag{6.8}$$

where ω is the required angular frequency of ghe vibrations.

Inserting (6.8) in the system (6.7) we obtain, after cancelling the common factor $\sin\omega t$, the system of ordinary differential equations

$$EJ_x\eta^{IV}+\frac{\gamma J_x}{g}\omega^2\eta''-\frac{\gamma F}{g}\omega^2\eta+(M_y\theta)''+k_\eta(v)\,\theta=0,\tag{6.9}$$

$$M_y\eta'' + EJ_\omega\theta^{IV} - \left(GJ_d - \frac{\gamma J_\varphi}{g}\omega^2\right)\theta'' - \left[\frac{\gamma F r^2}{g}\omega^2 - q_x e_x - k_\theta(v)\right]\theta = 0 \quad (6.9)$$

To integrate (6.9) we employ the method of virtual displacements. Assuming for that purpose

$$\eta(z) = B\psi(z), \quad \theta(z) = C\chi(z), \tag{6.10}$$

where $\psi(z)$ and $\chi(z)$ are chosen functions satisfying the boundary conditions of the problem. They can serve as flexure functions of a cantilever subjected to a uniformly distributed load (see (VIII.8.5)) (up to a constant factor). Taking the functions $\psi(z)$ and $\chi(z)$ also as virtual displacements and inserting (6.10) in (6.9), we obtain for the variation of the transverse load and of the torsional moment,

$$\delta q_y = B\left[EJ_x\psi^{IV} + \frac{\gamma J_x}{g}\omega^2\psi'' - \frac{\gamma F}{g}\omega^2\psi\right] + C\left[(M_y\chi)'' + k_\eta(v)\chi\right],$$

$$\delta m = BM_y\psi'' + C\left\{EJ_\omega\chi^{IV} - \left(GJ_d - \frac{\gamma J_\varphi}{g}\omega^2\right)\chi'' - \left[\frac{\gamma F r^2}{g}\omega^2 - q_x e_x - k_\theta(v)\right]\chi\right\}.$$

Applying integration by parts to some terms and using the natural boundary conditions at the ends of the beam, Lagrange's equations for these forces and the indicated virtual displacements assume the form

$$B\left[EJ_x\int_0^l(\psi'')^2\,dz - \frac{\gamma J_x}{g}\omega^2\int_0^l(\psi')^2\,dz - \frac{\gamma F}{g}\omega^2\int_0^l\psi^2 dz\right] + $$
$$+ C\left[\int_0^l M_y\chi\psi'' dz + \int_0^l k_\eta(v)\chi^2 dz\right] = 0,$$

$$B\int_0^l M_y\chi\psi'' dz + C\left\{EJ_\omega\int_0^l(\chi'')^2\,dz + \left(GJ_d - \frac{\gamma J_\varphi}{g}\omega^2\right)\int_0^l(\chi')^2\,dz - \right.$$
$$\left. - \left[\frac{\gamma F r^2}{g}\omega^2 - q_x e_x - k_\theta(v)\right]\int_0^l\chi^2 dz\right\} = 0.$$

Equating to zero the determinant of the coefficients of the unknowns B and C, we obtain for the parameter of the airplane velocity entering in the expressions of q_x, M_y and for the aerodynamic coefficients $k_\eta(v)$ and $k_\theta(v)$ the equation:

$$\begin{vmatrix} EJ_x\int_0^l(\psi'')^2\,dz - \frac{\gamma J_x}{g}\omega^2\int_0^l(\psi')^2 dz - \frac{\gamma F}{g}\omega^2\int_0^l\psi^2 dz & \int_0^l M_y\chi\psi'' dz + k_\eta(v)\int_0^l\chi^2 dz \\ \int_0^l M_y\chi\psi'' dz & EJ_\omega\int_0^l(\chi'')^2 dz + \left(GJ_d - \frac{\gamma J_\varphi}{g}\omega^2\right)\int_0^l(\chi')^2\,dz - \left[\frac{\gamma F r^2}{g}\omega^2 - q_x e_x - k_\theta(v)\right]\int_0^l\chi^2 dz \end{vmatrix} = 0. \tag{6.11}$$

In this equation the required velocity v is related to the angular frequency of vibration ω; for the here used relations of q_x, $\bar{M}_y$, $k_\eta(v)$ and $k_\theta(v)$ the equation will be of the fourth order.

Applying the extremum condition to (6.11), we can determine the critical velocity under which the airfoil will fail.

3. In addition to the indicated factors, it is possible to include also initial thermal stresses or the stresses due to pretensioned reinforcements. These initial stresses are taken into consideration by replacing the pure torsional rigidity $\overline{GJ_d}$ by the reduced rigidity GJ_d, using (VII.1.9) and (VII.5.1).

The present theory can be also extended to problems of aerodynamic stability of supersonic airfoil structures. In this case the terms of equations (6.7) which depend on the aerodynamic loads can be determined by a method evolved by A. A. Il'yushin /93/.

Chapter X

BEAMS OF SOLID SECTION

§ 1. General theory. Fundamental equations

1. We shall examine an elastic body having the form of a cylindrical or prismatical bar. We shall refer this body to a rectangular left-handed system of coordinates $Oxyz$ taking the Oz axis parallel to the generator (Figure 203a). A theory of warping is given here for beams of solid section, based on the same hypothesis as the theory of thin-walled open beams,

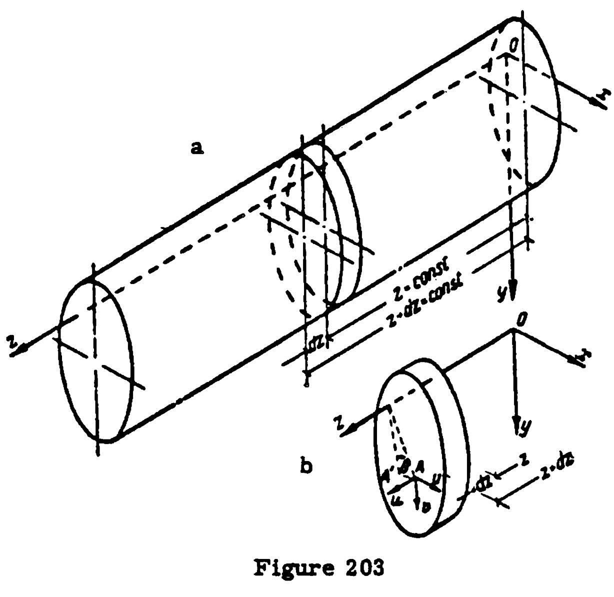


Figure 203

viz. on the hypothesis of no deformation of the beam in the plane of the cross section. The geometrical meaning of this hypothesis consists in that an elementary slice of the beam, enclosed between the sections $z=$const and $z+dz=$const, is considered in its plane as a stiff absolutely rigid body. Under this hypothesis the elementary slice has three degrees of freedom with respect to displacements in the plane of the cross section, corresponding to two translations u and v, parallel to the coordinate axes Ox and Oy, and one rotation θ about the longitudinal axis Oz (Figure 203b).

The elementary section dz can be deformed away from its plane and is regarded as an elastic body generally possessing an infinite number of degrees of freedom of which three refer to rigid displacements away from its plane (a translation along the Oz axis and two rotations about the axes Ox and Oy). The remaining infinity of degrees of freedom relate to the components of warping of the section which appear as a result of elastic deformations away from its plane. For the latter deformation the four-term relation is practically sufficient, three components of which represent deformations of the elementary section according to the law of plane sections and the fourth warping. Starting from the hypothesis adopted here and using the basic results of the theory of thin-walled beams, we may represent the displacements $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ of any point of the beam in the form

$$\left.\begin{aligned} u(x, y, z) &= u(z) - \theta(z) y, \\ v(x, y, z) &= v(z) + \theta(z) x, \\ w(x, y, z) &= w_1(z) + w_2(z) x + w_3(z) y + w_4(z) \varphi(x, y), \end{aligned}\right\} \tag{1.1}$$

where $u(z)$, $v(z)$, $\theta(z)$, $w_1(z)$, $w_2(z)$, $w_3(z)$ and $w_4(z)$ are the required functions, each depending on the coordinate z only. The quantities $u(z)$ and $v(z)$ represent the deflections of the axis Oz of the beam in the directions of the other two axes Ox and Oy. The quantity $\theta(z)$ is the angle of rotation of the cross section about the longitudinal axis Oz (the torsion angle).

The quantity $w_1(z)$ stands for translation of the cross section $z = \text{const}$ along the axis Oz. The quantities $w_2(z)$ and $w_3(z)$ stand for deflections of the beam according to the law of plane sections. The quantity $w_4(z)$ represents the degree of warping of the cross section, while the distribution of the longitudinal displacements $w(x, y, z)$ under this warping in the cross section $z = \text{const}$ is given by the function $\varphi(x, y)$.

In the theory of thin-walled open beams in restrained torsion the function $\varphi(x, y)$ was determined by the law of sectorial areas $\omega(s)$, and the degree of warping was taken as minus the z-derivative of the torsion angle, i.e. $(-\theta'(z))$. This was a result of the second hypothesis adopted, according to which the shear deformations vanish in the middle surface of the beam. In the present case this hypothesis, and thus also the law of sectorial areas, are meaningless. Instead of this law we shall henceforth use in the case of torsion the hyperbolic law of axial areas and we shall represent the function φ by the expression*

$$\varphi(x, y) = xy. \tag{1.2}$$

Generally, the beam warping under restrained torsion can in each particular case be determined also by another law, for example by the law given by Saint-Venant's theory of pure torsion.

In contradistinction to the theory of thin-walled open beams, the theory of warping of solid sections described here has a more general character. This theory is valid for solid section beams and for thin-walled beams of closed section, simply or multiply connected, and allows the warping factor of the section to be taken into account not only in torsion but also in bending or extension of the beam.

The function $\varphi(x, y)$, which we shall call the generalized warping coordinate of the cross section, together with the three elementary functions

* This law was advanced for the first time by the author for thin-walled beam-shells of closed section.

1, x, y representing the generalized coordinates in the law of plane sections, form a system of four linearly independent functions.

2. Expanding the displacements $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ in finite series (1.1), we have the following expressions for the strain components

$$\left.\begin{aligned}
e_{xx} &= \frac{\partial u(x, y, z)}{\partial x} = 0,\\
e_{yy} &= \frac{\partial v(x, y, z)}{\partial y} = 0,\\
e_{xy} &= \frac{\partial u(x, y, z)}{\partial y} + \frac{\partial v(x, y, z)}{\partial x} = 0,\\
e_{zz} &= \frac{\partial w(x, y, z)}{\partial z} = w_1'(z) + w_2'(z)x + w_3'(z)y + w_4'(z)\varphi(x, y),\\
e_{xz} &= \frac{\partial u(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial x} =\\
&= u'(z) + w_2(z) - \theta'(z)y + w_4(z)\frac{\partial\varphi(x, y)}{\partial x},\\
e_{yz} &= \frac{\partial v(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial y} =\\
&= v'(z) + w_3(z) + \theta'(z)x + w_4(z)\frac{\partial\varphi(x, y)}{\partial y}.
\end{aligned}\right\} \quad (1.3)$$

Of these six components the only ones that do not vanish in our model are the longitudinal extension e_{zz} and the shears e_{xz} and e_{yz} in the longitudinal planes Oxz and Oyz.

The other strains in the plane of the cross section vanish in accordance with the adopted hypothesis.

3. Knowing strains, we can determine the stresses by using the general relations of the theory of elasticity.

From the conditions

$$e_{xx} = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = 0,$$
$$e_{yy} = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = 0$$

we find

$$\sigma_x = \frac{\nu}{1-\nu}\sigma_z,$$
$$\sigma_y = \frac{\nu}{1-\nu}\sigma_z.$$

According to these relations the elastic law for the longitudinal extionsion e_{zz} has the form

$$\sigma_z = E_1 e_{zz}, \quad (1.4)$$

where

$$E_1 = \frac{1-\nu}{1-\nu-2\nu^2}E \quad (1.5)$$

(ν is Poisson's ratio).

The quantity E_1 defined by (1.5) represents the reduced Young's modulus of elasticity. In the following we shall denote this modulus by the letter E, determining it either by (1.5) or taking it, as in the case of a thin-walled open beam, to equal the ordinary Young's modulus.

Inserting the expression for e_{zz} according to (1.3) in expression (1.4), the latter becomes

$$\sigma_z = E\frac{\partial w}{\partial z} = E[w_1'(z) + w_2'(z)\,x + w_3'(z)\,y + w_4'(z)\,\varphi(x, y)]. \tag{1.6}$$

For the tangential stresses τ_{xz} and τ_{yz}, appearing in our model as the result of elastic strain, we find by using the relations (1.3) the expressions

$$\left.\begin{aligned} \tau_{xz} = Ge_{xz} = G\left[u'(z) + w_2(z) - \theta'(z)\,y + w_4(z)\frac{\partial\varphi(x, y)}{\partial x}\right], \\ \tau_{yz} = Ge_{yz} = G\left[v'(z) + w_3(z) + \theta'(z)\,x + w_4(z)\frac{\partial\varphi(x, y)}{\partial y}\right], \end{aligned}\right\} \tag{1.7}$$

where G is the shear [rigidity] modulus.

The positive directions of these stresses are shown on the elementary parallelepiped isolated from the beam with sides dx, dy, dz (Figure 204).

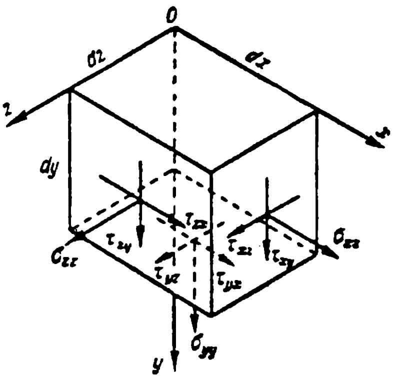


Figure 204

The tangential stresses are here determined from Hooke's law. This represents yet another difference between this theory and the theory of thin-walled beams of open section, where the assumed absence of shear deformations in the middle surface has led to the tangential stresses being determined from the equilibrium conditions.

4. We now pass to the derivation of the basic differential equations for the determination of the seven one-dimensional functions $u(z)$, $v(z)$, $\theta(z)$, $w_1(z)$, $w_2(z)$, $w_3(z)$ and $w_4(z)$. We shall use the author's general variational method, applying the principle of virtual displacements to an elementary transverse section of thickness dz, at an arbitrary fixed distance $z = \text{const}$ from the coordinate origin. This elementary section will experience external forces, some acting on the faces $z = \text{const}$ and $z + dz = \text{const}$ and due to the normal and tangential stresses which replace the cut-off parts of the beam, and others due to the given external body and surface forces. In addition, the section will be subjected to internal elastic stresses due to its elastic warping. Since the elementary section separated from the beam is in equilibrium, the total virtual work done by all external and internal forces acting on this section must vanish. As virtual displacements we shall take the longitudinal displacements corresponding to the four linearly independent generalized coordinates:

$$\psi_i(x, y) = 1,\ x,\ y,\ \varphi(x, y). \tag{1.8}$$

We denote the components along the coordinate axes of the body force by X, Y, Z and of the surface force by X_ν, Y_ν, Z_ν.

Hence, taking ψ_i as one of the generalized coordinates (1.8), it is possible to write the generalized equilibrium conditions of the elastic elementary section in the form

$$\iint \frac{\partial\sigma_z}{\partial z}\psi_i\,dx\,dy - \iint \tau_{xz}\frac{\partial\psi_i}{\partial x}\,dx\,dy - \iint \tau_{yz}\frac{\partial\psi_i}{\partial y}\,dx\,dy + \iint Z\psi_i\,dx\,dy + \int Z_\nu\psi_i\,ds = 0. \tag{1.9}$$

Table 45

		$w_1(z)$	$w_2(z)$	$w_3(z)$	$w_4(z)$	$u(z)$	$v(z)$	$\theta(z)$	Load terms	
Variational equations	1	FD^2	S_yD^2	S_xD^2	$S_\varphi D^2$	0	0	0	$\frac{1}{E}q_z$	0
	2	S_yD^2	$J_yD^2-\gamma F$	$J_{xy}D^2$	$J(x,\varphi)D^2-\gamma J\left(\frac{\partial\varphi}{\partial x}\right)$	$-\gamma FD$	0	γS_xD	$\frac{1}{E}r_x$	0
	3	S_xD^2	$J_{xy}D^2$	$J_xD^2-\gamma F$	$J(y,\varphi)D^2-\gamma J\left(\frac{\partial\varphi}{\partial y}\right)$	0	$-\gamma FD$	$-\gamma S_yD$	$\frac{1}{E}r_y$	0
	4	$S_\varphi D^2$	$J(x,\varphi)D^2-\gamma J\left(\frac{\partial\varphi}{\partial x}\right)$	$J(y,\varphi)D^2-\gamma J\left(\frac{\partial\varphi}{\partial y}\right)$	$J_\varphi D^2-\gamma\left[J\left(\frac{\partial\varphi}{\partial x}\right)^2+J\left(\frac{\partial\varphi}{\partial y}\right)^2\right]$	$-\gamma J\left(\frac{\partial\varphi}{\partial x}\right)D$	$-\gamma J\left(\frac{\partial\varphi}{\partial y}\right)D$	$\gamma\left[J\left(y\frac{\partial\varphi}{\partial x}\right)-J\left(x\frac{\partial\varphi}{\partial y}\right)\right]D$	$\frac{1}{E}r_\varphi$	0
Equilibrium equations	1	0	FD	0	$J\left(\frac{\partial\varphi}{\partial x}\right)D$	FD^2	0	$-S_xD^2$	$\frac{1}{G}q_x$	0
	2	0	0	FD	$J\left(\frac{\partial\varphi}{\partial y}\right)D$	0	FD^2	S_yD^2	$\frac{1}{G}q_y$	0
	3	0	$-S_xD$	S_yD	$\left[J\left(x\frac{\partial\varphi}{\partial y}\right)-J\left(y\frac{\partial\varphi}{\partial x}\right)\right]D$	$-S_xD^2$	S_yD^2	$(J_x+J_y)D^2$	$\frac{1}{G}m$	0

The double integration is performed over the whole cross section of the beam. The line integral is taken with respect to the coordinate s around the section contour in the plane Oxy.

The first term of the general variational equation (1.9) gives the total work done by all external elementary forces $\frac{\partial \sigma_z}{\partial z}\,dx\,dy$ in the corresponding virtual displacements ψ_i. The second and third terms give the work done by the internal tangential stresses in the corresponding shear deformations in the examined virtual state. The fourth term stands for the virtual work of all the longitudinal body forces Z acting on the given elementary section. The last term gives the total work done by all external longitudinal surface forces Z_v acting on the elementary section.

Assigning successively to the function ψ_i in equation (1.9) the values of the four generalized coordinates $1,\ x,\ y,\ \varphi(x, y)$, we obtain a system of four ordinary differential equations in the seven required one-dimensional functions $u(z),\ v(z),\ \theta(z),\ w_1(z),\ w_2(z),\ w_3(z)$ and $w_4(z)$.

We obtain the three missing differential equations from the equilibrium equations of the elementary section, regarded as a body rigid in its plane (the condition that the projections of all the forces on the axes Ox and Oy, and the moment with respect to the longitudinal axis Oz vanish):

$$\left.\begin{aligned}&\iint \frac{\partial \tau_{xz}}{\partial z}\,dx\,dy + \iint X\,dx\,dy + \int X_v ds = 0,\\&\iint \frac{\partial \tau_{yz}}{\partial z}\,dx\,dy + \iint Y\,dx\,dy + \int Y_v ds = 0,\\&\iint \left(\frac{\partial \tau_{yz}}{\partial z}x - \frac{\partial \tau_{xz}}{\partial z}y\right)dx\,dy + \iint (Yx - Xy)\,dx\,dy +\\&\qquad\qquad + \int (Y_v x - X_v y)\,ds = 0.\end{aligned}\right\} \tag{1.10}$$

The systems (1.9) and (1.10) are given in expanded form in Table 45 after using (1.6) and (1.7), where $\gamma = \frac{G}{E}$ and D denotes the differential operator with respect to the variable z, applied to the function in the uppermost row of the table

$$D = \frac{d}{dz},\quad D^2 = \frac{d^2}{dz^2}.$$

The following abbreviations are used in the table:

a) The notations of the cross sectional area, of the statical moments and of the moments of inertia as accepted in the theory of strength of materials:

$$\left.\begin{aligned}&\iint dx\,dy = F, &&\iint y^2\,dx\,dy = J_x,\\&\iint y\,dx\,dy = S_x, &&\iint x^2 dx\,dy = J_y,\\&\iint x\,dx\,dy = S_y, &&\iint xy\,dx\,dy = J_{xy};\end{aligned}\right\} \tag{1.11}$$

b) The geometrical characteristics, analogous to the sectorial statical moment S_ω and the sectorial moment of inertia J_ω of the theory of thin-walled open beams are termed, respectively, the statical bimoment and the bimoment of inertia:

$$\left.\begin{aligned}&\iint \varphi\,dx\,dy = S_\varphi,\\&\iint \varphi^2 dx\,dy = J_\varphi;\end{aligned}\right\} \tag{1.12}$$

c) The new geometrical characteristics, which in their structure recall moments of inertia and of which some have analogues in the theory of thin-walled beams and some are specific to solid beams, are symbolically written as "functions" J of an "argument" equal to the integrand:

$$\left.\begin{array}{ll}\iint \varphi x\,dx\,dy = J(x\varphi), & \iint x\frac{\partial\varphi}{\partial y}dx\,dy = J\left(x\frac{\partial\varphi}{\partial y}\right),\\ \iint \varphi y\,dx\,dy = J(y\varphi), & \iint y\frac{\partial\varphi}{\partial x}dx\,dy = J\left(y\frac{\partial\varphi}{\partial x}\right),\\ \iint \frac{\partial\varphi}{\partial x}dx\,dy = J\left(\frac{\partial\varphi}{\partial x}\right), & \iint \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial x}dx\,dy = J\left(\frac{\partial\varphi}{\partial x}\right)^2,\\ \iint \frac{\partial\varphi}{\partial y}dx\,dy = J\left(\frac{\partial\varphi}{\partial y}\right), & \iint \frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial y}dx\,dy = J\left(\frac{\partial\varphi}{\partial y}\right)^2;\end{array}\right\} \quad (1.13)$$

d) Finally, the following notations are employed for the terms depending on the external body and surface force:

$$\left.\begin{array}{l}\iint Z\,dx\,dy + \int Z_\nu ds = q_z,\\ \iint Zx\,dx\,dy + \int Z_\nu x ds = r_x,\\ \iint Zy\,dx\,dy + \int Z_\nu y ds = r_y,\\ \iint Z\varphi\,dx\,dy + \int Z_\nu \varphi ds = r_\varphi,\\ \iint X\,dx\,dy + \int X_\nu ds = q_x,\\ \iint Y\,dx\,dy + \int Y_\nu ds = q_y,\\ \iint (Yx - Xy)\,dx\,dy + \iint (Y_\nu x - X_\nu y)\,ds = m.\end{array}\right\} \quad (1.14)$$

5. The system of differential equations given in Table 45 is derived for an arbitrary rectangular coordinate system. The equations are much simpler if the principal central axes are taken as coordinate axes. As is well known, the following three characteristics vanish in this case,

$$S_x = S_y = J_{xy} = 0.$$

If for $\varphi(x, y)$ we take the function

$$\varphi(x, y) = xy - \frac{J(x^2y)}{J_y}x - \frac{J(xy^2)}{J_x}y, \quad (1.15)$$

which is orthogonal to the three generalized coordinates 1, x and y of the law of plane sections, another three geometrical characteristics will vanish, viz.

$$S_\varphi = J(x\varphi) = J(y\varphi) = 0.$$

Choosing $\varphi(x, y)$ in the form (1.15), the remaining geometrical characteristics which depend on this function will have the values

$$\left.\begin{array}{ll}J\left(\frac{\partial\varphi}{\partial x}\right) = -\frac{J(x^2y)}{J_y}F, & J\left(\frac{\partial\varphi}{\partial y}\right) = -\frac{J(xy^2)}{J_x}F,\\ J\left(x\frac{\partial\varphi}{\partial y}\right) = J_y, & J\left(y\frac{\partial\varphi}{\partial x}\right) = J_x,\\ J\left(\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial x}\right) = J_x + \left[\frac{J(x^2y)}{J_y}\right]^2 F, & J\left(\frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial y}\right) = J_y + \left[\frac{J(xy^2)}{J_x}\right]^2 F.\end{array}\right\} \quad (1.16)$$

The system of differential equations in the principal central axes,

with the function $\varphi(x, y)$ defined by (1.15), now becomes

$$\left.\begin{aligned}
&EFw_1'' + q_z = 0,\\
&EJ_y w_2'' - GFw_2 + GF\frac{J(x^2y)}{J_y} w_4 - GFu' + r_x = 0, && \text{(a)}\\
&EJ_x w_3'' - GFw_3 + GF\frac{J(xy^2)}{J_x} w_4 - GFv' + r_y = 0, && \text{(b)}\\
&GF\frac{J(x^2y)}{J_y} w_2 + GF\frac{J(xy^2)}{J_x} w_3 + EJ_\varphi w_4'' - G\Big\{J_x + J_y +\\
&\qquad + F\left[\frac{J(x^2y)}{J_y}\right]^2 + F\left[\frac{J(xy^2)}{J_x}\right]^2\Big\} w_4 + GF\frac{J(x^2y)}{J_y} u' +\\
&\qquad\qquad + GF\frac{J(xy^2)}{J_x} v' + G(J_x - J_y)\theta' + r_\varphi = 0, && \text{(c)}\\
&GFw_2' - GF\frac{J(x^2y)}{J_y} w_4' + GFu'' + q_x = 0, && \text{(d)}\\
&GFw_3' - GF\frac{J(xy^2)}{J_x} w_4' + GFv'' + q_y = 0, && \text{(e)}\\
&G(J_y - J_x) w_4' + G(J_x + J_y)\theta'' + m = 0. && \text{(f)}
\end{aligned}\right\} \quad (1.17)$$

The first equation of (1.17) is independent of the other six and does not take part in subsequent transformations. Eliminating the functions w_2, w_3 and w_4, the remaining six differential equations can be reduced to three equations for u, v and θ. The elimination can be performed in the following way. We determine w_2', w_3' and w_4' from the last three equations (d), (e) and (f) in terms of the second derivatives of u, v and θ and insert the obtained expressions in the three equations (a), (b) and (c) having first differentiated them once with respect to z. After some transformations we obtain the following three differential equations for the three sought functions $u(z)$, $v(z)$ and $\theta(z)$,

$$\left.\begin{aligned}
&EJ_y u^{\mathrm{IV}} + EJ(x^2y)\frac{J_x + J_y}{J_y - J_x}\theta^{\mathrm{IV}} = \bar{q}_x,\\
&EJ_x v^{\mathrm{IV}} + EJ(xy^2)\frac{J_x + J_y}{J_y - J_x}\theta^{\mathrm{IV}} = \bar{q}_y,\\
&EJ_\varphi \theta^{\mathrm{IV}} - G\frac{4J_xJ_y}{J_x + J_y}\theta'' = \bar{m},
\end{aligned}\right\} \quad (1.18)$$

where

$$\begin{aligned}
\bar{q}_x &= q_x + r_x' - \frac{EJ_y}{GF} q_x'' - \frac{EJ(x^2y)}{G(J_y - J_x)} m'',\\
\bar{q}_y &= q_y + r_y' - \frac{EJ_x}{GF} q_y'' - \frac{EJ(xy^2)}{G(J_y - J_x)} m'',\\
\bar{m} &= m - \frac{EJ_\varphi}{G(J_x + J_y)} m'' + \frac{J_y - J_x}{J_x + J_y}\left[r_\varphi' - \frac{J(x^2y)}{J_y} q_x - \frac{J(xy^2)}{J_x} q_y\right].
\end{aligned}$$

The first equation of the system (1.17), referring to longitudinal extension, is completely identical with its counterpart in the theory of thin-walled beams. The last equation of (1.18) is analogous to the equation of restrained torsion in the theory of thin-walled beams if we consider the expression $\frac{4J_xJ_y}{J_x + J_y}$ as an approximate value of the moment of inertia J_d for pure torsion.

The remaining two equations, i.e. the first two equations of the system (1.18), referring to bending in the principal planes, differ from the corresponding equations of the theory of thin-walled beams by the terms

containing the fourth derivatives of the torsion angle $\theta(z)$. These equations can also be reduced to a form completely analogous to the corresponding bending equations of thin-walled open beams in the principal planes. To do this we turn to the first two equations (1.1) which express the displacements of an arbitrary point of the cross section in terms of the displacement and rotation angle of the origin. To be definite, let the point A with the coordinates a_x and a_y be the point for which the displacements u_A and v_A are to be determined. The expressions (1.1) will then have the form

$$u_A = u - a_y\theta, \quad v_A = v + a_x\theta. \tag{1.19}$$

Solving (1.19) for u and v (the displacements of the coordinate origin), we obtain

$$u = u_A + a_y\theta, \quad v = v_A - a_x\theta. \tag{1.20}$$

If in (1.19) the torsion angle θ refers to an axis passing through the coordinate origin, we should refer the angle θ in (1.2) to an axis passing through the point $A(a_x, a_y)$.

Inserting (1.20) in the first two equations of (1.18), we obtain

$$\left.\begin{aligned} EJ_y u_A^{IV} + \left(EJ_y a_y + EJ(x^2y)\frac{J_x+J_y}{J_y-J_x}\right)\theta^{IV} &= \bar{q}_x, \\ EJ_x v_A^{IV} + \left(-EJ_x a_x + EJ(xy^2)\frac{J_x+J_y}{J_y-J_x}\right)\theta^{IV} &= \bar{q}_y. \end{aligned}\right\} \tag{1.21}$$

We shall choose the coordinates of the point A in such a way that the coefficients of θ^{IV} vanish. Equations (1.21) will then have a form entirely analogous to their counterparts for thin-walled open beams, viz.

$$EJ_y u_A^{IV} = \bar{q}_x, \quad EJ_x v_A^{IV} = \bar{q}_y. \tag{1.22}$$

The point A will be the shear center for solid beams and its coordinates will be determined by expressions analogous to those of thin-walled open beams,

$$\left.\begin{aligned} a_x &= \frac{J(xy^2)}{J_x}\cdot\frac{J_x+J_y}{J_y-J_x}, \\ a_y &= -\frac{J(x^2y)}{J_y}\cdot\frac{J_x+J_y}{J_y-J_x}. \end{aligned}\right\} \tag{1.23}$$

6. Starting from the formal likeness of the differential equations for restrained torsion of solid beams and for thin-walled open beams noted in the previous subsection, we may by analogy identify the coefficient $\frac{4J_xJ_y}{J_x+J_y}$ with the moment of inertia for pure torsion J_d.

Setting $J_d = \frac{4J_xJ_y}{J_x+J_y}$, we obtain a very simple approximate expression for the pure torsional rigidity of solid as well as hollow beams.

Table 46 below lists values of $\frac{4J_xJ_y}{J_x+J_y}$ calculated for a beam of rectangular section with various side ratios b/a. We show for comparison the value of J_d calculated for the same sections by the exact methods of the theory of elasticity (the coefficients of a^3b are given in the table) /136/.

The table shows that, notwithstanding the approximate character of the expression $\frac{4J_xJ_y}{J_x+J_y}$, the values obtained with the help of this expression are quite close to the exact values of J_d, especially for large values of the

ratio $\frac{b}{a}$. The most disparate case, with $\frac{b}{a}=1$ (square section), has a departure of 18%.

Table 46

$\frac{b}{a}$	1	2	3	4	5	10	∞
$\frac{4J_xJ_y}{J_x+J_y}$	0.167	0.267	0.300	0.319	0.321	0.330	0.333
J_d	0.141	0.229	0.263	0.281	0.291	0.313	0.333

7. The basic equations of the general theory of warping of beams discussed here can be also obtained by the direct reduction of the three-dimensional elastic problem to a one-dimensional problem. The equilibrium equation for an elementary parallelepiped in the direction of the axis Oz has the form

$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0.$$

Determining the stresses entering in this equation from the relations

$$\tau_{zx}=G\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),$$

$$\tau_{zy}=G\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),$$

$$\sigma_z=E\frac{\partial w}{\partial z},$$

and keeping in mind our basic assumption of the absence of strains e_{xx}, e_{yy}, e_{xy} in the plane perpendicular to the beam axis, we obtain an equation for the longitudinal displacement $w=w(x, y, z)$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{E}{G}\frac{\partial^2 w}{\partial z^2}+\frac{1}{G}Z=0, \tag{1.24}$$

where E is the reduced longitudinal Young's modulus.

In equation (1.24), let

$$w(x, y, z)=\sum W_k(z)\,\varphi_k(x, y)$$
$$k=1,2,3,\ldots, n$$

(where φ_k are given functions and $W_k(z)$ are sought); we multiply this equation, proceeding by the variational method of reduction, by $\varphi(x, y)\,dx\,dy$, take the integral of the left-hand side over the cross section of the beam, use where necessary integration by parts and equate the results to zero. For fixed i and a proper choice of the approximating functions we thus obtain the corresponding i-th variational equation of the system of Table 45.

This variational method of reduction to ordinary differential equations can be successfully applied, extensively drawing on basic physical hypotheses and conceptions, not only to the theory of beams, plates and shells but also to other problems of mathematical physics. Thus, for example, approximate solutions to numerous problems in the theory of

filtration, hydrodynamics, heat conduction, etc can be worked out on the basis of this method.

8. It should be noted that the above theory of bending and torsion of solid beams, which is represented by the fundamental equations (1.24), is different in principle from Saint-Venant's theory. The difference is that Saint-Venant's theory and the elementary theory of bending stipulate that the longitudinal fibers (layers) of the square beam do not exert pressure on one another in bending and torsion, whereas our theory, besides the stresses on the cross section, allows for normal stresses on the surface elements parallel to the axis Oz.

In view of the indicated distinction between the initial hypotheses, our fundamental equation (1.24), in addition to derivatives with respect to x and y, also contains a second partial derivative with respect to z. The corresponding equation of Saint-Venant's theory has partial derivatives of the sought function only with respect to the coordinates x and y of the cross section of the square beam.

9. We observe that the stability equations (V.1.10) and the vibration equations (IX.1.9) of the theory of thin-walled beams hold true for solid beams. This follows from the fact that the bending equations (1.22) and the equation of torsion of the theory of solid beams (the third equation of (1.18)) are completely identical with the corresponding equations of bending and torsion of the theory of thin-walled beams (I.7.3). The difference will consist only in the values of the geometrical characteristics. In calculating these characteristics the contour integrals of the theory of thin-walled beams are replaced by the double integrals of the theory of solid beams. The sectorial bimoment of inertia J_ω of the warping law (1.15) is replaced by the characteristic $J_\varphi = \int\int_F \varphi^2\,dx\,dy$. The moment of inertia for pure torsion J_d is calculated from the formula $J_d = \frac{4J_xJ_y}{J_x+J_y}$ and the coordinates of the shear center are determined from (1.23).

§ 2. Beams with two axes of symmetry

The system (1.17) is considerably simplified if we deal with beams possessing one or two axes of symmetry in the cross section.

We shall examine, by way of illustration, a beam with two axes of symmetry. This may be a beam of rectangular section, an I-beam with equal flanges, a beam of elliptical section which can be either solid or hollow with a bore of elliptical form, etc. The geometrical characteristics $J(x^2y)$ and $J(xy^2)$ vanish in this case. Leaving problems of bending in the principal planes and extensions aside (these are sufficiently well known from the theory of strength of materials) we turn to the problem of restrained torsion in the beam. The system of equations (1.17) is separated and the problem of restrained torsion is represented by the two differential equations (c) and (f) of the system (1.17),

$$\left.\begin{aligned} EJ_\varphi w_4'' - G(J_x+J_y)\,w_4 + G(J_x-J_y)\,\theta' + r_\varphi = 0, \\ -G(J_x-J_y)\,w_4' + G(J_x+J_y)\,\theta'' + m = 0. \end{aligned}\right\} \qquad (2.1)$$

The two simultaneous equations (2.1) can be reduced to a single

differential equation in the torsion angle θ by eliminating the function w_4 in the same way as before. It is advantageous here to use another method, introducing a solving function. This is convenient since then we do not have to differentiate equations (2.1) and at the same time work with the derivatives, instead of the functions themselves which depend on the loads r_φ and m and have a clear physical meaning.

The introduction of a solving function is especially simple when one of the equations of (2.1) is homogeneous. We shall, therefore, examine two cases: a) We retain $r_\varphi \neq 0$ for an external longitudinal load and we assume $m=0$. b) We assume $r_\varphi = 0$ and $m \neq 0$ for an external transverse load.

a) Here we have the system

$$\left.\begin{aligned} &EJ_\varphi w_4'' - G(J_x + J_y)\, w_4 + G(J_x - J_y)\,\theta' + r_\varphi = 0, \\ &-G(J_x - J_y)\, w_4' + G(J_x + J_y)\,\theta'' = 0. \end{aligned}\right\} \tag{2.2}$$

We shall choose the solving function $F = F(z)$ so that the second equation of (2.2), which is homogeneous, will be identically satisfied. To this end it is necessary to assume

$$w_4 = (J_x + J_y)\, F'', \quad \theta = (J_x - J_y)\, F'. \tag{2.3}$$

Introducing (2.3) in the first equation of (2.2), we obtain the solving equation for the system (2.2) in the following form,

$$EJ_\varphi (J_x + J_y)\, F^{\mathrm{IV}} - 4GJ_xJ_yF'' + r_\varphi = 0. \tag{2.4}$$

In certain cases it is convenient to relate this equation to the bimoment. As we know well now, the bimoment is given by definition in the following form

$$B = \iint \sigma(z, x, y)\, \varphi(x, y)\, dx\, dy.$$

Inserting the values of σ from (1.6), we obtain

$$B = EJ_\varphi w_4',$$

and, using the first expression (2.3), we have

$$B = EJ_\varphi (J_x + J_y)\, F'''. \tag{2.5}$$

Differentiating equation (2.4) once with respect to z and using (2.5), we obtain

$$B'' - \frac{G}{E} \cdot \frac{4J_xJ_y}{J_\varphi (J_x + J_y)}\, B + r_\varphi' = 0.$$

b) Here, where the external load consists of transverse forces, i.e. $m \neq 0$, and $r_\varphi = 0$, our system of differential equations will have the form

$$\left.\begin{aligned} &EJ_\varphi w_4'' - G(J_x + J_y)\, w_4 + G(J_x - J_y)\,\theta' = 0, \\ &-G(J_x - J_y)\, w_4' + G(J_x + J_y)\,\theta'' + m = 0. \end{aligned}\right\} \tag{2.6}$$

In this system the first equation is homogeneous; accordingly we look for the solving function in a form satisfying it identically:

$$\left.\begin{aligned} w_4 &= -G(J_x - J_y)\, F', \\ \theta &= EJ_\varphi F'' - G(J_x + J_y)\, F. \end{aligned}\right\} \tag{2.7}$$

Inserting (2.7) in the second equation of (2.6), we obtain the solving

equation

$$EJ_\varphi (J_x + J_y) F^{IV} - 4GJ_xJ_yF'' + \frac{m}{G} = 0. \tag{2.8}$$

But for the free term this equation coincides with (2.4).

Dividing (2.8) by $(J_x + J_y)$, it becomes

$$EJ_\varphi F^{IV} - G\frac{4J_xJ_y}{J_x + J_y}F'' + \frac{m}{G(J_x + J_y)} = 0.$$

Alternatively, we can obtain the required differential equation also in $\theta(z)$. For that it is necessary to differentiate the first equation of (2.1) once with respect to z and to eliminate the function $w_4'(z)$ between it and the second equation (2.1). Denoting the resulting free term by $\overline{m}(z)$, we have

$$EJ_\varphi \theta^{IV} - G\frac{4J_xJ_y}{J_x + J_y}\theta'' - \overline{m} = 0.$$

This equation has the same structure as the equation for restrained torsion of thin-walled open beams.

§ 3. Beams with a single axis of symmetry

The design problem of solid beams or hollow thick-walled structures having a single axis of symmetry in the cross section, such as a rail (Figure 205a) or the hull of a vessel with the cross section shown in Figure 205b, is of a great practical interest. In problems of this kind we study combined bending away from the plane of symmetry and restrained torsion associated with warping. We set aside the simpler problem of longitudinal extension and of bending in the plane of symmetry.

Of the seven sought functions (displacement components) we exclude three, viz. $w_1(z)$, $w_3(z)$ and $v(z)$. The other four, $u(z)$, $\theta(z)$, $w_2(z)$ and $w_4(z)$, we have to determine. Relations (1.1) have here the following form,

$$\left.\begin{aligned} u(z, x, y) &= u(z) - \theta(z)\,y, \\ v(z, x, y) &= \theta(z)\,x, \\ w(z, x, y) &= w_2(z)\,x + w_4(z)\,\varphi(x, y). \end{aligned}\right\} \tag{3.1}$$

The system of differential equations of Table 45 will also be simpler, since we drop the first and third variational equations and the second equilibrium equation. Referring the section to the principal central axes, we obtain the system of differential equations given in Table 47.

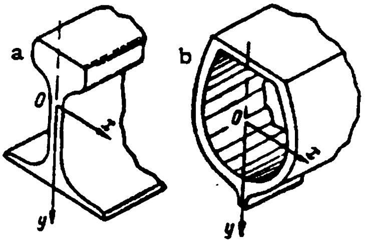


Figure 205

We take the distribution of the warping over the cross section in the form of the product xy; the function $\varphi(x,y)$ is assumed to have the form

$$\varphi(x, y) = xy + ax. \tag{3.2}$$

Table 47

		$w_2(z)$	$w_4(z)$	$u(z)$	$\theta(z)$	Load term	Right-hand side
Variational equations	2	$EJ_yD^2 - GF$	$EJ(x\varphi)D^2 - GJ\left(\frac{\partial\varphi}{\partial x}\right)$	$-GFD$	—	r_x	0
	4	$EJ(x\varphi)D^2 - GJ\left(\frac{\partial\varphi}{\partial x}\right)$	$EJ_\varphi D^2 - G\left[J\left(\frac{\partial\varphi}{\partial x}\right)^2 + J\left(\frac{\partial\varphi}{\partial y}\right)^2\right]$	$-GJ\left(\frac{\partial\varphi}{\partial x}\right)D$	$G\left[J\left(y\frac{\partial\varphi}{\partial x}\right) - J\left(x\frac{\partial\varphi}{\partial y}\right)\right]D$	r_φ	0
Equilibrium equations	1	GFD	$GJ\left(\frac{\partial\varphi}{\partial x}\right)D$	GFD^2	—	q_x	0
	3	—	$G\left[J\left(x\frac{\partial\varphi}{\partial y}\right) - J\left(y\frac{\partial\varphi}{\partial x}\right)\right]D$	—	$G(J_x + J_y)D^2$	m	0

Stipulating orthogonality of the functions $\varphi(x, y)$ and x, we obtain for a

$$a = -\frac{J(x^2y)}{J_y}.$$

For the above choice of the function $\varphi(x, y)$ the quantities entering in the coefficients of the equations of Table 47 will have the form

$$\left.\begin{aligned}
&J(x\varphi)=0,\\
&J_\varphi = J(x^2y^2) - \frac{J^2(x^2y)}{J_y},\\
&J\left(\frac{\partial\varphi}{\partial x}\right) = aF = -F\frac{J(x^2y)}{J_y},\\
&J\left(\frac{\partial\varphi}{\partial x}\right)^2 = J_x + a^2F = J_x + \frac{J^2(x^2y)}{J_y^2}F,\\
&J\left(\frac{\partial\varphi}{\partial y}\right)^2 = J_y,\\
&J\left(y\frac{\partial\varphi}{\partial x}\right) = J_x,\\
&J\left(x\frac{\partial\varphi}{\partial y}\right) = J_y.
\end{aligned}\right\} \tag{3.3}$$

After some simple transformations the equation system of Table 47 can be replaced by another which is more conveneint, viz.

$$\left.\begin{aligned}
&w_1' = \frac{J_x+J_y}{J_x-J_y}\theta'' + \frac{m}{G(J_x-J_y)},\\
&w_2' = -u'' + \frac{J(x^2y)}{J_y}\cdot\frac{J_x+J_y}{J_x-J_y}\theta'' + \bar{q}_x,\\
&-EJ_yu^{\mathrm{IV}} + EJ(x^2y)\frac{J_x+J_y}{J_x-J_y}\theta^{\mathrm{IV}} + \bar{r}_x = 0,\\
&EJ_\varphi\theta^{\mathrm{IV}} - G\frac{4J_xJ_y}{J_x+J_y}\theta'' + \bar{m} = 0,
\end{aligned}\right\} \tag{3.4}$$

where $\bar{q}_x$, $\bar{r}_x$ and $\bar{m}$ denote

$$\bar{q}_x = \frac{J(x^2y)}{GJ_y(J_x-J_y)}m - \frac{1}{GF}q_x,$$

$$\bar{r}_x = q_x + r_x' + \frac{E}{G}\cdot\frac{J(x^2y)}{J_x-J_y}m'' - \frac{EJ_y}{GF}q_x'',$$

$$\bar{m} = -m + \frac{J_x-J_y}{J_x+J_y}\left[r_\varphi' - \frac{J(x^2y)}{J_y}q_x\right] + \frac{EJ_\varphi}{G(J_x+J_y)}m''.$$

The third equation of the system (3.4) can be written in another form if we introduce, instead of the function $u(z)$ for the displacement of the coordinate origin on the axis Ox, another function, $u_A(z)$, denoting the displacement of a certain point A in the direction of the axis Ox and situated at the distance

$$a_y = \frac{J(x^2y)}{J_y}\cdot\frac{J_x+J_y}{J_x-J_y}. \tag{3.5}$$

from the origin* .

The third equation of (3.4) takes the form of the familiar equation of bending

$$-EJ_y u_A^{IV} + \bar{r}_x = 0. \tag{3.6}$$

After the determination of θ from the last equation of (3.4) and of $u(z)$ or $u_A(z)$ from (3.6), the displacements w_2 and w_4 are easily determined from the first two equations of (3.4). Consequently, the last equation of (3.4) is the fundamental solving equation. The first, as was already indicated, is analogous in structure to the equation for restrained torsion in the theory of thin-walled open beams.

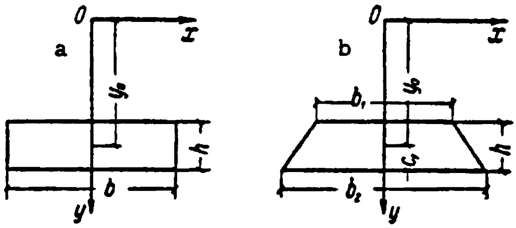


Figure 206

Some points must now be noted, concerning the calculation of the geometrical characteristics needed to construct the coefficients of the differential equations. The coordinates of the centroid, the area of the cross section and the moments of inertia J_x and J_y are conveniently listed in the majority of engineering handbooks. We therefore pay attention only to the calculation of the two new geometrical characteristics, $J(x^2y)$ and $J(x^2y^2)$, entering in the moment of inertia J_φ.

Since the exact calculation of the integrals $J(x^2y)$ and $J(x^2y^2)$ is rather difficult and even not at all necessary (in view of the approximate character of the present theory) we suggest to use for the calculation of $J(x^2y)$ and $J(x^2y^2)$ the method of dividing complicated sections (such as a rail etc) into elementary rectangles and trapezia with bases parallel to the Ox axis, which are inscribed in the given circumference. The expressions for the rectangle and trapezium are quite simple. For a rectangle with the base b and height h, whose centroid is situated a distance of y_0 from the Ox axis (Figure 206a) we have

$$J(x^2y) = \int\limits_{y_0-\frac{h}{2}}^{y_0+\frac{h}{2}} y \int\limits_{-\frac{b}{2}}^{+\frac{b}{2}} x^2\,dx\,dy = \frac{hb^3}{12} y_0, \tag{3.7}$$

* From the first relation (3.1) we have

$$u_A = u(z) - \theta(z)\,a_y,$$

whence

$$u = u_A + a_y\theta.$$

Substituting the last expression in the third equation of the system (3.4) and setting the coefficient θ^{IV} equal to zero, we arrive at (3.5).

$$J(x^2y^2) = \int\limits_{y_0-\frac{h}{2}}^{y_0+\frac{h}{2}} y^2 \int\limits_{-\frac{b}{2}}^{+\frac{b}{2}} x^2\,dx\,dy = \frac{hb^3}{12}\left(y_0^2+\frac{h^2}{12}\right). \tag{3.7}$$

For an equilateral trapezium with bases b_1, b_2, height h and centroid removed a distance y_0 from the Ox axis (Figure 206b) we obtain the more complicated expressions

$$J(x^2y) = \int\limits_{y_0+c_1-h}^{y_0+c_1} y \int\limits_{-\frac{1}{2}\left[b_2-\frac{b_2-b_1}{h}(y_0+c_1-y)\right]}^{+\frac{1}{2}\left[b_2-\frac{b_2-b_1}{h}(y_0+c_1-y)\right]} x^2\,dx\,dy =$$

$$= \frac{y_0h}{48}(b_1^2+b_2^2)(b_1+b_2)+\frac{h^2}{360}(b_2-b_1)(b_1^2+3b_1b_2+b_2^2), \tag{3.8}$$

$$J(x^2y^2) = \int\limits_{y_0+c_1-h}^{y_0+c_1} y^2 \int\limits_{-\frac{1}{2}\left[b_2-\frac{b_2-b_1}{h}(y_0+c_1-y)\right]}^{+\frac{1}{2}\left[b_2-\frac{b_2-b_1}{h}(y_0+c_1-y)\right]} x^2\,dx\,dy =$$

$$= \frac{h}{12}\left\{\left(\frac{y_0}{2}\right)^2(b_1+b_2)(b_1^2+b_2^2)+\frac{y_0h}{15}(b_2-b_1)(b_1^2+3b_1b_2+b_2^2)+\right.$$
$$\left.+\frac{h^2}{180(b_1+b_2)}\left[2(b_1+b_2)(b_1^2+b_2^2)+5b_1b_2(b_1+b_2)^2+b_1^2b_2^2\right]\right\}, \tag{3.9}$$

where $c_1=\frac{h}{3}-\frac{2b_1+b_2}{b_1+b_2}$.

For the calculation of the geometrical characteristics of a rail it is possible to divide the cross section in a first approximation into a few simple portions, e.g. as in Figure 207. The result will be, of course, closer to reality if the contour is divided into a larger number of rectangles and trapezia which fill-in the contour of the cross section more closely; $J(x^2y)$ and $J(x^2y^2)$ for the whole section are obtained by adding up the separate contributions (3.7), (3.8), (3.9) for the rectangles and trapezia which make up the cross section. This is clear that, if necessary, any other geometrical characteristics, such as F, S_x, S_y, J_x and J_y, can be also calculated in the same way.

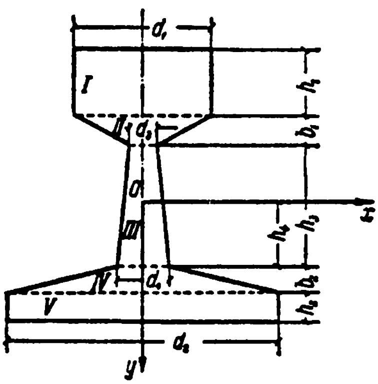
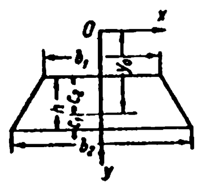
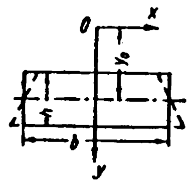
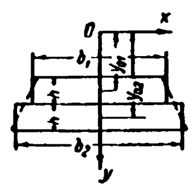
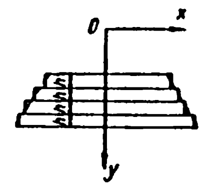


Figure 207

In order to have a clear notion of the influence of the number of parts into which the area is divided for calculation on the accuracy of the result, we give the comparative Table 48. This table lists values of the integrals $J(x^2y)$ and $J(x^2y^2)$ for an equilateral trapezium, as calculated from expressions (3.8) and (3.9) and as calculated with the help of expressions (3.7) by replacing the trapezium consecutively by one, two and four rectangles.

The values of the same integrals for an upside-down trapezium with the same dimensions are given in brackets.

Table 48

			
$h=2$ $b_1=3;\ b_2=5$ $c_1=\frac{11}{12}$ $y_0=3\frac{1}{12}=\frac{37}{12}$	$h=2$ $b=4$ $y_0=3$	$h=1$ $b_1=3.5;\ \ b_2=4.5$ $y_{01}=2.5;\ \ y_{02}=3.5$	$h=\frac{1}{2}$ $b_i=3\frac{1}{4};\ 3\frac{3}{4};\ 4\frac{1}{4};\ 4\frac{3}{4}$ $y_{0i}=2\frac{1}{2};\ 2\frac{3}{4};\ 3\frac{1}{4};\ 3\frac{3}{4}$
$J(x^2y)=36.7\ (31.3)$	$J(x^2y)=32.0$	$J(x^2y)=35.5\ (31.49)$	$J(x^2y)=36.4\ (31.35)$
$J(x^2y^2)=122.16\ (89.76)$	$J(x^2y^2)=99.6$	$J(x^2y^2)=116.3\ (92.16)$	$J(x^2y^2)=120.69\ (90.36)$

The above design method for structures having a single axis of symmetry in the cross section is equally applicable, as we have emphasized, to the design of solid beams and to thick-walled hollow structures such as ships' hulls (Figure 205b). However, in the last case the calculation of the geometrical characteristics entering in the coefficients of the differential equations is rather complicated in practice. Therefore, in the design of hollow or, generally speaking, not solid (whether open or closed) structures the author recommends to proceed in the same way as in the theory of thin-walled beams and to refer the cross section to the middle (contour) line. This allows the use of graphic-analytical methods for the calculation of the geometrical characteristics.

§ 4. Note on Saint-Venant's principle

We have repeatedly shown that in thin-walled, open or closed, beams with deformable contour the longitudinal normal stresses, due to the longitudinal bimoment loads applied at any section of the beam, can be considerable even in quite far from the place of application of the load. In other words, Saint-Venant's principle is valid over a limited range for beams of this kind, applying without restriction only to very slender thin-walled beams. Solid beams, like thin-walled beams with a closed rigid contour, differ in this respect from thin-walled beams of open or closed deformable contour and Saint-Venant's principle applies to them fully. We shall now perform a small comparative calculation to verify this.

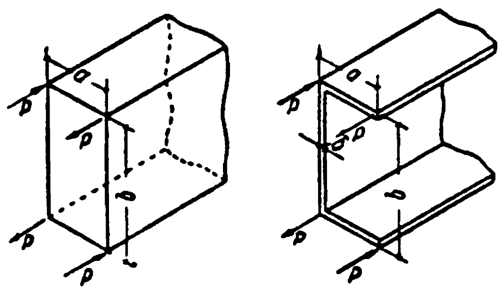


Figure 208

We consider a rectangular solid beam with sides a and $b=2a$ and a thin-walled channel beam of web depth $b=2a$, flange width a and overall wall thickness of $\delta=0.1a$. Let (Figure 208) a load in equilibrium (bimoment) be applied at one end of each of the two beams. For the solid beam

$$B_0=\sum_{k=1}^{4} P x_k y_k,$$

where x_k and y_k are the coordinates of the vertices of the rectangle.

For the thin-walled channel beam

$$B_0=\sum_{k=1}^{4} P\omega_k,$$

where ω_k are the sectorial coordinates of the corresponding points of the cross section.

We take the differential equation in the form

$$B'' - k^2 B = 0. \tag{4.1}$$

This equation is homogeneous, since we deal with an external edge load for which we account by imposing boundary conditions. It is convenient for our purpose to regard the beam as infinitely long. The second boundary condition will then require finiteness of the stresses at the end of the beam remote from the one to which the load B_0 is applied.

The coefficient k^2 has the following values:

for a solid section beam $k^2 = \frac{G}{E} \frac{4J_x J_y}{J_\varphi (J_x + J_y)}$;

for a thin-walled beam $k^2 = \frac{GJ_d}{EJ_\omega}$.

We take the solution of equation (4.1) in the form

$$B = C_1 e^{-kz} + C_2 e^{kz}.$$

From the condition of finiteness of $B(z)$, when $z \to \infty$, we find that $C_2 = 0$. Agreeing that the bimoment B_0 is applied at the end $z = 0$, we have $C_1 = B_0$. Therefore the bimoment in any section due to the action of the external bimoment B_0 applied at $z = 0$, will be given by

$$B(z) = B_0 e^{-kz}. \tag{4.2}$$

We shall take $E = 2G$ (for a Poisson's ratio $\nu = 0$). With the dimensions as chosen we obtain for the solid beam $k = 2.19a^{-1}$. For the thin-walled beam $J_d = \frac{4}{3000} a^4$, $J_\omega = \frac{43}{240} a^6$ and $k = 0.061a^{-1}$.

Inserting the found value of k in (4.2), we find where does the bimoment equal $\frac{B_0}{2}$:

a) For the solid beam at the section $z = 0.3a$;

b) For the thin-walled beam at the section $z = 11a$.

The bimoment falls to a tenth of its original value:

a) For the solid beam at the section $z = a$;

b) For the thin-walled beam at the section $z = 38a$.

We see from this comparison that the decay of the bimoment along solid beams is more than 30 times faster than in thin-walled open beams; the bimoment load has thus clearly a local characrer.

§ 5. Warping of a beam in extension

In the theory of thin-walled open beams the warping of the cross section is connected with torsion. The sectorial area $\omega(s)$ serves as the generalized warping coordinate, and minus the derivative of the torsion angle $(-\theta'(z))$ represents the degree of warping. In solid beams and thin-walled closed beams the warping may be associated just as well with other forms of deformation, to wit, bending or extension, depending on the character of the problem. As a result, the generalized warping coordinate

$\varphi(x, y)$ cannot be given once and for all in any definite analytical form and its configuration is determined by the character of the problem; a successful choice may turn out to depend mainly on the intuition of the investigator.

Example. We shall examine a circular solid beam of length $2l$ and radius $r=a$, extended by concentrated longitudinal forces P applied at the end faces and directed along the beam axis (Figure 209).

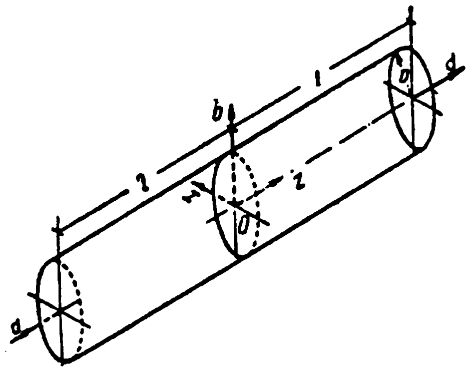


Figure 209

Since here we have axial symmetry, $u=v=\theta=w_2=w_3=0$ and the components of the longitudinal displacement $w_1(z)$ and $w_4(z)$ are to be determined.

The displacement $w_1(z)$ is determined by the methods of strength of materials from the first equation of Table 45 which, for $q_z=0$ and $S_\varphi=0$, reads $w_1''=0$. For this part of the solution the longitudinal normal stress is given by the formula

$$\sigma_1(z)=\frac{P}{F}. \tag{5.1}$$

The longitudinal normal stress corresponding to the second term of the longitudinal displacement $w_4(z)\varphi(x, y)$, is given by

$$\sigma_4=Ew_4'(z)\varphi(x, y). \tag{5.2}$$

We determine the function $w_4(z)$ from the differential equation

$$EJ_\varphi w_4''-G\left[J\left(\frac{\partial\varphi}{\partial x}\right)^2+J\left(\frac{\partial\varphi}{\partial y}\right)^2\right]w_4=0, \tag{5.3}$$

which is obtained from the fourth equation of Table 45 for $r_\varphi=0$ and $S_\varphi=0$.

The function $\varphi(x, y)$ should be chosen so as to satisfy the condition of axial symmetry and be orthogonal to the function $\varphi_1(x, y)=1$ or, in other words, to satisfy the equality $S_\varphi=0$.

We take $\varphi(x, y)=x^2+y^2+C=r^2+C$ and propose to determine the quantity C from the orthogonality relation

$$S_\varphi=\iint \varphi 1\, dx\, dy=0.$$

We obtain

$$S_\varphi=\iint (x^2+y^2+C)\, dx\, dy=\iint (x^2+y^2)\, dx\, dy+C\iint dx\, dy=0.$$

It is more convenient to calculate the first integral in polar coordinates $(x^2+y^2=r^2)$

$$\iint (x^2+y^2)\,dx\,dy=\int_0^a\int_0^{2\pi} r^3 dr\,d\vartheta=\frac{\pi a^4}{2}.$$

The second integral gives the area of the cross section,

$$\iint dx\,dy=F=\pi a^2.$$

Therefore $C=-\frac{\pi a^4}{2\pi a^2}=-\frac{a^2}{2}$ and the function $\varphi(x, y)$ has the form (Figure 210)

$$\varphi(x, y)=x^2+y^2-\frac{a^2}{2}. \tag{5.4}$$

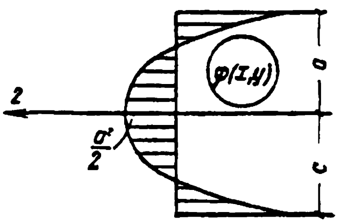


Figure 210

Taking $\varphi(x, y)$ in the form (5.4), equation (5.3) becomes

$$EJ_\varphi w_4'' - 4G(J_x+J_y)\,w_4=0. \tag{5.5}$$

The solution of (5.5) has the form

$$w_4=C_1\,\mathrm{sh}\,kz+C_2\,\mathrm{ch}\,kz, \tag{5.6}$$

where

$$k=\sqrt{\frac{4G(J_x+J_y)}{EJ_\varphi}}.$$

We place the z-origin in the middle of the beam. It is then sufficient to examine only one half of the beam, from $z=0$ to $z=l$.

For $z=0$ the condition $w_4(0)=0$ has to be satisfied, owing to symmetry. Whence $C_2=0$.

We find the second constant from the variational boundary condition at $z=l$, which we obtain by equating to zero the total work done by the longitudinal normal forces and the external load in the longitudinal unit displacements $\varphi(x, y)$,

$$\left[\iint \sigma(z, x, y)\,\varphi(x, y)\,dx\,dy-\iint P\varphi(0)\,dx\,dy\right]_{z=l}=0\,^{*}).$$

* In the general case, the variational boundary conditions are more complicated. If $p(x, y)$ is the magnitude of the external longitudinal load $z=l$ and $\sigma(z, x, y)$ is the internal longitudinal stress, then obviously for $z=l$ the following equality must hold

$$\sigma(l, x, y)-p(x, y)=0. \tag{a}$$

In determining $\sigma(z, x, y)$ we always start from some assumptions, conditions and hypotheses which simplify the determination of $\sigma(z, x, y)$ and relate to the distribution of these stresses over the cross section. These assumptions may be either the law of plane sections or any of the warping laws suggested by the character of the problem. The magnitude of the external load $p(x, y)$ is independent of any preliminary conditions or hypotheses and hence equality (a) cannot be exactly satisfied in the general case. Instead of (a) we have the variational integral condition, requiring that at $z=l$ the total virtual work done by all forces, internal and external, must vanish. We shall be closer to reality if we can give here the longitudinal normal stresses in the form of a sum

$$\sigma(l, x, y)=\sum_{j=1,2,3,\ldots,n} K_j\varphi_j(x, y), \tag{b}$$

i.e. a linear combination of the functions $\varphi_j(x, y)$, each of which describes an initial distribution over the cross section, following from the character of the problem. K_j are constants to be determined.

(cont'd on next page)

Inserting σ from (5.2) and $\varphi(0) = -\frac{a^2}{2}$ from (5.4), we obtain

$$EJ_\varphi w_4' + P\frac{a^2}{2} = 0. \tag{5.7}$$

Differentiating w_4, as given by (5.6), once with respect to z and keeping in mind that $C_2 = 0$, we have for $z = l$ after substitution in (5.7)

$$EJ_\varphi k C_1 \operatorname{ch} kl + P\frac{a^2}{2} = 0,$$

whence

$$C_1 = -\frac{P}{2EJ_\varphi} \cdot \frac{a^2}{k \operatorname{ch} kl}.$$

We now obtain an expression for σ_4,

$$\sigma_4(z, x, y) = -\frac{Pa^2}{2J_\varphi} \cdot \frac{\operatorname{ch} kz}{\operatorname{ch} kl}\left(x^2 + y^2 - \frac{a^2}{2}\right). \tag{5.8}$$

Combining (5.1) and (5.8) we obtain a more accurate expression for the longitudinal normal stresses for a circular beam extended by forces P applied along the axis,

$$\sigma = \frac{P}{F} - \frac{Pa^2}{2J_\varphi}\left(x^2 + y^2 - \frac{a^2}{2}\right)\frac{\operatorname{ch} kz}{\operatorname{ch} kl}. \tag{5.9}$$

Since

$$F = \pi a^2, \quad J_\varphi = \iint\left(r^2 - \frac{a^2}{2}\right) r\, dr\, d\varphi = \frac{\pi a^4}{12},$$

we can write (5.9) also in the form

$$\sigma = \frac{P}{\pi a^2}\left[1 - \frac{6}{a^2}\left(r^2 - \frac{a^2}{2}\right)\frac{\operatorname{ch} kz}{\operatorname{ch} kl}\right].$$

§ 6. Warping of a strut in compression and bending

We shall examine a single span, solid beam with rigid hinge supports at both ends. We shall consider these supports to be placed at the level of the lower boundaries and to restrain each of the supported points

Any of the functions $\varphi_i(x, y)$ $(i = 1, 2, 3, \ldots, j, \ldots, n)$ can be taken as virtual displacements. The variational boundary condition may then be written as

$$\iint\left[\sum_{j=1,2,3,\ldots,n} K_j\varphi_j(x, y) - p(x, y)\right]\varphi_i(x, y)\, dx\, dy = 0. \tag{c}$$

Letting i have the values 1, 2, 3, ..., n according to the number of the chosen functions $\varphi_i(x, y)$, we obtain n equations from which all the coefficients K_i can be determined. We see here a complete analogy with the expansion of the function in a Fourier series. If the functions $\varphi_j(x, y)$ are orthogonal, the system (c) is resolved into n simple equations. In the case treated in the text the distribution of the stresses σ over the section is described by two functions: the function $\varphi_1(x, y) = 1$, expressing a uniform distribution (an elementary problem in the theory of strength of materials) and $\varphi_4(x, y) = x^2 + y^2 - \frac{a^2}{2}$ expressing the variation of warping for the case of axial symmetry. Since in this case $\varphi_4(x, y)$ is taken orthogonal to the function $\varphi_1(x, y) = 1$, the boundary conditions are satisfied for each term separately.

from vertical and horizontal displacements (Figure 211). Let the beam have an I-section with two axes of symmetry (Figure 212).

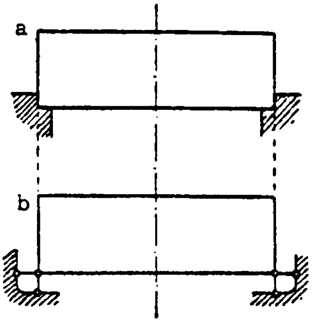


Figure 211

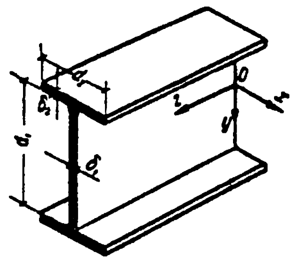


Figure 212

We shall assume that the beam is subjected to a vertical load in the symmetry plane Oyz and passing, therefore, through the line of shear centers. With the immobile hinges placed lower than the axis at the terminal sections, such a load causes not only bending but also eccentric compression due to the horizontal component of the support reaction called the t h r u s t. This thrust, applied at the lower flange only causes, besides the stresses of bending and compression, additional stresses connected with warping of the cross sections.

Since the load acts in the plane of symmetry Oyz and the boundary conditions are here also symmetrical with respect to the plane Oyz, formula (1.6) for the longitudinal normal stresses becomes

$$\sigma_z = E[w_1'(z) + w_3'(z)y + w_4'(z)\varphi(y)], \tag{6.1}$$

where, due to symmetry, the transverse distribution function of the warping $\varphi(y)$ depends only on the single variable y.

Assuming that the functions 1, y and $\varphi(y)$ satisfy the orthogonality relations, we multiply (6.1) by 1, y and $\varphi(y)$ in turn and integrate over the cross section. The following expressions are then obtained for the generalized forces in the cross section of the beam. For the longitudinal compressive force P, for the transverse bending moment M_x and for the bimoment B,

$$\left.\begin{aligned} P &= -EFw_1'(z), \\ M_x &= EJ_x w_3'(z), \\ B &= EJ_\varphi w_4'(z), \end{aligned}\right\} \tag{6.2}$$

where the bimoment of inertia J_φ is given by

$$J_\varphi = \int_F \varphi^2 dF.$$

Inserting (6.2) in (6.1), we obtain the following three-term expression for the longitudinal normal stress,

$$\sigma = -\frac{P}{F} + \frac{M_x}{J_x}y + \frac{B}{J_\varphi}\varphi. \tag{6.3}$$

We shall now deal with the generalized warping $\varphi(y)$. We shall assume that along the web of our I-beam the warping of the cross section φ is parabolic, reaching constant equal values on both flanges. In this case φ will obviously satisfy the condition of orthogonality to y (Figure 213b) and we are left with the orthogonalization of the function φ with respect to the uniform extension function, identically equal to unity (Figure 213a). Assuming the values of the ordinates of the diagram of φ on the flanges of the I-profile equal to $\varphi\left(\frac{d_1}{2}\right)=\frac{2F_1}{3F}$ and at the middle of the I-section web $\varphi(0)=1-\frac{2F_1}{3F}$, where F is the area of the entire section and F_1 is the area of the web section, this condition is evidently fulfilled. The corresponding diagram of $\varphi(y)$ is shown in Figure 213c.

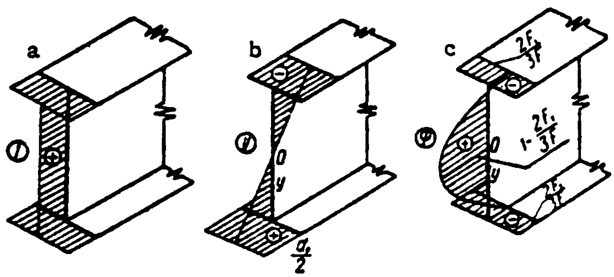


Figure 213

The function $\varphi(y)$ has the analytical form

$$\varphi(y)=1-\frac{2F_1}{3F}-\frac{4y^2}{d_1^2}. \tag{6.4}$$

For an I-section the principal geometrical characteristics F, J_x, J_φ are calculated from the expressions

$$\left.\begin{aligned} F&=F_1+2F_2,\\ J_x&=\frac{d_1^2}{12}(F_1+3F_2),\\ J_\varphi&=\frac{F_1}{3}\left[\frac{4}{3}\left(\frac{F_1}{F}\right)^2-\frac{8}{3}\frac{F_1}{F}+F\right]+\frac{8}{9}\frac{F_1^2F_2}{F^2}. \end{aligned}\right\} \tag{6.5}$$

In the present problem the thrust H is a statically indeterminate quantity. We have to determine the thrust from the condition that the horizontal displacements vanish at the supports. Applying the method of forces to the solution of this problem, we choose as fundamental system a statically-determinate beam with one fixed and one mobile support. Such a system is obtained from the given one by doing away with the horizontal connection which corresponds to the thrust H. The equation of the method of forces for the thrust H will be

$$\delta_{HH}H+\delta_{Hq}=0, \tag{6.6}$$

where δ_{HH} is the horizontal displacement of the point of application of the thrust H per unit thrust. δ_{Hq} is the horizontal displacement of the same point in the fundamental system due to the given load q.

Relating (6.3) to the stress σ_H arising in the fundamental system as

the result of the thrust H only, we find

$$\sigma_H = -\frac{H}{F} - \frac{Hd_1}{2J_x} y + \frac{B(z)}{J_\varphi} \varphi. \tag{6.7}$$

For the determination of the bimoment $B(z)$ we shall use the variational equations (1.9). Computing the work done by all external and internal forces acting on an elementary section in the virtual displacements $\varphi(y)$ determined by (6.4), body forces and distributed longitudinal loads being absent, we may write the corresponding equation (1.9) in the form

$$\iint \frac{\partial \sigma_z}{\partial z} \varphi \, dx \, dy - \iint \tau_{yz} \frac{\partial \varphi}{\partial y} dx \, dy = 0, \tag{6.8}$$

where the tangential stresses τ_{yz} are given by (1.7):

$$\tau_{yz} = G \left[v'(z) + w_3(z) + w_4(z) \frac{\partial \varphi}{\partial y} \right]. \tag{6.9}$$

Inserting (6.1) and (6.9) in (6.8) and integrating this equation we obtain, after using the orthogonality relations,

$$EJ_\varphi w_4''(z) - GJ_{\varphi'} w_4(z) = 0. \tag{6.10}$$

Here the bimoment of inertia J_φ is given by (6.5). For the characteristic $J_{\varphi'}$ we obtain

$$J_{\varphi'} = \int_F \left(\frac{\partial \varphi}{\partial y} \right)^2 dF = \frac{16\delta_1}{3d_1}.$$

Differentiating (6.10) once with respect to z and using the expression (6.2) for the bimoment $B(z)$, we obtain the differential equation

$$B''(z) - k^2 B(z) = 0, \tag{6.11}$$

where the elastic characteristic k^2 has the following value

$$k^2 = \frac{GJ_{\varphi'}}{EJ_\varphi}.$$

Integrating (6.11), we obtain

$$B(z) = C_1 \operatorname{ch} kz + C_2 \operatorname{sh} kz.$$

Shifting the z-origin to the middle of the beam, denoting the full length by $2l$ and having in mind the boundary conditions

$$\text{for} \quad z = \pm l \quad B = -H\varphi_H = \frac{2F_1}{3F} H,$$

we find

$$B = \frac{2F_1}{3F} H \operatorname{ch} kz \cdot \varphi(y). \tag{6.12}$$

Inserting (6.12) in (6.7), we obtain

$$\sigma_H = -H \left[\frac{1}{F} + \frac{d_1}{2J_x} y - \frac{2F_1}{3F} \operatorname{ch} kz \cdot \varphi(y) \right]. \tag{6.13}$$

In particular, the stress $\sigma = \sigma(z, y)$ at any point of an I-beam, due to a longitudinal eccentric compressive force H, can be determined from this expression. This stress is calculated on the assumption of a parabolic

[quadratic] variation of the warping. The stress $\sigma=\sigma(z, y)$ being known as a function of z and y, we can integrate Hooke's equation

$$w'=\frac{\sigma}{E} \tag{6.14}$$

to obtain an expression for the longitudinal displacement $w=w(z, y)$. Considering that the displacement w in the middle of the span can be taken equal to zero by symmetry, we obtain

$$w(z, y)=-\frac{H}{E}\left(\frac{z}{F}+\frac{d_1^2 z}{4J_x}y-\frac{1}{k}\frac{2F_1}{3F}\operatorname{sh}kz\cdot\varphi\right).$$

Assuming in this expression $H=1$ for the points of support $\left(z=l,\ y=\frac{d_1}{2}\right)$ we obtain

$$\delta_{HH}=-\frac{1}{E}\left[\frac{l}{F}+\frac{d_1^2 l}{4J_x}+\frac{\operatorname{sh}kl}{k}+\left(\frac{2F_1}{3F}\right)^2\right].$$

We obtain the free term δ_{Hq} of (6.6) by examining the deformation of the fundamental system subjected to the load q.

We shall examine the case of a uniformly distributed load. For the bending moment $M_x=M_x(z)$ in any section $z=$const we have

$$M_x=\frac{q(l^2-z^2)}{2}.$$

The stress σ_q in the fundamental system is given by the one-term formula

$$\sigma_q=\frac{M_x}{J_x}y=\frac{q(l^2-z^2)}{2J_x}y. \tag{6.15}$$

From (6.13) and (6.15) we obtain the following expression for the w_q longitudinal displacement

$$w_q=\frac{qz}{2EJ_x}\left(l^2-\frac{z^2}{3}\right)y.$$

For $z=l,\ y=\frac{d_1}{2}$ this yields

$$\delta_{Hq}=\frac{qd_1l^3}{6EJ_x}.$$

Inserting in (6.6) the falues found for δ_{HH} and δ_{Hq}, we obtain an expression for the thrust H,

$$H=\frac{qd_1l^3}{6}\frac{1}{\frac{J_xl}{F}+\frac{d_1^2 l}{4}+\frac{J_x\operatorname{sh}kl}{k}\left(\frac{2F_1}{3F}\right)^2} \tag{6.16}$$

Having determined the thrust H, the additional normal stresses σ_H at any point are found from (6.7). The total stress in the strut undergoing warping is obtained by adding the following stresses: the basic σ_q calculated by the usual expression (6.15) of strength of materials disregarding the thrust, and the additional σ_H determined by (6.13) and (6.16).

If we discard in (6.13) and (6.16) the bimomental terms relating to warping, we shall have a simpler approximate solution of the problem of the stresses in a strut or of the so-called arch beam.

Chapter XI

BIMOMENT THEORY OF THERMAL STRESSES

§ 1. Basic equations

1. We have previously discussed the variational method of reducing two- and three-dimensional problems of the theory of flexible bodies of cylindrical or prismatic shape to one-dimensional problems of our general bimoment theory of beams. This reduction widens the scope of the theory of strength of materials and of the applied theory of elasticity and allows the solution of many practically important new problems of structural strength by relatively simple analytical tools, under the corresponding physical assumptions. These problems comprise, in particular, the problem of initial stresses and strains appearing in beams as a result of variation of the temperature field.

We shall present here a theory comparatively simple but sufficiently accurate for practical purposes. As before, we assume that in thin-walled and solid beams subjected to the action of temperature, the longitudinal extensions and the shearing strain in the longitudinal planes are the only significant components of strain. The extension and shear in the plane of the cross section are assumed to vanish.

Let x, y, z once more denote the coordinates of any point of the beam. We shall denote by $w = w(x, y, z)$ the total longitudinal displacement of the point. In the general case this displacement is a function of all three coordinates x, y, z and is composed of the elastic displacement and the displacement due to thermal effects. Applying our variational method, we seek the total displacement $w(x, y, z)$ in the form of a sum of terms, each being a product of the required function which depends only on the longitudinal coordinate z and the given function which depends on the two cross-sectional coordinates x, y.

Limiting ourselves (as in the theory of thin-walled beams) to a four-term expression and letting the warping law remain arbitrary, we may write

$$w(x, y, z) = \zeta(z) - \xi'(z)x - \eta'(z)y + w(z)\varphi(x, y). \tag{1.1}$$

The sum of the first three terms is the contribution of the law of plane sections. The last (fourth) term determines the longitudinal displacement due to warping of the section. The function φ describing this warping is chosen on the basis of physical considerations relevant to the character of the problem at hand.

We know by now that for a thin-walled open beam warping of the sections in flexural torsion obeys the law of sectorial areas. In the case of

a thin-walled closed rectangular or elliptical section the warping in torsion is well described by the function $\varphi = xy$. We call this relationship the law of axial areas. This law also formed the basis of our analysis of warping of solid beams in flexural (restrained) torsion.

In flexural torsion of a thin-walled (or solid) beam this function φ can serve as the free torsion function of the beam, obtained in Saint-Venant's solution of the corresponding problem.

As will be shown below, the law of warping can be also obtained from the given temperature distribution over the cross section. We shall consider that the expression (1.1), being general, is valid not only for a thin-walled beam (open or closed) but also for a solid beam. The difference will consist in that for a thin-walled beam φ will depend only on the coordinate s, determining in the cross section the position of the point on the contour, whereas for a solid beam this function will depend on the two coordinates x, y, determining the position of the point in the two-dimensional domain of the section. We shall henceforth assume that the four linearly independent functions $1, x, y, \varphi$ of (1.1) satisfy the orthogonality relations,

$$\left.\begin{aligned} &\int_F x\,dF = \int_F y\,dF = \int_F xy\,dF = 0, \\ &\int_F \varphi\,dF = \int_F \varphi x\,dF = \int_F \varphi y\,dF = 0. \end{aligned}\right\} \tag{1.2}$$

2. Let $T = T(x, y, z)$ be a given function of x, y, z, characterizing the temperature field in the space occupied by the beam. We represent this function as a sum of four terms, with separate terms depending (linearly) on x and on y, thus corresponding to the law of plane sections. Assuming

$$T = t_0(z) + t_1(z)\,x + t_2(z)\,y + t(z)\,\varphi(x, y), \tag{1.3}$$

multiplying both sides of (1.3) in turn by $1, x, y, \varphi(x, y)$, integrating both sides over the cross section and using the orthogonality relations (1.2), we obtain expressions for the expansion coefficients (1.3):

$$\left.\begin{aligned} t_0 &= \frac{\int_F T\,dF}{F}, & t_2 &= \frac{\int_F Ty\,dF}{J_y}, \\ t_1 &= \frac{\int_F Tx\,dF}{J_x}, & t &= \frac{\int_F T\varphi\,dF}{J_\varphi}, \end{aligned}\right\} \tag{1.4}$$

where F is the section area; J_x, J_y are the principal moments of inertia; J_φ is the principal bimoment of inertia. All these parameters are principal characteristics of the section, which has four degrees of freedom.

We shall now determine the part of the elastic strain in the relative longitudinal extension of the beam at some point. This strain obviously equals the difference between the total relative extension $\varepsilon_n = \frac{\partial w}{\partial z}$ and the linear thermal expansion $\varepsilon_t = \alpha T$ (α is the coefficient of linear expansion). Using (1.3), we then have

$$\varepsilon = \zeta' - \alpha t_0 - (\xi'' + \alpha t_1)\,x - (\eta'' + \alpha t_2)\,y + (\theta' - \alpha t)\varphi. \tag{1.5}$$

Multiplying the elastic strain ε by the Young's modulus E we shall have

a four-term expression for the longitudinal normal stress

$$\sigma = E[\zeta' - \alpha t_0 - (\xi'' + \alpha t_1)x - (\eta'' + \alpha t_2)y + (w' - \alpha t)\varphi]. \quad (1.6)$$

In this expression the first three terms represent the longitudinal stresses distributed over the section according to the law of plane sections. These stresses are statically equivalent for the whole cross section to a longitudinal centrally applied force N and to bending moments M_x and M_y. If the beam (whether thin-walled or solid) is of the form of a statically-determinate square beam, all the internal generalized forces corresponding to the law of plane sections and examined in the elementary theory of beams will vanish under thermal variation. In particular, equating to zero the longitudinal force N and the bending moments M_x, M_y, we obtain for the functions $\zeta(z)$, $\xi(z)$, $\eta(z)$ the equations

$$\zeta' = \alpha t_0, \quad \xi'' = -\alpha t_1, \quad \eta'' = -\alpha t_2.$$

These equations (with the functions $t_0 = t_0(z)$, $t_1 = t_1(z)$, $t_2 = t_2(z)$ given by (1.4)) determine the free thermal strain in axial extension and bending of the beam, as described by the law of plane sections. The last term of (1.6) gives the normal stresses σ for the externally statically-determinate beam,

$$\sigma = E(w' - \alpha t)\varphi. \quad (1.7)$$

These stresses are due to warping and result in a longitudinal bimoment, expressed in the principal generalized coordinates 1, x, y, φ by

$$B = \int_F \sigma\varphi dF = EJ_\varphi(w' - \alpha t). \quad (1.8)$$

The expression for the longitudinal normal stresses due to a bimoment can be also written in another form,

$$\sigma = \frac{B}{J_\varphi}\varphi. \quad (1.9)$$

In addition to normal stresses σ, tangential stresses τ_{xz} and τ_{yz} also appear in the cross sections of the beam. Without external loads these stresses similarly lead to a system of interbalanced forces in any section. The expressions for the tangential stresses will have the form

$$\tau_{xz} = Gw\frac{\partial\varphi}{\partial x}, \quad \tau_{yz} = Gw\frac{\partial\varphi}{\partial y}, \quad (1.10)$$

where G is the shear modulus.

3. Separating from the beam an elementary transverse section removed a certain distance $z = \text{const}$ from the z-origin (as we did in § 1, Chapter X), imparting to this section, in accordince with the variational method, a virtual displacement $\varphi = \varphi(x, y)$ corresponding to our chosen law of warping, we find from the virtual displacements and the strains the work done by all external and internal forces operating in the section. Equating this work to zero, we obtain the variational equation

$$\int_F \frac{\partial\sigma}{\partial z}\varphi dF - \int_F \tau_{xz}\frac{\partial\varphi}{\partial x}dF - \int_F \tau_{yz}\frac{\partial\varphi}{\partial y}dF = 0. \quad (1.11)$$

Introducing in (1.11) the stresses σ, τ_{xz}, τ_{yz} from (1.7) and (1.10) and

performing certain transformations, we obtain an equation for the function $w(z)$

$$w'' - k^2 w = f, \tag{1.12}$$

where k^2 is the generalized elastic warping characteristic of the beam, defined as the ratio of the generalized transverse rigidity $GJ_{\varphi'}$ (referring to shear) and the longitudinal rigidity EJ_{φ} (referring to extension)

$$k^2 = \frac{GJ_{\varphi'}}{EJ_{\varphi}}. \tag{1.13}$$

For a given warping function φ the generalized geometrical characteristics of the section $J_{\varphi'}$ and J_{φ} are calculated from the expressions

$$\left.\begin{aligned} J_{\varphi'} &= \int_F \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] dF, \\ J_{\varphi} &= \int_F \varphi^2 \, dF. \end{aligned}\right\} \tag{1.14}$$

For a composite section with $\varphi = \varphi(x, y)$ and $dF = dx\,dy$ the definite integrals in (1.14) will obviously be double integrals. For a thin-walled beam, where $\varphi = \varphi(s)$ and $dF = \delta \cdot ds$, they reduce to simple integrals.

The function $f = f(z)$ on the right-hand side of (1.12) depends on the temperature and is determined by the formula

$$f = \alpha \frac{\partial t}{\partial z} \int_F \varphi^2 dF. \tag{1.15}$$

Together with the associated boundary conditions equation (1.12) determines the function $w = w(z)$ and hence through (1.7) also the normal stress $\sigma = \sigma(x, y, z)$ at an arbitrary point of the beam.

4. We shall assume that the temperature in the beam, T, is independent of z and is a given function of the two other coordinates x, y. The function $f = f(z)$, determined by (1.15) vanishes in this case and (1.12) will be homogeneous,

$$w'' - k^2 w = 0. \tag{1.16}$$

The general integral of (1.16) may be written in the form

$$w(z) = C_1 e^{kz} + C_2 e^{-kz}. \tag{1.17}$$

The expression for the normal stress σ is obtained from the general formula (1.7),

$$\sigma = E\left[k\left(C_1 e^{kz} - C_2 e^{-kz}\right) - \alpha t\right]\varphi. \tag{1.18}$$

§ 2. Thermal stresses in a semi-infinite beam

1. We shall examine a very long beam. The coordinate z will be measured from the free end of the beam. With increasing z, i.e. on moving away from the free end, the longitudinal displacement $w(z)$ contributed by warping of the sections must remain finite. For an infinitely long beam ($z \to \infty$) this function tends to zero. This condition implies that in (1.18)

$$C_1 = 0.$$

We determine the second integration constant C_2 from the condition that the normal stress σ must vanish at the free end. Under this condition the constant C_2 becomes

$$C_2 = -\frac{\sigma t}{k}.$$

With these values of the integration constants (1.18) becomes

$$\sigma = -E\alpha t\,(1 - e^{-kz})\,\varphi. \tag{2.1}$$

Formula (2.1) shows that when the temperature is independent of z the longitudinal normal stresses due to thermal warping reach the highest values in sections farthest from the beam ends. In the middle part of a long beam these stresses are approximately given by the formula

$$\sigma = -E\alpha t\varphi.$$

Inserting $t\varphi$ from (1.3), we obtain

$$\sigma = -E\alpha\,(T - t_0 - t_1x - t_2y). \tag{2.2}$$

In (2.2) $T = T(x, y)$ is the given temperature distribution; t_0, t_1, t_2 are constants calculated in the principal coordinates by (1.4). It is easy to show that, for an infinitely long beam with free (unfixed) ends, (2.2) yields under the above statical hypotheses an exact solution of the present problem of thermal stresses. The quantity

$$t\varphi = T - t_0 - t_1x - t_2y$$

represents the purely thermal warping, which differs from the basic warping function φ in the constant factor t. For any temperature distribution $T = T(x, y)$ a function proportional to the thermal warping

$$\varphi = \frac{1}{t}(T - t_0 - t_1x - t_2y).$$

can accordingly be chosen to serve as the $\varphi = \varphi(x, y)$ which represents the elastic warping.

2. To illustrate the outlined method for the determination of thermal stresses, we shall now examine a thin-walled metal I-beam with equal flanges (Figure 212). We assume that during the variation of the temperature field all points of the flanges are constantly kept at the same temperature. We denote this temperature by t_n. Over the web the temperature distribution follows a quadratic law, having an extremum (maximum or minimum) t_0 at the center of the section. The quantities t_n and t_0 may be either positive or negative. The temperature distribution over the I-section for the case $t_0 > t_n > 0$ is shown in Figure 214. Since the diagram of the temperature T is symmetrical with respect to both axes of symmetry of the section, the second and third terms of the four-term relations (1.1) and (1.3) must vanish. The required function $w = w(x, y, z)$ is represented by a two-term expression

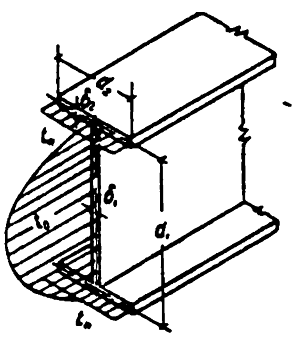


Figure 214

$$w(z, s) = \zeta(z)\,1 + w(z)\,\varphi(s).$$

The ordinate diagrams of the basic functions 1 and φ are shown in Figure 213a and 213c. The first of these functions has the

constant value of unity at all points of the section. The second, $\varphi(s)$, is represented on the web of the I-beam by a parabola with the middle ordinate $1-\frac{2F_1}{3F}$ (F_1 is the web section area, F is the area of the entire section). On the flange the function $\varphi(s)$ has a constant negative value, equal to $\left(-\frac{2F_1}{3F}\right)$ for both flanges.

The warping function $\varphi(s)$ is governed by the parabolic vertical temperature distribution. This function is orthogonal to the function $\varphi_0=1$ of the uniform temperature distribution,

$$\int_F \varphi dF=0.$$

In addition, the $\varphi(s)$ diagram is normalized so that the rise of the parabola equals one. The diagram of the given temperature T, shown in Figure 214 in the principal coordinates 1 and $\varphi(s)$, is obtained by superposing the two corresponding diagrams according to

$$T(s)=t_{av}+(t_0-t_n)\varphi(s), \tag{2.3}$$

where t_{av} is the temperature average over the section,

$$t_{av}=\frac{\int_F TdF}{F}.$$

Since on the web the rise of $\varphi(s)$ is equal to one and the rise of the temperature curve $T(s)$ there is equal to the temperature difference t_0-t_n the factor t_0-t_n must precede $\varphi(s)$ in (2.3).

Having found the warping function for the given temperature, we can determine $J_{\varphi'}$ and J_φ from (1.14),

$$J_{\varphi'}=\int_F (\varphi')^2\delta ds=\frac{16\delta_1}{3d_1},$$

$$J_\varphi=\int_F \varphi^2\delta ds=\frac{F_1}{3}\left[\frac{4}{3}\left(\frac{F_1}{F}\right)^2-\frac{8}{3}\frac{F_1}{F}+F\right]+\frac{8}{9}\frac{F_1^2F_2}{F^2},$$

where d_1 and δ_1 are the depth and thickness of the web; F_1, F_2 are respectively the area of the cross sections of the web and of one flange; F is the total section area of the I-beam. Having determined these geometrical characteristics we find, for given physical moduli E and G, the generalized elastic characteristic k from (1.13). The stresses σ at the points of the sections close to the beam end are calculated from (2.1). In this expression the temperature parameter t equals now

$$t=\frac{\int T\varphi ds}{J_\varphi}=t_0-t_n. \tag{2.4}$$

In cross sections distant from the beam ends (2.1) becomes very simple,

$$\sigma=E\alpha(t_n-t_0)\varphi.$$

It is seen from this expression that the internal bimoment stresses are proportional to the temperature difference between the flange and the middle point of the web. For the stresses σ_n and σ_0 in the flanges and at the section center we obtain the following values from the $\varphi=\varphi(s)$

diagram shown in Figure 213c,

$$\sigma_n = -\frac{2}{3}\frac{E\alpha F_1}{F}(t_n - t_0),$$
$$\sigma_0 = \left(1 - \frac{2F_1}{3F}\right)E\alpha(t_n - t_0).$$

§ 3. Thermal stresses in a finite beam

1. We shall determine the thermal bimoment stresses in a beam of finite length. We assume, as before, that the temperature $T = T(x, y)$ does not depend on z. We shall start from the homogeneous equation (1.16).

The general integrals for the functions $w = w(z)$ and $B = B(z)$, where $B(z)$ is determined by (1.8), are written in the form

$$\left.\begin{aligned} w &= C_1 \operatorname{sh} kz + C_2 \operatorname{ch} kz, \\ B &= EJ_\varphi[k(C_1 \operatorname{ch} kz + C_2 \operatorname{sh} kz) - \alpha t]. \end{aligned}\right\} \tag{3.1}$$

Since the temperature $T = T(x, y)$ remains constant along the beam, the strain and stress field in the beam is symmetrical with respect to the middle of its length. Placing the z-origin in midspan and denoting it by $2l$, we obtain the boundary conditions for one half of the beam

$$\left.\begin{aligned} &\text{for } z = 0 \quad w = 0, \\ &\text{for } z = l \quad B = 0. \end{aligned}\right\} \tag{3.2}$$

The first condition of (3.2) is purely geometrical. It is a consequence of the absence of longitudinal displacements in the middle cross section $z = 0$. This section, which was plane prior to deformation, must stay plane after it because of symmetry.

The second condition of (3.2) is purely statical. It results from the condition of vanishing normal stress at the free end.

With the boundary conditions (3.2) the integration constants C_1, C_2 assume the values

$$C_1 = \frac{\alpha t}{k \operatorname{ch} kl}, \quad C_2 = 0.$$

Formula (1.9) for the normal stresses becomes

$$\sigma = -E\alpha t\left(1 - \frac{\operatorname{ch} kz}{\operatorname{ch} kl}\right)\varphi. \tag{3.3}$$

These stresses are maximal at points of the middle cross section, i.e. for $z = 0$,

$$\sigma_{max} = -E\alpha t\left(1 - \frac{1}{\operatorname{ch} kl}\right)\varphi.$$

For the case of the above-examined I-beam t is calculated from (2.3). The function $\varphi = \varphi(s)$ is shown in Figure 213c.

2. It is also interesting to determine the tangential thermal stresses. For a thin-walled beam these stresses are determined from the equilibrium condition by integrating the equation

$$\frac{\partial(\sigma\delta)}{\partial z} + \frac{\partial(\tau\delta)}{\partial s} = 0.$$

Referring this equation to the web of the I-beam, measuring s from the middle of the web, noting that the tangential stress at the middle of the web vanishes because of symmetry, we find from formula (3.3) for σ and the $\varphi=\varphi(s)$ diagram shown in Figure 213c that

$$\tau=-E\alpha t\frac{k\,\mathrm{sh}\,kz}{\mathrm{ch}\,kl}\left[\left(1-\frac{2F_1}{3F}\right)s-\frac{4s^3}{3d_1^2}\right]. \tag{3.4}$$

This result enables us to calculate the tangential stress τ at an arbitrary point of the I-beam web. The tangenial stress τ is largest at the joints of the flanges and web, i.e. for $s=\frac{d_1}{2}$, where d_1 is the depth

$$\tau_{\max}=E\alpha tk\frac{d_1}{3}\left(1-\frac{F_1}{F}\right)\frac{\mathrm{sh}\,kz}{\mathrm{ch}\,kl}.$$

These maximum stresses vary along the beam (referred to the middle section) as the odd function $\mathrm{sh}\,kz$.

3. It is also possible to determine from the second equilibrium equation

$$\frac{\partial(\sigma_s\delta)}{\partial s}+\frac{\partial(\tau\delta)}{\partial z}=0 \tag{3.5}$$

with the function (3.4) for $\tau=\tau(z,s)$, the normal stress σ_s across the longitudinal section of the I-beam web. Assuming that at the joint of the I-beam web and flange there are tangential stresses only (neglecting the normal stresses between web and flange) we obtain for $\sigma_s=\sigma_s(z,s)$ the expression

$$\sigma_s=E\alpha tk^2\frac{\mathrm{ch}\,kz}{\mathrm{ch}\,kl}\left[\left(1-\frac{2F_1}{3F}\right)\left(\frac{s^2}{2}-\frac{d_1^2}{8}\right)-\frac{1}{d_1^2}\left(s^4-\frac{d_1^4}{16}\right)\right].$$

It is possible to calculate by this expression the stress $\sigma_s(z,s)$ at an arbitrary point of the I-beam web. For $z=0$ and $s=0$, i.e. for a central point of the middle section of the web, σ_s is given by

$$\sigma_s=-\frac{E\alpha tk^2d_1^2}{16\,\mathrm{ch}\,kl}\left(1-\frac{4F_1}{3F}\right).$$

4. Note that in determining the tangential stresses τ and the transverse normal stresses σ_s we use here the equations of statics, while in the determination of the longitudinal normal stresses we use Hooke's law. In deriving the basic variational equation (1.11) the tangential stresses were similarly determined from Hooke's law by (1.10). It would be more correct to determine the tangential stresses from the equations of statics by (3.4), since we operate with approximate variational methods, but we certainly could also find them from (1.10). To prove this assertion, consider a cantilever beam loaded by a uniformly distributed load q.

We take the beam span to be considerably larger than its depth h so that the law of plane sections holds and thus also the fundamental theorems of strength of materials:

$$\frac{d^2M}{dz^2}=q,\quad \frac{dQ}{dz}=-q.$$

Integrating these expressions and considering the bending moment to be proportional to the longitudinal normal stresses and the transverse force

proportional to the tangential stresses, we obtain the following estimate for these quantities

$$\sigma \sim \iint q dz^2, \quad \tau \sim \int q\, dz, \quad \sigma_s \sim q, \tag{3.6}$$

using (3.5) to find the estimate of the transverse normal stress σ_s.

The relationships (3.6) show that only the longitudinal normal stresses are essential in the stressed state of the beam. The tangential stresses and, moreover, the transverse normal stresses, are factors of a higher order of smallness. Therefore it is natural to defer the determination of these secondary factors from the equations of statics until we find the distribution of the longitudinal normal stresses.

We use the method of successive approximations for the tangential and transverse normal stresses.

5. The above-given method for the determination of thermal stresses in an I-beam can be also applied to a narrow rectangular plate in which the temperature varies according to a quadratic law over the width of the plate. The stresses in the plate are obtained from the expressions derived above for an I-beam if, assuming the sectional area of the flanges to vanish, we set there $F = F_1$.

Chapter XII

PLANE AND TORTUOUS THIN-WALLED CURVED BEAMS

§ 1. Bending and torsion of a plane beam whose axis forms a circular arc of small curvature

The theory given in the preceding chapters can be also extended to thin-walled curved beams of arbitrary inflexible section contour.

We shall examine a beam whose axis is curved into a circular arc. We shall refer the cross section of the beam to the principal central axes, so that the Ox axis lies in the plane of the circular beam axis and forms together with the axes Oy and Oz a left-handed system of coordinates (Figure 215).

Separating from the beam, by two close sections, an elementary transverse strip corresponding to a unit length of the line of centroids, we can write the six equilibrium equations for this strip. We obtain the following expressions for the projection on the coordinate axes of the total force acting on the strip, allowing for the curvature of the beam:

$$\left.\begin{aligned} \frac{dQ_x}{dt}+\frac{N}{R}+q_x=0,\\ \frac{dQ_y}{dt}+q_y=0,\\ \frac{dN}{dt}-\frac{Q_x}{R}+q_z=0. \end{aligned}\right\} \tag{1.1}$$

Here N is the normal force; Q_x, Q_y are the forces in the directions of the axes Ox, Oy, Oz; q_x, q_y, q_z are the projections on the axes x, y, z of the external loads per unit length; t is the arc-length of the line of centroids and R is the radius of curvature of this line (Figure 216a).

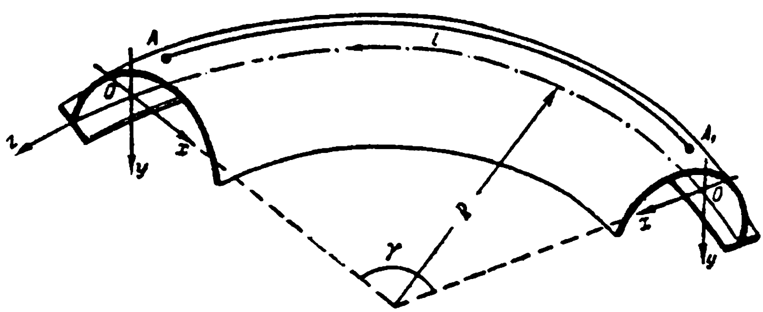


Figure 215

The projections will have the form of the total moment

$$\left.\begin{aligned} \frac{dM_x}{dt} - Q_y + \frac{H}{R} &= 0, \\ \frac{dM_y}{dt} + Q_x &= 0, \\ \frac{dH}{dt} - \frac{M_x}{R} + m_z &= 0, \end{aligned}\right\} \tag{1.2}$$

where M_x and M_y are the bending moments in the cross section of the beam; H and m are respectively the torsional moment in the section and the externally applied torsional moment. As before, we consider the moments M_x, M_y, H and m as positive if, observed from the side of positive x, y, z (respectively) they produce a clockwise rotation. Equations (1.2) allow for the curvature of the beam. In design, therefore, we use the vector representation of the moments M_x, M_y and H (Figure 216b).

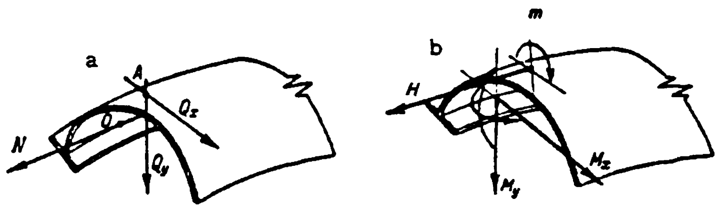


Figure 216

Note that the statical equations (1.2) have an approximate character. The essential point is that according to the theory of thin-walled beams of open section we refer some force factors to the line of centroids and some to the line of shear centers. Thus, for example, we refer the bending moments M_x and M_y to the principal central axis of the cross section, but the torsional moment H to the line of shear centers. Correspondingly, the radius R in equations (1.2) will differ from the radius of curvature of the line of centroids by the amount a_x, where a_x is the coordinate of the shear center. We shall consider beams of small initial curvature with the ratio of the largest sectional dimension to the radius of curvature of the axis of the order of 1/10 and less. In this case we neglect in (1.2) the quantity $\frac{a_x}{R}$ as compared to unity. This assumption means that in deriving the statical equations we refer all the force factors to the line of centroids. Obviously, it is also possible to refer all force factors to the line of shear centers, and generally to an arbitrary axis of the beam in the plane Oxz situated at a distance of the order of a_x from the line of centroids. The error will be insignificant and will be neglected in the following.

Eliminating the forces N, Q_x and Q_y between (1.1) and (1.2), we obtain

$$\left.\begin{aligned} M_y''' + \frac{M_y'}{R^2} - q_x' + \frac{q_x}{R} &= 0, \\ M_x'' + \frac{H'}{R} + q_y &= 0, \\ -\frac{M_x}{R} + H' + m_z &= 0, \end{aligned}\right\} \tag{1.3}$$

where the derivatives are taken with respect to the arc-length t of the beam axis.

Let, as usual, $\zeta=\zeta(t)$ be the longitudinal displacement of the beam, $\xi(t)$, $\eta(t)$ the deflections of the line of shear centers in the direction of the axes Ox and Oy respectively and $\theta=\theta(t)$ the angle of rotation of the section $t=$ const about the shear center A. As before, θ is considered positive if, looking at the cross section from the side to which the positive tangent to the line of shear centers points, it rotates clockwise.

As in the case of a rectangular beam, the strained state of a beam having an initially curved axis and rigid section contour is determined by four quantities, viz. the relative longitudinal extension $\varepsilon(t)$, the flexures $\varkappa_1(t)$ and $\varkappa_2(t)$ of this axis in the planes parallel to the principal planes Oyz, Ozx and the twist $\tau(t)$.

We shall now derive the expressions for these deformations. As before, we choose the line of centroids to serve as beam axis and take the derivatives along this line, since a_x is dropped, being negligible compared with R.

For the longitudinal deformation ε we obtain the following obvious expression

$$\varepsilon=\zeta'-\frac{\xi}{R}. \tag{1.4}$$

The curvatures of the deformed beam are k_x, k_y and k_z (k_x and k_y are the curvatures in the planes Oyz and Ozx respectively, and k_z is the torsion [second curvature] of the tortuous axis). Their projections on the axis in the initial undeformed state are

$$k_x=-\eta'',$$
$$k_y=\frac{1}{R}+\xi''+\frac{\xi}{R^2},$$
$$k_z=\theta'.$$

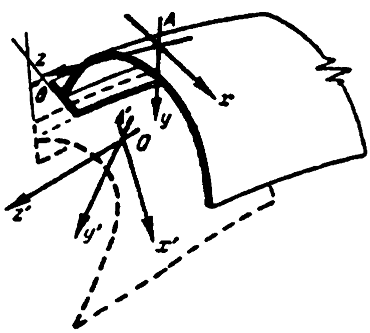


Figure 217

The first term in the second expression gives the initial curvature of the beam. The last term represents the variation of the curvature k_y due to the expansion of the circular beam through the displacement $\xi(t)$. It should be noted that in our considerations we have used an auxiliary coordinate system with the origin in the shear center and the axes Ox and Oy parallel to the principal central axes (Figure 217).

The expressions for the curvatures referred to the axes Ox', Oy' and Oz' in the strained state (Figure 217) are

$$k_{x'}=k_x\cos(xx')+k_y\cos(yx')+k_z\cos(zx'),$$

$$k_{y'}=k_x\cos(xy')+k_y\cos(yy')+k_z\cos(zy'),$$

$$k_{z'}=k_x\cos(xz')+k_y\cos(yz')+k_z\cos(zz'),$$

where the direction-cosines of the axes in the initial state with respect to those in the strained state are given in Table 49 for small displacements ξ, η and small rotation angles θ.

Table 49

	x	y	z
x'	1	θ	$-\xi'$
y'	$-\theta$	1	$-\eta'$
z'	ξ'	η'	1

From this table we obtain the following expressions for the curvatures $k_{x'}$, $k_{y'}$ and $k_{z'}$,

$$k_{x'} = -\eta'' + \left(\frac{1}{R} + \xi'' + \frac{\xi}{R^2}\right)\theta - \theta'\xi',$$

$$k_{y'} = -\eta''\theta + \frac{1}{R} + \xi'' + \frac{\xi}{R^2} - \theta'\eta',$$

$$k_{z'} = -\eta''\xi' + \left(\frac{1}{R} + \xi'' + \frac{\xi}{R^2}\right)\eta' + \theta'.$$

Neglecting here all products of ξ, η, θ and their derivatives, and dropping the term $\frac{1}{R}$ which stands for the initial curvature, we obtain for the flexures and the twist the following final expressions:

$$\varkappa_1 = -\eta'' + \frac{\theta}{R}, \quad \varkappa_2 = \xi'' + \frac{\xi}{R^2}, \quad \tau = \theta' + \frac{\eta'}{R}. \tag{1.5}$$

The first and the second expressions determine the variation of the curvatures of the beam axis $\varkappa_1(t)$ and $\varkappa_2(t)$ parallel to the principal planes Oyz, Ozx. The last expression gives the torsion [second curvature] $\tau(t)$.

In view of the relations

$$\left.\begin{aligned} N &= EF\zeta', \\ M_x &= -EJ_x\eta'', \\ M_y &= EJ_y\xi'', \\ H &= -EJ_\omega\theta''' + GJ_d\theta', \end{aligned}\right\} \tag{1.6}$$

previously obtained for a beam with a straight axis ((I.8.3) and (I.8.11) we have, using (1.4) and (1.5), the following expressions for the generalized statical quantities N, M_x, M_y and H,

$$\left.\begin{aligned} N &= EF\left(\zeta' - \frac{\xi}{R}\right), \\ M_x &= -EJ_x\left(\eta'' - \frac{\theta}{R}\right), \\ M_y &= EJ_y\left(\xi'' + \frac{\xi}{R^2}\right), \\ H &= -EJ_\omega\left(\theta''' + \frac{\eta'''}{R}\right) + GJ_d\left(\theta' + \frac{\eta'}{R}\right). \end{aligned}\right\} \tag{1.7}$$

Relation (I.8.3) for the bimoment B becomes

$$B = -EJ_\omega\left(\theta'' + \frac{\eta''}{R}\right). \tag{1.8}$$

For the normal stresses we take the four-term expression (I.8. 5),

$$\sigma = \frac{N}{F} - \frac{M_y}{J_y}x + \frac{M_x}{J_x}y + \frac{B}{J_\omega}\omega, \tag{1.9}$$

in which $x = x(s)$, $y = y(s)$ are the coordinates of the section contour point in the principal central axes Ox, Oy; J_x and J_y are the principal moments of inertia of the section; J_ω is the principal sectorial moment of inertia.

Introducing (1.7) into (1.3), we obtain

$$\left.\begin{aligned} EJ_y\left(\xi^{\mathrm{V}} + 2\frac{\xi'''}{R^2} + \frac{\xi'}{R^4}\right) - q_x' + \frac{q_z}{R} = 0,\\ -E\left(\frac{J_\omega}{R^2} + J_x\right)\eta^{\mathrm{IV}} + \frac{GJ_d}{R^2}\eta'' - \frac{EJ_\omega}{R}\theta^{\mathrm{IV}} + \frac{EJ_x + GJ_d}{R}\theta'' + q_y = 0,\\ -\frac{EJ_\omega}{R}\eta^{\mathrm{IV}} + \frac{EJ_x + GJ_d}{R}\eta'' - EJ_\omega\theta^{\mathrm{IV}} + GJ_d\theta'' - \frac{EJ_x}{R^2}\theta + m_z = 0. \end{aligned}\right\} \tag{1.10}$$

The first of equations (1.10) determines (independently of the other two) the deflections $\xi = \xi(t)$ in the plane of the beam. Like the expression of (1.7) for M_y, this equation is identical with the well-known Boussinesq equation. We note that in deriving the expression for x_z (and hence for M_y) we did not use the hypothesis of nonextensibility of the beam axis. Equation (1.10) for $\xi = \xi(t)$ thus describes the bending of the circular beam in its plane, allowing for elongation of the axis.

The second and third equations (1.10) are two simultaneous symmetrical differential equations for the functions $\eta = \eta(t)$, $\theta = \theta(t)$ determining the deformation of the circular beam away from its plane. As the author has proved in a number of works on shells /40, 51/, the symmetry properties of equations (1.10), as expressed by the reciprocity of the operators of the differential matrix, is a general property of the equilibrium equations of elastic beams and shells of arbitrary form and accords with the fundamental energy theorems of statical elasticity.

By eliminating from these equations the deflections $\eta = \eta(t)$, we obtain the basic differential equation for the torsion of a circular thin-walled beam in whose sections there appear normal stresses due not only to bending but also to twisting,

$$\begin{aligned} EJ_\omega\theta^{\mathrm{VI}} + \left(\frac{2EJ_\omega}{R^2} - GJ_d\right)\theta^{\mathrm{IV}} + \frac{1}{R^2}\left(\frac{EJ_\omega}{R^2} - 2GJ_d\right)\theta'' - \frac{GJ_d}{R^4}\theta =\\ = -\frac{J_\omega}{RJ_x}q_y'' + \frac{J_\omega + R^2J_x}{R^2J_x}m_z'' + \frac{EJ_x + GJ_d}{REJ_x}q_y - \frac{GJ_d}{ER^2J_x}m_z. \end{aligned} \tag{1.11}$$

The normal stresses are determined, in accordance with the fourth term of (1.9), by the law of sectorial areas. Equating to zero (1.11) all the terms containing J_ω, we obtain as a particular case an equation for the torsion of a circular beam in which the normal stresses in the sections are due to a single bending moment,

$$\theta^{\mathrm{IV}} + \frac{2\theta''}{R^2} + \frac{\theta}{R^4} = \frac{m_z}{ER^2J_x} - \frac{m_z''}{GJ_d} - \frac{1}{R}\left(\frac{1}{GJ_d} + \frac{1}{EJ_x}\right)q_y. \tag{1.12}$$

Note that in the design of a beam with circular axis for bending and torsion it is also possible to start from the equation of the bimoments.

In view of the relations

$$H' = B'' - \frac{GJ_d}{EJ_\omega} B,$$

the last two equations (1.3) may be written

$$\left.\begin{aligned} M_x'' + \frac{M_x}{R^2} &= -q_y + \frac{m_z}{R}, \\ B^{\mathrm{IV}} + (1-k^2)\frac{B'}{R^2} - k^2\frac{B}{R^4} &= -\left(m_z'' + \frac{q_y}{R}\right), \end{aligned}\right\} \tag{1.13}$$

where k^2 is an elastic characteristic defined by

$$k^2 = \frac{GJ_d R^2}{EJ_\omega}.$$

From equations (1.13) we can easily determine the moment $M_x = M_x(t)$ and the bimoment $B = B(t)$. Having found these as functions of t we can go on and determine from

$$\left.\begin{aligned} \eta'' - \frac{\theta}{R} &= -\frac{M_x}{EJ_x}, \\ \theta'' + \frac{\eta''}{R} &= -\frac{B}{EJ_\omega} \end{aligned}\right\} \tag{1.14}$$

the kinematical quantities $\eta = \eta(t)$, and $\theta = \theta(t)$ characterizing the deformation of the circular beam away from the plane of its axis. The derivatives in (1.13) and (1.14) are taken with respect to the coordinate t. If we take the central angle $\gamma = \frac{t}{R}$ as the unknown variable, equations (1.13) and (1.14) become

$$M_x'' + M_x = -R^2\left(q_y - \frac{m_z}{R}\right),$$

$$B^{\mathrm{IV}} + (1-k^2)B'' - k^2 B = -R^3\left(\frac{m_z''}{R} + q_y\right),$$

$$\eta'' - R\theta = -\frac{R^2}{EJ_x} M_x,$$

$$R\theta'' + \eta'' = -\frac{R^3}{EJ_\omega} B.$$

We must adduce to the general integral of the forces and displacements the boundary conditions; these are readily verified to be eight in number (four conditions at each end of the beam). These conditions represent the generalization of the conditions for a beam with a straight axis being bent in the plane Oxz and simultaneously twisted. The statical factors in any cross section consist of the moment M_x and the force Q_y causing bending, and of the bimoment B and the general torsional moment H which cause twisting. The displacement $\eta = \eta(t)$ of the section in the direction of the axis Oy, the angle of rotation of the section η' about the axis Ox, the angle of rotation $\theta = \theta(t)$ about the tangent to the beam axis (torsion angle) and, finally, the warping of the section represented by the second curvature $\tau = \theta' + \frac{\eta'}{R}$ serve as independent kinematical factors for any section $t = \text{const}$.

If one of the free ends of the beam is fixed in such a way that the terminal section is restrained from displacements η and θ and the normal

stresses in this section vanish, we have a hinge constraint. The boundary conditions are expressed in this case by the vanishing of the four quantities η, θ, $\eta'' - \frac{\theta}{R}$ and $\theta'' + \frac{\eta''}{R}$ at the end of the beam. In the case of clamping, the boundary values of η, θ, η', θ' all vanish.

The above theory includes a series of practically important problems concerning the bending and torsion of thin-walled curved plane beams and arches of small initial curvature, i.e. where the largest dimension of the cross section in the initial plane is small compared with the radius R of the beam axis, not exceeding about 1/10 of this radius.

Arches of an I-, channel, box-like and other symmetrical sections, in various positions with respect to the initial plane, may be examined on the basis of this theory.

The geometrical characteristics entering in the equations derived here, which determine the various rigidities, are calculated just as in the case of a beam with a straight axis.

§ 2. Spatial stability of circular beams, arches and torus-shells of rigid section. Fundamental equations

We shall assume that a thin-walled circular beam of arbitrary section, with the principal axis Ox in the plane of the beam axis, is subjected to moments M_y applied at the ends of the beam and causing bending in the initial plane, and also to a normal pressure in the form of a load of specific intensity $q_y = \text{const}$ which is applied on a circle passing through the point $(e_x, 0)$ of the cross section and produces in the beam a compressive force P, where

$$P = (R - e_x)\, q, \tag{2.1}$$

R being the radius of the beam axis. In this expression e_x is taken as positive; for brevity the specific load q_y is denoted by q.

The stability equation for the part of the circular beam which is subjected to the moment $M_y = \text{const}$ and the normal pressure $q = \text{const}$, is obtained from (1.3) by substituting there for the moments the expressions (1.6) and for the load terms the expressions (V.1.5), where according to (1.5) we take $\xi'' = \varkappa_2$, $\eta'' = -\varkappa_1$ and $\theta' = \tau$. Since upon flexural-torsional instability the hydrostatic load q applied at the point E gives rise (for rotation of the section through the angle θ about the shear center A) to a complementary external reduced torsional moment $m_q^* = -q(e_x - a_x)\theta$, we have

$$\left.\begin{aligned}
EJ_y\left(\varkappa_2'' + \frac{\varkappa_2'}{R^2}\right) + P\varkappa_2' + a_y P\tau'' &= 0,\\
EJ_x\varkappa_1'' + P\varkappa_1 - \frac{EJ_\omega}{R}\tau''' + \left(\frac{GJ_d}{R} + a_x P - M_y\right)\tau' &= 0,\\
a_y P\varkappa_2 + \left(\frac{EJ_x}{R} + a_x P - M_y\right)\varkappa_1 + EJ_\omega\tau''' + {}&\\
+ (r^2 P + 2\beta_x M_y - GJ_d)\tau' + q(e_x - a_x)\theta &= 0.
\end{aligned}\right\} \tag{2.2}$$

Equations (2.2) and (1.5) are the general stability equations of a part of the

circular beam (or arch) of arbitrary form with the only limitation that one of the principal axes of the section (in our case the axis Ox) is to lie in the initial plane of the beam. The quantities J_x, J_y, J_ω, J_d, a_x, a_y, r^2, β_x entering in these equations, which are the geometrical characteristics of a thin-walled open section, are determined by the corresponding expressions of the theory of thin-walled beams with a straight axis.

§ 3. Radially loaded circular ring. Particular cases. Generalization of Maurice Levy's problem

If only a radial load q, applied at some point $(e_x, 0)$ of the axis Ox, acts on a ring, then $M_y = 0$ and equations (2.2) become

$$\left.\begin{aligned} &EJ_y\left(x_2''' + \frac{x_2'}{R^2}\right) + Px_2' + a_yP\tau'' = 0,\\ &EJ_xx_1'' + Px_1 - \frac{EJ_\omega}{R}\tau''' + \left(\frac{GJ_d}{R} + a_xP\right)\tau' = 0,\\ &a_yPx_2 + \left(\frac{EJ_x}{R} + a_xP\right)x_1 + EJ_\omega\tau''' +\\ &\qquad + (r^2P - GJ_d)\tau' + q(a_x - e_x)\theta = 0. \end{aligned}\right\} \tag{3.1}$$

One of the possible forms of instability for a closed circular ring can be characterized by the rotation of all the sections of the ring through the same angle θ.

Bearing (1.5) in mind and assuming $\xi = \eta = 0$ and $\theta = \text{const}$ and using (2.1), we obtain from the last equation of (3.1).

$$q = -\frac{EJ_x}{Re_x(R - a_x)} \approx -\frac{EJ_x}{R^2e_x}. \tag{3.2}$$

Expression (3.2) determines the critical value of the normal pressure applied on the axis Ox at an arbitrary point e_x.

The minus sign in (3.2) shows that for a positive value of the denominator the critical load points along the axis Ox toward the negative values, i.e. away from the center of curvature of the beam. For $e_x = 0$ instability of the ring through rotation of all sections by the same angle is impossible.

Other forms, determined by the system

$$\left.\begin{aligned} EJ_y\left(x_2''' + \frac{x_2'}{R^2}\right) + Px_2' + a_yP\tau'' &= 0,\\ EJ_xx_1'' + Px_1 - \frac{EJ_\omega}{R}\tau''' + \left(\frac{GJ_d}{R} + a_xP\right)\tau' &= 0,\\ a_yPx_2 + \left(\frac{EJ_x}{R} + a_xR\right)x_1 + EJ_\omega\tau''' + (r^2P - GJ_d)\tau' &= 0. \end{aligned}\right\} \tag{3.3}$$

can be realized in this case.

The eigenfunctions of this system for a closed circular ring will be trigonometric functions of the argument $\frac{nt}{R}$, where n is a positive integer and t is the coordinate along the line of centroids, here a circle of radius R. Setting in (3.3)

$$x_2 = A\sin\frac{nt}{R},$$

$$x_1 = B\sin\frac{nt}{R},$$
$$\tau = C\cos\frac{nt}{R},$$

we obtain for the coefficients A, B and C a system of three linear homogeneous equations. The critical forces are found by stipulating the existence of nontrivial solutions of this system, i.e., requiring that A, B and C do not vanish.

It is easy to show that the case $n=1$ drops out. The least values for the critical forces are obtained for $n=2$, i.e. for a flexural-torsional instability in which all deformations and displacements on the entire circle vanish only at four symmetrically situated points. The equation for the critical forces for $n=2$ is

$$\begin{vmatrix} P-\frac{3EJ_y}{R^3} & 0 & -\frac{2a_yP}{R} \\ 0 & P-\frac{4EJ_x}{R^3} & -\frac{2}{R}\left(\frac{4EJ_\omega}{R^3}+\frac{GJ_d}{R}+a_xP\right) \\ a_yP & \frac{EJ_x}{R^3}+a_xP & \frac{2}{R}\left(\frac{4EJ_\omega}{R^3}-r^2P+GJ_d\right) \end{vmatrix}=0. \tag{3.4}$$

A ring of arbitrary assymetrical section will have three flexural-torsional modes of instability, corresponding to the three roots P_1, P_2 and P_3 of (3.4).

Each of these modes is characterized by rotation of all sections about some circular axis. As in the case of a straight beam, we shall thus have in the cross section three simultaneous centers of rotation.

Equation (3.4) is general and allows the determination of the critical forces P for a thin-walled ring or a torus-shell of arbitrary rigid contour.

If the shear center lies on the Ox axis, for example in case there is one axis of symmetry in the initial plane of the ring, then $a_y=0$ and the cubic equation (3.4) gives, for $P=qR$, one value for the critical load

$$q_3=\frac{3EJ_y}{R^3}, \tag{3.5}$$

and two others, q_1 and q_2, determined from the second degree equation

$$\left(1-\frac{a_x^2}{r^2}\right)q^2-\left[\left(1+\frac{a_xR}{4r^2}\right)q_x+\left(1+\frac{a_x}{R}\right)q_\omega\right]q+\frac{3}{4}q_\omega q_x=0, \tag{3.6}$$

in which

$$\left.\begin{aligned} q_x&=\frac{4EJ_x}{R^3}, \\ q_\omega&=\frac{1}{Rr^2}\left(\frac{4EJ_\omega}{R^2}+GJ_d\right), \\ r^2&=\frac{J_x+J_y}{F}+a_x^2. \end{aligned}\right\} \tag{3.7}$$

In the case of two axes of symmetry (3.6) becomes

$$q^2-(q_x+q_\omega)q+\frac{3}{4}q_xq_\omega=0.$$

Expression (3.5) is identical with the well-known formula of Maurice Levy /251/ and gives the critical load q_3 corresponding to pure flexural instability of the ring in its initial plane.

The critical loads q_1 and q_2, determined for given elastic and geometrical characteristics of a thin-walled circular beam by equations (3.6) and (3.7), correspond to tortuous modes of instability, in which the instantaneous centers of rotation lie on circles in the plane of the ring.

§ 4. Stability of radially loaded arches. Generalization of Timoshenko's problem

Let a thin-walled arch have an arbitrary symmetrical section with the symmetry axis Ox and be subjected to a normal load q applied on the axis of the arch and directed toward the center.

For $a_y=0$ and $P=qR$ equations (3.3) then become

$$\left.\begin{aligned} EJ_y\left(\varkappa_2''' + \frac{\varkappa_2'}{R^2}\right) + qR\varkappa_2' = 0, \\ EJ_x\varkappa_1'' + qR\varkappa_1 - \frac{EJ_\omega}{R}\tau''' + \left(\frac{GJ_d}{R} + Ra_xq\right)\tau' = 0, \\ EJ_\omega\tau''' + (r^2Rq - GJ_d)\tau' + \left(\frac{EJ_x}{R} + a_xRq\right)\varkappa_1 = 0. \end{aligned}\right\} \tag{4.1}$$

If the arch has hinged ends, we set in (4.1)

$$\varkappa_2 = A\sin\frac{n\pi t}{l}, \quad \varkappa_1 = B\sin\frac{n\pi t}{l}, \quad \tau = C\cos\frac{n\pi t}{l},$$

where l is the length of the entire arc; t is the distance from the support along the arc to the examined point and n is a positive integer ($n=1, 2, 3, \dots$). Then we have

$$\left.\begin{aligned} q - \frac{EJ_y}{R}\left[\left(\frac{n\pi}{l}\right)^2 - \frac{1}{R^2}\right] = 0, \\ \left(1 - \frac{a_x^2}{r^2}\right)q^2 - \left[\left(1 + \frac{a_xl^2}{n^2\pi^2r^2R}\right)q_x + \left(1 + \frac{a_x}{R}\right)q_\omega\right]q + \\ + \left(1 - \frac{l^2}{n^2\pi^2R^2}\right)q_xq_\omega = 0, \end{aligned}\right\} \tag{4.2}$$

where q_x and q_ω are given by the expressions

$$\left.\begin{aligned} q_x = \frac{EJ_x}{R}\left(\frac{n\pi}{l}\right)^2, \\ q_\omega = \frac{1}{Rr^2}\left[EJ_\omega\left(\frac{n\pi}{l}\right)^2 + GJ_d\right]. \end{aligned}\right\} \tag{4.3}$$

For $n=2$ (the case of an arch with immobile hinge supports) the first of equations (4.2) yields for q an equation identical with Timoshenko's well-known equation /178/ for the stability of an arch in the plane of its axis. Together with (4.3) (for the same n ($n=2, 3, 4, \dots$)) the second equation of (4.2) gives two values, q_1 and q_2, corresponding to flexural-torsional instability modes of a thin-walled arch. The least critical forces are obtained for $n=1$. An analysis of this solution shows that as a rule flexural-torsional instability is the dangerous mode for a thin-walled circular arch. It gives for the critical force a value smaller than that determined for $n=2$ by the first of equations (4.2).

§ 5. Stability in plane bending of a beam with a circular axis. Generalization of another problem of Timoshenko

Setting in equations (2.2) $P=q=0$, $M_y=M$, we obtain the stability equation for a circular beam of arbitrary section in plane bending, with the axis Ox in the plane of curvature

$$\left.\begin{aligned} EJ_x x_1'' + \left(\frac{GJ_d}{R} - M\right)\tau' = 0, \\ \left(\frac{EJ_x}{R} - M\right)x_1 + EJ_\omega \tau'' + \left(2\beta_x M - GJ_d\right)\tau' = 0. \end{aligned}\right\} \tag{5.1}$$

If the beam has hinged ends,

$$x_1 = B\sin\frac{n\pi t}{l}, \quad \tau = C\cos\frac{n\pi t}{l},$$

where $n=1, 2, 3, \ldots$. The equation for the critical moments becomes

$$\begin{vmatrix} -P_x & -\frac{n\pi}{lR}P_\omega + \frac{n\pi}{l}M \\ \frac{n^2\pi^2}{l^2 R}P_x - M & \frac{n\pi r^2}{l}P_\omega - \frac{2n\pi}{l}\beta_x M \end{vmatrix} = 0, \tag{5.2}$$

where

$$P_x = \frac{EJ_x n^2\pi^2}{l^2}, \quad P_\omega = \frac{1}{r^2}\left(EJ_\omega \frac{n^2\pi^2}{l^2} + GJ_d\right).$$

The least values of M occur for $n=1$, i.e. for a sinusoidal buckling shape with one half-wave extending over the entire length l of the arch. Determining M from (5.2) and setting $n=1$, we obtain

$$M = RP_x\left\{\frac{1}{2}\left(\frac{r^2}{R^2}\frac{P_\omega}{P_x} + \frac{l^2}{\pi^2 R^2} - \frac{2\beta_x}{R}\right) \pm \right.$$
$$\left. \pm\sqrt{\frac{1}{4}\left(\frac{r^2}{R^2}\frac{P_\omega}{P_x} + \frac{l^2}{\pi^2 R^2} - \frac{2\beta_x}{R}\right)^2 + \frac{r^2}{R^2}\left(1 - \frac{l^2}{\pi^2 R^2}\right)\frac{P_\omega}{P_x}}\right\}, \tag{5.3}$$

where

$$P_x = \frac{EJ_x\pi^2}{l^2}, \quad P_\omega = \frac{1}{r^2}\left(\frac{EJ_\omega \pi^2}{l^2} + GJ_d\right). \tag{5.4}$$

This solution of the stability problem for plane bending of a thin-walled arch is of general character and allows the determination of the critical moments M for an arch of arbitrary section. The geometrical characteristics J_x, J_d, J_ω, β_x and r^2 are calculated just as for a thin-walled beam with a straight axis. For given form, dimensions of the cross section and length and radius of the arch, expressions (5.3) and (5.4) determine two moments which differ not only in their sign (this is perfectly obvious physically) but also in their absolute values $\left(\text{due to the curvature } \frac{1}{R}\right)$

For a section with two axes of symmetry, it is necessary to set in (5.3) and (5.4) $\beta_x = 0$, $r^2 = \frac{J_x + J_y}{F}$, where F is the section area. For a section in the form of a very narrow rectangle we may assume that $\beta_x = 0$ and $J_\omega = 0$. Then (5.3) becomes

$$M = \frac{1}{2R}\left(EJ_x + GJ_d\right) \pm \sqrt{\frac{1}{4R^2}\left(EJ_x + GJ_d\right)^2 + \left(\frac{\pi^2}{l^2} - \frac{1}{R^2}\right)EJ_x GJ_d}.$$

This formula, which is a particular case of the more general formula (5.3), is identical with the one given by Timoshenko /178/.

§ 6. The tortuous beam. The law of sectorial areas for the bimoments

1. Let $X=X(t)$, $Y=Y(t)$, $Z=Z(t)$ represent a space curve in fixed Cartesian coordinates. The distance along the curve from the origin $t=0$ which coincides with one end of the curve to the examined point with the running coordinate t—or some other parameter—is taken as the independent variable. Figure 218 shows a helix with its three projections on the coordinate planes. This curve is wrapped on the surface of a circular cylinder; its parametric equations, giving the coordinates of any of its points T, are

$$\left.\begin{aligned} X&=X_0+a\cos\gamma,\\ Y&=Y_0+a\sin\gamma,\\ Z&=Z_0+b\gamma,\end{aligned}\right\} \tag{6.1}$$

where X_0 and Y_0 are the coordinates of the center of the circular base of the cylinder; a is the radius of this circle; γ is the polar angle in the plane OXY; b is a coefficient proportional to the pitch of the helix.

The angular coordinate γ serves here as independent variable. For $\gamma=0$ (6.1) gives the coordinates X_K, Y_K, Z_K of the initial point K of the helix,

$$X_K=X_0+a,\quad Y_K=Y_0,\quad Z_K=Z_0.$$

For $\gamma=3\pi$ we obtain the coordinates of the terminal point L of the examined part of the helix. The equations of the projections of the helix on the coordinate planes are obtained from (6.1) by eliminating γ,

$$\begin{gathered}(X-X_0)^2+(Y-Y_0)^2=a^2,\\ X=X_0+a\cos\frac{Z-Z_0}{b},\\ Y=Y_0+a\sin\frac{Z-Z_0}{b}.\end{gathered}$$

These equations show that the projection of the helix on the plane OXY is a circle, the projection on the plane OYZ is a sine curve and the projection on the plane OZX is a cosine curve.

Let a normal plane pass at a point of the curve. We shall describe on this plane the section contour of a thin-walled beam in such a manner that the shear center A of a beam with a straight axis lies on the helix (Figure 219). We direct the axes Ax and Ay of the local triad parallel to the principal central axes of the beam section (they lie along the binormal and the normal to the space curve, respectively). The axis Az lies along the tangent to this curve in the direction of increasing t. As the normal plane moves along the curve the contour line, which lies in this plane, describes a certain surface in space. This surface will be the middle surface of a thin-walled tortuous beam. For beam axis we accordingly have the line of shear centers, i.e. the locus of the points A whose positions in the normal plane are given by (I.7.5), referred to the mobile axes Ox and Oy.

We shall assume that the beam behaves as a thin-walled statically determinate system with respect to the bimoments in an arbitrary element dt, enclosed between two neighboring planes normal to the beam axis and passing through the points $t=\text{const}$ and $t+dt=\text{const}$. In other words, we consider a beam composed of very thin strips, whose sections are subjected to tangential stresses uniformly distributed across the wall. The

beam must then have zero rigidity in pure torsion ($GJ_d = 0$). Such a beam, but with a straight axis, was examined in § 9, Chapter II.

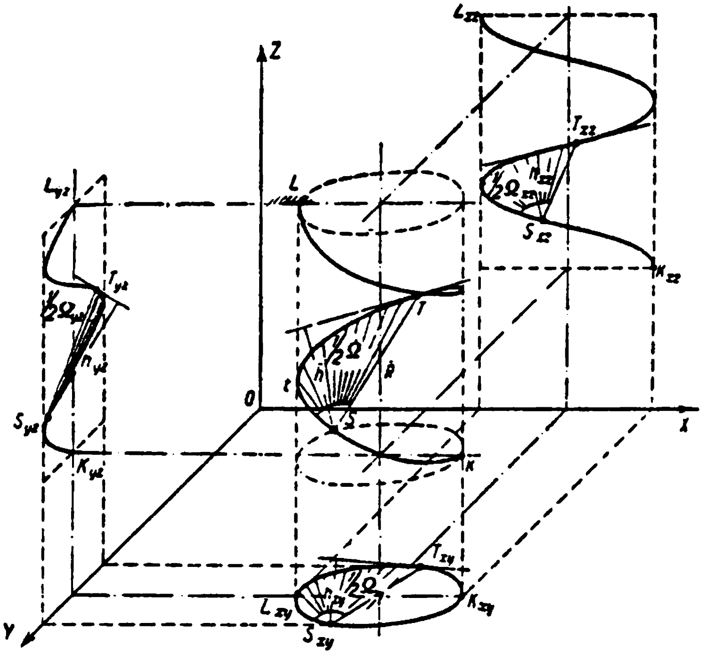


Figure 128

We shall call a beam with the above-indicated properties, if its section contour is unaltered after deformation, an ideal thin-walled beam. If the curvature of the beam axis is small, i.e. the dimensions of the section contour are very small compared with the radius of curvature, we have

$$\frac{dB}{dt} = H, \tag{6.2}$$

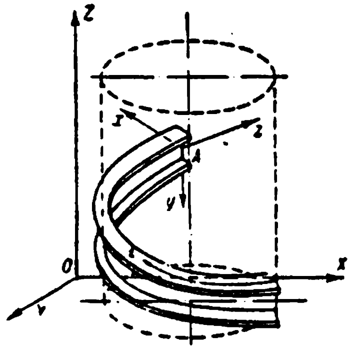


Figure 219

where H and B are respectively the torsional moment and the bimoment in the section $t =$ const.

Equation (6.2) is exact for an ideal thin-walled beam with a straight axis and expresses our theorem that the derivative of the bimoment with respect to the arc-length t equals the torsional moment. This theorem is also a generalization of the Schwedler-Zhuravskii theorem.

For the tortuous beam we append (6.2) to the regular six equations of statics to obtain a complete system of seven equilibrium equations for the seven generalized internal statical factors, viz. three forces, three moments and one bimoment in any section $t =$ const.

Suppose the beam to be rigidly fixed at the end L so that its end section there is restrained with respect to all seven generalized displacements defined above (Figure 218). This means that at the fixed end each

of the following vanishes separately: the three components of the linear displacement vector of the end of the beam axis, the three components of the angular displacement vector of the cross section, and the warping of this section. We shall consider the other end of the beam K free from any of the seven constraints indicated above. Let the following external statical factors operate at a point S on the axis of this sort of "cantilever": the force $\bar{P}_S$, given by its components P_{SX}, P_{SY} and P_{SZ}, along the axes OX, OY and OZ, the moment $\bar{M}_S$ with the components M_{SX}, M_{SY} and M_{SZ} along the same axes, and the bimoment B_S.

Under the action of this load internal forces appear in the beam wherever its sections undergo sectorial warping. These consist of the seven generalized statical factors, comprising the usual six (three forces and three moments) and the bimoment. On the portion KS of the beam axis between the free (unfixed) end K and the point of application S, all the internal forces vanish, which statement applies for an ideal beam also to the bimoment. At each point T on SL, the internal forces are found from the equilibrium conditions. In any section $t=$const the force vector and the moment vector are found from the six usual equilibrium conditions of the beam, considered as a rigid three-dimensional body. For the torsional moment $H=H(t)$ we obtain, in particular

for $t_S \leqslant t \leqslant t_L$

$$H(t)=P_{SX}h_{YZ}(t)+P_{SY}h_{ZX}(t)+P_{SZ}h_{XY}(t)+ \\ +M_{SX}X'(t)+M_{SY}Y'(t)+M_{SZ}Z'(t), \qquad (6.3)$$

where $h_{YZ}(t)$, $h_{ZX}(t)$ and $h_{XY}(t)$ are the projections on the coordinate planes OYZ, OZX and OXY of the perpendicular $h(t)$ drawn from the application point S of the force P_S to the tangent to the beam axis at the point with the coordinate t; $X'(t)$, $Y'(t)$ and $Z'(t)$ are the derivatives of the coordinates with respect to the arc-length t. Inserting (6.3) in (6.2) and integrating, we obtain

$$B(t)=P_{SX}\Omega_{YZ}(t)+P_{SY}\Omega_{ZX}(t)+P_{SZ}\Omega_{XY}(t)+ \\ +M_{SX}X(t)+M_{SY}Y(t)+M_{SZ}Z(t)+B_S. \qquad (6.4)$$

Here $\Omega_{YZ}(t)$, $\Omega_{ZX}(t)$ and $\Omega_{XY}(t)$ are the projections on the coordinate planes of the vector $\bar{\Omega}(t)$ whose magnitude is double the area of the ruled surface traced by the mobile radius-vector which issues from the fixed point S (where the force $\bar{P}$ is applied), as its other end travels along the beam axis from S to the point T at which the bimoment is to be determined. The quantities $\Omega_{XY}(t)$, $\Omega_{YZ}(t)$ and $\Omega_{ZX}(t)$ are double the areas of the segment-like figures in the coordinate planes bounded by the projection of the arc ST of the beam axis and the projection of the chord joining its ends.

The quantities $X(t)$, $Y(t)$ and $Z(t)$ in (6.4) represent the projections of the vector $\bar{R}(t)$ (the distance between S and T) on the axes OX, OY and OZ (the coordinate origin coincides with the point of application, S, of the moment $\bar{M}_S$).

The last term of (6.4) refers to the external concentrated bimoment B_S, applied at the point S. Writing (6.4) in the vectorial form

$$B(t)=B_S+\bar{M}_S\bar{R}(t)+\bar{P}_S\bar{\Omega}(t), \qquad (6.5)$$

where the last two terms are the scalar products of the corresponding vectors, we may express the result by the following theorem.

[C a s e I]. Only an external bimoment B_S acts on an ideal thin-walled tortuous beam of inflexible open section with one end free and the other fixed; the bimoment in any cross section, situated between the point of of application of the external concentrated forces and the fixed end, remains constant and equal to the external bimoment.

[C a s e II]. Only a concentrated moment $\overline{M}_S$ is acting; the bimoment equals the scalar product of the vector of this concentrated moment and the radius-vector $\overline{R}(t)$ going from the point of application S of the moment to the point T for which the bimoment $B(t)$ is determined.

[C a s e III]. Only a concentrated force $\overline{P}_S$ is acting; the internal bimoment $B(t)$ at any point of the beam axis between the point of application S of the force and the fixed end equals the scalar product of the force vector $\overline{P}_S$ and the vector $\overline{\Omega}(t)$ which presents twice the area of the surface swept out by the radius-vector as it moves along the beam axis between S and the point T where the bimoment is determined. On the portion between the free end and the point of application of the external forces the bimoment $B(t)$ vanishes.

2. For a beam having an axis in the form of a plane curve (or a broken line), (6.4) becomes

$$B(t) = B_S + M_{SX}X(t) + M_{SY}Y(t) + P_{SZ}\Omega_{XY}(t). \tag{6.6}$$

This expression represents the bimoment of a plane curved cantilever beam, in analogy with the law of sectorial areas for longitudinal displacements and normal stresses in a beam with a straight axis whose cross sections undergo sectorial warping.

Note that in contrast to the internal bending and torsional moments determined from the usual equations of statics, the bimoment $B(t)$ due to a concentrated force P depends in a curved beam not only on the point of application of this force but also on the form of the curved beam axis between the point of application and the point with the coordinate t, which determines the examined section. Thus, for example, in the case of the plane circular, ideal thin-walled "cantilever" shown in Figure 220, the bimoment at the support equals twice the product of the area of the corresponding segment and the vertical force P_Z. For the "cantilever" having a zig-zag shape in its plane, shown in Figure 221, the bimoment at the support must vanish, since the sectorial area in the examined part equals zero. An analogous conclusion is reached for the displacement u away from the plane of the section of the beam.

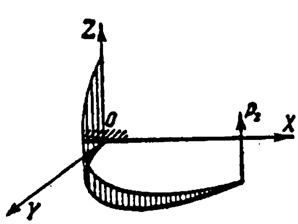


Figure 220

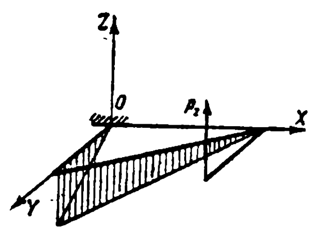


Figure 221

The general expression (I.3.16) can also be extended to plane curved thin-walled beams if the contour line, shown in Figure 220, is understood to represent not the circumference of the cross section but the axis of this beam. The quantity u will represent the displacement of a point of the beam axis from its plane; ξ' and η' are the angles of rotation and θ' is the degree of warping. These are now considered as functions of the coordinate t and not of z.

3. In case the rigidity of the beam in pure torsion GJ_d differs from zero, we shve instead of (6.6) the differential equation

$$-EJ_{\omega}\theta'' + GJ_d\theta = P_{SX}\Omega_{YZ}(t) + P_{SY}\Omega_{ZX}(t) + P_{SZ}\Omega_{XY}(t) + \\ + M_{SX}X(t) + M_{SY}Y(t) + M_{SZ}Z(t) + B_S. \quad (6.7)$$

Together with the boundary conditions, this equation determines the torsion angle $\theta = \theta(t)$ due to concentrated external statical factors, applied at any point O of the beam. We note that over $t_A \leq t \leq t_e$ the right-hand member of (6.7) must vanish.

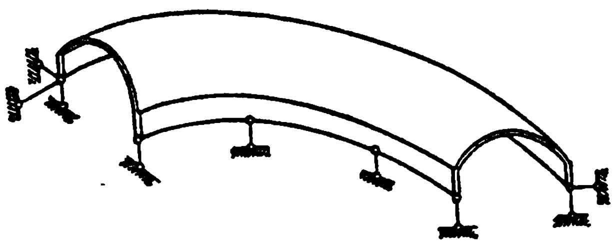


Figure 222

By a certain generalization to statically-indeterminate systems, the fundamental results given here form the basis of the modern theory of the design of plane and space frames consisting of thin-walled beams and curved beams of any kind. By this theory we can design, allowing for sectorial warping, wagon frames composed of channel beams and of any kind of plane and space systems of thin-walled, curved, open beams.

Figure 222 shows, as an example of such a system, the structure of a torus-shaped multispan roof reinforced by curved side-members.

If all three basic dimensions of such a structure are of different orders, it can be designed by the methods of the above bimoment theory of thin-walled curved beams of open section.

The reactions of the imagniary support connections can be found as statically-indeterminate quantities from the condition of the displacements vanishing in the directions of the corresponding supports, in the same way as shown before in § 1, Chapter III for a multispan beam-shell with a straight axis.

A SHORT HISTORICAL SKETCH AND LITERATURE SURVEY

It has been established by theoretical and experimental studies that the distribution of the stresses and strains in a square-end beam under transverse bending depends ont only on the magnitude of the bending moment but also on the position of the plane of action of the external forces (the bending plane). The hypothesis of plane sections, which underlies the elementary theory of bending, holds true only in one particular case of external transverse loading, when the load passes through the so-called shear center.

The departure from the law of plane sections of the classical theory in the case of a transverse load not passing through the shear center was first experimentally observed by Bach in 1909 /211/.

Experimenting with metal channel section beams, Bach established that a transverse load acting perpendicularly to the symmetry plane of the channel and passing through its centroid causes torsional deformations in addition to bending. For a general position of the load the elongations of the four extreme fibers of the channel do not follow the law of plane sections.

The torsional deformations in Bach's experiments were considerably less when the transverse load was made to pass through the axis of the channel web rather than through the centroid. Bach ascribed this departure from the law of plane sections to the asymmetry of the section.

Professor S. P. Timoshenko dealt with the problem of bending and torsion of thin-walled beams in connection with his work on the stability of the plane form of bending of an I-beam/176/. He established experimentally the value of the rigidity in pure torsion of I-beams and studied in detail the problem of torsion in which, beside tangential stresses, normal stresses also appear in the cross sections. The torsion angles measured in Timoshenko's experiments were in good agreement with the theoretical values calculated by his formulas.

No new matter was published on the problem of torsion of a thin-walled beam associated with bending of the separate elements for several years after the works of Bach and Timoshenko.

In 1921, 12 years after Bach's experiments, Maillart published a work /253/ devoted to the problem of bending and torsion of thin-walled metal beams. There he analyzed Bach's experiments, commenting that the departure from the law of plane sections under torsion associated with bending of separate elements can also take place in symmetrical profiles.

In his subsequent papers, published in 1922 and 1924, Maillart gave design data for the determination of the shear center /254/, in addition to the results of experimental studies. He obtained these data by using Timoshenko's method. Following Timoshenko, Maillart determined the shear center as the intersection point of the resultants of the elementary tangential stresses when the beam is bent in its principal planes.

In 1927 appeared S. A. Bernshtein's work /17/ in which the author remarked on the considerable deviation from the law of plane sections of the distribution of normal stresses in the chord cross sections of open bridge trusses and called this phenomenon "deplanation"*.

In the period from 1921 to 1926 the works of Zimmerman /288/, Sonntag, Eggensschwyler /226/ and Weber /285/ were published in the foreign technical literature. The most interesting of these is Weber's work in which the author, beside the method of determination of the shear center, presented a generalization of Timoshenko's results for the torsion of the I-beam and the method of determination of the complementary normal stresses in torsion for various 2-flange sections (I with unequal flanges, channel and Z). In this work, the author paid attention to the connection between the shear center and the torsion center (the cross-section point that remains fixed under torsion). He proved that these two points coincide under torsion associated with bending of the flanges.

The problem of finding the shear center for beams of solid section was dealt with by the following Soviet scientists: Academician B. G. Galerkin /60/, Academician L. S. Leibenzon /111/ and G. E. Proktor /143/. Original studies on the determination of the shear center for solid sections were conducted during the last years by N. Kh. Arutyunyan and N. O. Gulkanyan /7/, and by M. E. Berman /16/.

The departure from the law of plane sections, shown by thin-walled beams in torsion associated with bending of the separate elements, plays an essential role not only in problems of strength but also in problems of stability. Experimental studies, conducted by many workers in the USSR and abroad, showed that in many cases the limiting forms of instability i. e., the forms giving the lowest values for the critical force are the torsional or, in the more general case, the flexural torsional.

Thus, for example, experiments on duraluminum aircraft spars, conducted at TsAGI in 1934 /18/ have shown that rectangular beams of open sections as a rule buckle when twisted and instability occurs for critical forces considerably smaller than the theoretical values given by Euler's expressions.

The German engineer Wagner treated the problem of stability of thin-walled aircraft beams. Jointly with Pretscher he published in 1934 a theoretical work in which they gave equations for the determination of the critical forces for torsional instability of aircraft beams /284/. In the derivation of his expressions for the complementary normal stresses due to torsion, Wagner used a law analogous to the law of sectorial areas introduced by the author in 1936 for profiles of arbitrary form /41/. It should be noted that, in examining torsional deformations, Wagner assumed that at instability the torsion center coincides with the shear center. In reality the torsion center, as was shown by the author's studies, generally does not coincide with the shear center. Coincidence occurs only for one kind of section, namely, when the shear center coincides with the centroid of the section. Wagner's expression is therefore applicable only to beams having two symmetry axes in their section. Apparently the first to have drawn attention to the incorrectness of Wagner's results was Ostenfeld /269/ (1931), who obtained exact solutions for T- and channel sections.

* ["Deplanatsiya" remains the standard Russian term for warping].

The problems of bending, torsion and stability of a beam of polygonal section were examined by Kappus /242/ (1937).

Lundqvist and Fligg /252/ (1937) determined the position of the center of rotation corresponding to minimum critical load.

Among Soviet researchers, the problems of stability of aircraft beams was dealt with by P. M. Znamenskii (in connection with experimental studies at TsAGI, disproving the theory of lateral buckling). In 1934 he published a paper /92/ in which he used the Ritz-Timoshenko method to derive approximate expressions for the critical force for torsion. However, Znamenskii's expressions have the same range of applicability as Wagner's since Znamenskii starts from the assumption that the torsion center coincides at the moment of instability with the shear center.

In 1936, a work by F. and H. Bleich was published which was devoted to the problem of torsion and stability of thin-walled sections /214/. Using the energy method, the authors obtained in this paper a system of three differential equations for the case of axial compression.

However, these authors started from the law of plane sections and replaced the normal stresses in the cross section by their resultant, taken as a concentrated force applied at the centroid. As a result of this substitution the last term in one of the equations, the equilibrium of the beam under rotation about the longitudinal axis (corresponding to our third equation) does not contain the longitudinal force. This led to the loss of one of the three roots of the corresponding solving equation and gave for the two others incorrect results.

In 1931 the author worked out a technical theory of orthotropic cylindrical and prismatical shells of intermediate length with arbitrary cross section. This theory, first published in /38/ and later in the monographs /39, 40/ is based on a representation of shells by spatial systems capable of resisting at every point not only extensions or compressions along two mutually perpendicular directions but also bending in a single transverse direction.

Of the internal forces in the shell the transverse bending moments and forces on the longitudinal sections of the shell are taken into consideration in addition to the normal and shear forces. The longitudinal bending and torsion moments are taken equal to zero since they are quantities playing no essential role in the spatial behavior of the structure. Also discarded are the transverse extensions and the shear deformations of the middle surface corresponding to these moments (in the sense of the mathematical analogy). Under the mentioned statical and geometrical hypotheses the equilibrium problem of an elastic shell is reduced to the integration of a system of two symmetrical partial differential equations for the two required functions: the longitudinal normal stress and the transverse bending moment. Each of these equations is of fourth order in the contour coordinate (transverse) and of second order in the coordinate along the generator. The solution of this two-dimensional boundary-value problem, when applied to a shell roof, is given on the basis of the author's new variational method which allows, in combination with the methods of structural mechanics for statically indeterminate beam systems, the reduction of complicated partial differential equations to a system of ordinary differential equations with a symmetrical matrix.

For a cylindrical shell considered as a spatial system consisting of

an infinite number of narrow longitudinal plates and an infinite number of transverse frame-strips, which we term discrete-continuous, these equations assume an eight-term structure. In accordance with the physical meaning, the eight-term equations of the combined method for a prismatical shell are divided into two groups: equilibrium equations and compatibility equations for the strains. For the integration of the equations of the theory of cylindrical and prismatical shells, the author has worked out a method of expanding the required solutions in eigenfunctions of transverse vibrations of a homogeneous solid beam. It has been thus possible to evolve practical methods for the design of cylindrical and prismatical shells under an arbitrary load and under arbitrary boundary conditions. This theory was afterwards extended by the author to shells with arbitrary multiply connected flexible contour.

Of non-Russian works containing some technical applications of the combined variational method for designing prismatical shells and the application of this method to the design of structural shell-roofs the following may be cited (for example) /223, 272, 274, 279/.

Numerous new practically important problems in construction, in aviation, in ship building and other fields of modern technology have been solved on the basis of our theory for the calculation and design of thin-walled structures. The technical theory of orthotropic shells constitutes a new branch of the structural mechanics of spatial systems.

A wooden ribbed shell structure, unique in its dimensions, was calculated, designed and built according to the methods of the above mentioned theory for the first time in the USSR in 1935. It covered an area of 6,000 m^2 and consisted of two cylindrical shells (100 m $\times$ 30 m each) connected to one another and resting on transverse reinforced concrete frames placed at the end sections at a distance of 100 m from one another.

The design results of this structure, given partially in the doctoral thesis of the author /40/, turned out to be completely unexpected and remarkable in the departure from the conventional precepts of the classical theory of strength of materials and the structural mechanics of beam systems.

Studies have shown that the state of stress of a single-span shell under the action of a transverse load depends essentially on the position of this load in the plane of the cross section. Thus, for example, under a unilateral, vertical, uniformly distributed load applied along a longitudinal edge, the bending stresses determined by the methods of beam theory based on the law of plane sections were accompanied by additional normal stresses in the cross sections of the shell which did not follow this law and were due to the action of an external twisting load. This load per unit length is determined as the torsion moment due to the given distributed vertical load with respect to the line of shear centers. In the present case of a unilateral load, the additional normal stresses considerably exceed the stresses determined by the usual methods of strength of materials which are based on the law of plane sections and which do not allow for the influence of the eccentricity of the transverse load with respect to the vertical symmetry axis. These stresses are responsible for the new mode of behavior of a ribbed shell, which is characterized by flexural torsion. In contrast to pure torsion, examined in Saint-Venant's theory, flexural torsion is associated with the appearance in the shell not only of tangential

but also of normal stresses. The application of the theory of orthotropic cylindrical shells to the structure of a ribbed roof described above has considerably changed the ideas on the behavior of thin-walled beams of channel section and the like, and revealed new factors of spatial deformation connected with the warping of the cross sections mainly due to unilateral asymmetrical loading. The results of the calculation play a significant role in determining the basic dimensions of the structure ensuring its strength. In particular, it has become clear that under a unilateral load large tangential stresses (besides the normal stresses) arise in the shell, attaining maximum values at the extreme upper parts of the ceiling adjoining the skylight. These stresses called for reinforcement of the structure through increasing the thickness of the corresponding parts.

The studies have also shown that for cylindrical open shells of sufficient length, or with sufficiently rigid transverse ribs, the transverse bending deformation connected with the variation in the form of the shell section (due to transverse bending moments) is not significant for the spatial behavior of the structure. This deformation has practically no influence on the magnitude and character of the normal and tangential stress distribution in the cross sections of the shell. Ignoring the small transverse bending deformation or, which is the same, introducing an additional assumption as to the invariability of the section contour, the author has considerably simplified his earlier, more rigorous theory and has obtained in 1935 a new law for the longitudinal deformations and the longitudinal normal stresses and of their distribution over the cross section; this law he termed the l a w o f s e c t o r i a l a r e a s .

The law of sectorial areas, containing as a particular case the Bernoulli-Navier law of plane sections and describing plane flexure as well as spatial flexural-torsional deformation of a shell, was taken as the basis of a practical method of designing ribbed arch-shells and later also as the basis of the general theory of strength, stability and vibrations of thin-walled beams of open section.

Thus, the modern theory of thin-walled beams, initially set forth in the works /41, 42, 43/ and later in the author's general theory, was based on considering the thin-walled beam as a spatial system of the cylindrical or prismatical shell type with rigid section contour. In combination with the variational principles of the mechanics of continua this conception allowed the introduction of the new ideas of generalized coordinates of a warped section and of the flexural-torsional bimoment as a new generalized longitudinal force, corresponding (in the sense of the virtual work) to warping of the cross section and differing in principle from the forces and moments examined in the statics of a solid body. The result turned out to be more general and more fruitful than the conceptions of beam theory and of the classical theory of strength of materials which were the starting point of by and large all the author's predecessors listed above, beginning with Timoshenko.

The scientific and methodological significance of the theory of thin-walled beam-shells lies also in that, resting on hypotheses more general than those of the classical theory bending and rejecting the hypothesis of plane sections and Saint-Venant's principle (which essentially restates it in terms of internal generalized forces and moments), this theory considerably widens the scope of strength of materials.

The mathematics of the author's bimoment theory is based on the reduction to ordinary differential equations with one or several generalized elastic characteristics. A proper choice of the physical assumptions concerning the stress and strain distribution in the plane of the cross section allows then the elaboration of more precise design methods for thin-walled beams and shells of open or closed multiply connected section, as well as for solid beams, taking into account the warping and alteration of the section contour.

It is appropriate to mention here that S. P. Timoshenko published in 1945 an extensive paper on the theory of thin-walled beams /281/ (with a reference to the author's monograph /45/), in which, defining the fundamental problem of flexural torsion of a beam, he also starts from the idea of examining the beam as a shell. This article has been translated into French and published in Belgium /257/ in 1947.

Timoshenko's article, published in Russian as a supplement to the second edition of his book /181/, is in a sense a brief exposition of the principal chapters of our monograph /45/ only in a different notation. The theory of thin-walled beams, following the author's ideas and retaining his terminology (without actually referring to him), has been discussed in a series of other works of foreign authors, for example, Goodier and Beskin /213, 232/.

Of the Soviet works on the theory of thin-walled beams we should mention in the first place the monograph of A. A. Umanskii /182/, who proposed an original method of designing thin-walled beams with an inflexible closed section and examined on the basis of the bimoment theory a series of new problems on the design of plane twin beam structures (which he calls "bistructures").

The book by D. V. Bychkov and A. K. Mroshchinskii /29/ contains a systematic exposition of the theory of thin-walled beams as applied to the design of structural metal baems. Bychkov's book /30/ contains methods of designing frames made of thin-walled beams. The books by B. N. Gorbunov and A. I. Strel'bitskii /73, 74/ examine the problems of the application of our theory to the design of thin-walled wagon frames.

An original exposition of the theory of thin-walled beams, based on the application of the Euler-Lagrange variational equations is given in the work of G. Yu. Dzhanelidze /80/. It is also of theoretical interest, containing, in addition to the variational formulation of our theory, also methods of designing beams of closed section.

In the book by Dzhanelidze and Panovko /82/, besides basic results on the bimoment theory of flexural torsion of solid beams, methods of determining the stresses and deformations of a closed section beam in restrained torsion are also described in considerable detail.

A rigorous mathematical foundation of the physical hypotheses at the basis of the theory of thin-walled beams, and also an examination of the range of applicability of this theory was given in the work of A. L. Gol'denveizer /70/. Special chapters in the doctoral dissertation of Yu. N. Rabotnov /146/ are also devoted to this very important problem.

A large number of works, performed in the laboratories of numerous research institutes and universities have been devoted to the important problem of experimental study of the deformation of thin-walled beams (girders, columns) in flexural torsion and to the experimental verification

of the fundamental assumptions of our theory, especially of the law of sectorial areas. To such works belong, in particular, the studies of N.A. Boloban /18/, D.V. Bychkov /26, 27, 28/, M.I. Dlugach /86/, N.G. Dobudoglo /88, 89, 90/, A.R Rzhanitsyn /153/, S.I. Stel'makh /161, 162/. Yu. I. Yagn /207/ and other authors.

The experiments conducted by K.F. Kovalov /104/, have confirmed the conclusions of the author's general theory of the design of closed beam-shells with allowance for contour deformation /51/. They have shown that restrained torsion of such thin-walled beams is associated, as a rule, with a considerable distortion of the contour. The form of the warped cross section is similar to its form under pure torsion. On the basis of the experimental results, K.F. Kovalov confirms our conclusion of the necessity of allowing for contour deformation in the design of closed thin-walled sections not reinforced by transverse ribs.

The works of G.Yu. Dzhanelidze /81, 82/ and Ya. G. Panovko /133, 134/, in which open and closed beams were examined, contributed to the wide popularization of the bimoment theory of thin-walled beams. Problems of designing thin-walled beams taking shear deformations into account were examined by R.A. Adadurov /1, 2/.

An application of the theory to the strength computation of thin-walled aircraft structures is contained in the works of S.N. Kan /94, 95, 96/ and V.I. Klimov /102/.

Thin-walled slightly conical beams, as used in airplane construction, were studied by L.I. Balabukh /10, 11/ and B.P. Tsibulya /193, 194/.

The theory of thin-walled beams was applied in the design of hydraulic segment gates by B.S. Vasil'kov and I.E. Mileikovskii /34/. Hydraulic plane gates were examined by V.N. Pastushikhin /137/ as thin-walled spatial structures of the prismatical shell type.

The design problems of thin-walled beams with transverse connections were elucidated in the works of G.P. Sobolevskii /160/, M.I. Dlugach /86/, and A.M. Shanshiashvili /204, 205/.

The extension of the bimoment theory of flexural torsion to thin-walled plane curved beams was presented in the works of N.Ya. Gryunberg /76/, G.Yu. Dzhanelidze /81/, A.R. Rzhanitsyn /156/ and A.A. Umanskii /185, 186/.

An extension of this theory to thin-walled spatial beams is given by the author in /48/. It also examines the problem of spatial stability and spatial torsional equilibrium shapes of a thin-walled plane curved beam whose axis has the form of a circular arc.

Of great practical importance is the problem of the spatial design of thin-walled beams beyond the elastic limit and the determination of the crippling load of thin-walled beam structures. This problem has been also studied in the Soviet Union by the methods of the bimoment theory, using the law of sectorial areas.

The problem of designing thin-walled beams beyond the elastic limit was treated in the works of R.A. Mezhlumyan /122/, E.A. Raevskaya /148/, N.D. Rein /151/, A.R. Rzhanitsyn /154/ and A.I. Strel'bitskaya /171, 173/ whose many studies, confirmed by experiment, are of great practical importance. By using, besides the law of sectorial areas, also other geometrical relationships of the author's general theory of warping of shells which serves as the basis for the eight-term equations and starting

from the variational method of reducing the partial differential system to ordinary differential equations, A.R. Rzhanitsyn has examined in the last years a series of problems concerning the design of shells beyond the elastic limit and the determination of their crippling load.

The stability in pure bending of a rectangular beam made of strain-hardened material was studied by Hill and Clark /235/ (1950) and by Wittrick /287/, (1952). The stability of an I-beam in pure bending under the assumption of purely plastic character of the process of buckling was examined by Flint (229) (1953).

Some problems in the stability of elastic-plastic thin-walled beams were studied by L.M. Kachanov /100/ (1956); he started from the assumption of the absence of stress release at the moment of buckling.

Various cases of stability in plane bending according to the theory of deformation and taking into account the stress release of the materials were examined by L.M. Kachanov /99/, (1951) and by K.S. Chobanyan /196/, (1953).

The problem of stability of thin-walled structures is important no less than the problem of strength. The problem of elastic stability of thin-walled beams under various boundary conditions was treated by S.A. Ambartsumyan /5/, A.L. Gol'denveizer /68/, Z.N. Mazurmovich /117/ and by many others.

The stability of the upper chord of an open bridge, whose elements are in the form of thin-walled beams, was examined by P.A. Lukash /113/.

The stability of eccentrically compressed thin-walled beams was studied by G.M. Chuvikin /197, 198, 200/. The application of the stability theory to the design of aircraft structures was carried out by I.F. Obraztsov /129, 130/.

The problem of spatial stability, worked out by the author /45/ for elastic beams (struts, girders), was later developed successfully towards the determination of the critical forces for beams losing their stability under stresses exceeding the elastic limit.

To the works devoted to the stability problems of thin-walled beams beyond the elastic limit belong the investigations of B.M. Broude /22, 23, 24/ who studied the stability of beams beyond the elastic limit under axial and eccentric loading and also the general treatments of instability of beams eccentrically compressed beyond the elastic limit by L. B. Bunatyan /25/, A.V. Gemmerling and G.V. Longvinovich /64/, R.A. Mezhlumyan /122/, V.V. Pinadzhyan /139, 140/, A.R. Rzhanitsyn /157/, and others. Extensive experimental studies on the stability of beams and columns beyond the elastic limit were conducted by G.M. Chuvikin /24, 199/.

Of foreign studies, /219, 235/ are devoted to the problem of designing thin-walled beams beyond the elastic limit.

The study of the behavior of thin-walled structures under the action of dynamic loads also is of special interest.

The problems of designing thin-walled beams and ribbed arch-shells for natural vibrations and transient loads was dealt with by B.M. Terenin /47, 50/. The practically important problem of the free and forced vibrations of flat hydraulic gates was solved by V.N. Pastushikhin /137/.

A study on the stresses and deformations in thin-walled beams due to a mobile load was made by V.G. Aleksandrov /3/.

The problem of dynamic stability and spatial flexural-torsional vibration in the wreckage of the Tacoma bridge was examined on the basis of the general equations of our theory in the doctoral theses of V.V. Bolotin /19/, I.I. Gol'denblat /67/ and F.D. Dmitriev /87/.

Of great theoretical and practical importance is also the problem of spatial design of thin-walled flexible beams undergoing finite deformations. This problem is also being successfully tackled in the Soviet Union, in which the original studies of L.P. Kobets /103/ and S.P. Vyaz'menskii /58/ are to be especially noted.

Of practical interest is the method of deformational design of beams in combined compression and flexure, worked out on the basis of the author's theory by Yu.D. Kopeikin /105/, A.A. Pikovskii and K. A. Mikhailichenko /138/.

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Explanatory List of Russian Abbreviations Appearing in the Text and Bibliography

Abbreviation	Full name (transliterated)	Translation
AN SSSR	Akademiya Nauk SSSR	Academy of Sciences of the USSR
AN USSR	Akademiya Nauk Ukrainskoi SSR	Academy of Sciences of the Ukrainian SSR
DAN SSSR	Doklady Akademii Nauk SSSR	Reports of the Academy of Sciences of the USSR
Gosstroiizdat	Gosudarstvennoe izdatel'stvo stroitel'noi literatury	State Publishing House of Literature on Building
Gostekhizdat	Gosudarstvennoe izdatel'stvo tekhno-teoreticheskoi literatury	State Publishing House of Theoretical and Technical Literature
GPI	Gosudarstvennyi proektnyi institut	State Design and Planning Institute
LGU	Leningradskii Gosudarstvennyi Universitet	Leningrad State University
MAI	Moskovskii aviatsionnyi ordena Lenina institut (im. S. Ordzhonikidze)	Moscow "Order of Lenin" Aviation Institute (im. S. Ordzonikidze)
Mashgiz	Gosudarstvennoe nauchno-tekhnicheskoe izdatel'stvo mashinostroitel'noi literatury	State Scientific and Technical Publishing House of Literature on Machinery
MEMIIT	Moskovskii elektromekhanicheskii institut inzhenerov zheleznodorozhnogo transporta	Moscow Electromechanical Institute of Railroad Engineers
MISI	Moskovskii ordena Trudovogo Krasnogo Znameni inzhenerno-stroitel'nyi institut (im. V.V. Kuibysheva)	Moscow "Order of Red Banner of Labor" Civil Engineering Institute (im. V.V. Kuibyshev)
Oborongiz	Gosudarstvennoe izdatel'stvo oboronnoi promyshlennosti	State Publishing House of the Defense Industry

ONTI	Ob"edinenie nauchno-tekhnicheskikh izdatel'stv	United Scientific and Technical Publishing Houses
OTN	Otdelenie tekhnicheskikh nauk (Akademii Nauk SSSR)	Department of Technical Sciences (of the Academy of Sciences of the USSR)
Proektstal'konstruktsiya	Gosudarstvennyi institut po proektirovaniyu, issledovaniyu i ispytaniyu stal'nykh konstruktsii i mostov	State Institute for the Design, Study and Testing of Fabricated Steel and Bridges
Stroiizdat	Gosudarstvennoe izdatel'stvo literatury po stroitel'stvu i arkhitekture	State Publishing House for Literature on Construction and Architecture
Sudpromgiz	Gosudarstvennoe soyuznoe izdatel'stvo sudostroitel'noi promyshlennosti	State All-Union Publishing House of the Shipbuilding Industry
Transpechat'	Upravlenie po rasprostraneniyu i ekspedirovaniyu pechati Transzheldorizdata	Administration for the Distribution of Printed Matter of the Transheldorizdat
Tranzheldorizdat	Gosudarstvennoe transportnoe zheleznodorozhnoe izdatel'stvo	State Publishing House for Railroad Transportation Literature
TsAGI	Tsentral'nyi aero-gidrodinamicheskii institut im. N. E. Zhukovskogo	Central Aero-Hydrodynamical Institute im. N. E. Zhukovskii
TsNIPS	Tsentral'nyi nauchno-issledovatel'skii institut promyshlennykh sooruzhenii	Central Scientific Research Institute of Industrial Structures
VIA	Voenno-inzhenernaya akademiya	Military Engineering Academy
VIA RKKA	Voenno-inzhenernaya akademiya Raboche-Krest'yanskoi Krasnoi Armii	Military Engineering Academy of the Workers' and Peasants' Red Army
VMF	Voenno-morskoi flot	Navy
VVA	Voenno-vozdushnaya akademiya	Air Force Academy
VVIA	Voenno-vozdushnaya inzhenernaya akademiya	Air Force Engineering Academy